For the ρ pole, the residue is given by

$$-\frac{2\lambda_1}{\alpha'}S_\mu p_{3\mu}(\alpha_{23}-\alpha_{31})-\frac{4\lambda_1'}{\alpha'}S_\mu(p_2-p_1)_\mu=+4\lambda_1 S_\mu p_{3\mu}p_3\cdot(p_2-p_1)-\frac{4\lambda_1'}{\alpha'}S_\mu(p_2-p_1)_\mu. \tag{A10}$$

For the two independent $A\rho\pi$ couplings, we take

$$(g_1 S_\mu \rho_\mu \times g_2 S_\mu p_{3\mu} p_{S\nu}\rho_\nu)(\boldsymbol{\eta}_\rho\cdot\boldsymbol{\eta}_S\times\boldsymbol{\eta}_3). \tag{A11}$$

Using (A11) and (A4), we calculate the contribution of the ρ pole to $\pi\pi\to\pi A_1$ and find for the residue of the ρ pole

$$+2g_1 g_{\rho\pi\pi}S_\mu(p_2-p_1)_\mu+2g_2 g_{\rho\pi\pi}S_\mu p_{3\mu}p_3\cdot(p_2-p_1). \tag{A12}$$

Comparing (A10) and (A12), we see that

$$\lambda_1=\tfrac{1}{2}g_2 g_{\rho\pi\pi},\quad \lambda_1'=-\tfrac{1}{2}\alpha' g_1 g_{\rho\pi\pi}. \tag{A13}$$

PHYSICAL REVIEW VOLUME 182, NUMBER 5 25 JUNE 1969

Pion Gauge Invariance and Low-Energy Theorems*

G. KRAMER† AND W. F. PALMER

High Energy Physics Division, Argonne National Laboratory, Argonne, Illinois 60439

(Received 23 December 1968)

Zero-mass pion theories invariant under c-number translations ("chiral transformations") of the pion field are studied in a general framework. The operator which induces the chiral transformation is defined in Fock space (in which it is not unitary) and in von Neumann's infinite-tensor-product space (in which it is unitary). The transformed (noninvariant) Fock-space vacuum is recognized as a coherent state in the tensor-product space. The generator of the chiral transformation—a constant of the motion in gauge-invariant theories—is diagonalized, and its eigenvectors, the "chiral states," are employed in one of two derivations of a low-energy theorem for zero-mass pion emission and absorption, assuming gauge invariance of the theory. The other method of derivation is also used to rederive the electromagnetic gauge conditions. Then Lagrangian models (gradient-coupling, c-number, and operator theory) are studied in which the invariance is realized provided the current is suitably restricted. Implications of the low-energy theorem are checked (exactly for the c-number theory, in lowest-order perturbation theory for the operator theory). A larger class of models is then considered in which, it is shown, the complicated set of transformations under which the Lagrangian is invariant reduce, by virtue of the field equations and the asymptotic condition, to a simple pion translation when expressed in terms of the asymptotic fields, and hence obey the supposition of our theorem, which we again check in lowest-order perturbation theory.

1. INTRODUCTION

In quantum electrodynamics, the invariance of the vector potential against local guage transformations $A_{\mu\,\mathrm{in\,(out)}}(x)\to A_{\mu\,\mathrm{in\,(out)}}(x)+\partial_\mu\Lambda(x)$ is necessary because only then is the theory a Lorentz-invariant description of zero-mass (spin-one) particles.[1] A gauge principal for pion interactions, $\phi_{\mathrm{in\,(out)}}\to\phi_{\mathrm{in\,(out)}}+c$, however, is apparently neither natural nor necessary. No classical limit exists (as in quantum electrodynamics) which guides one to such an invariance; moreover the invariance requires pions of zero mass and is therefore physically interesting only when it is broken. However, many theories of current interest may be pion gauge-invariant in the limit of vanishing pion mass. We have in mind the various phenomenological Lagrangians of the past few years, which are invariant under a set of transformations (of the interacting fields) which contains $\phi\to\phi+c$. Thus, we expect amplitudes calculated from such Lagrangians to obey pion gauge conditions in the limit of zero-pion mass. Since the Lagrangians (which appear to incorporate the current algebra results) include many ingredients beside pion gauge invariance in the zero-mass limit, it is interesting to determine which if any of their predictions are due solely to the pion gauge conditions. The Adler[2] consistency condition, for example, requires that the πN forward scattering amplitude vanishes when one of the pion's four-momentum goes to zero. This result is essentially based on the hypothesis of partially con-

* Work performed under the auspices of the U. S. Atomic Energy Commission.

† Permanent address: University of Hamburg, Hamburg, Germany.

[1] D. Zwanziger, Phys. Rev. **133**, B1036 (1964); S. Weinberg, *ibid.* **135**, B1049 (1964); **138**, B988 (1965).

[2] Stephen L. Adler, Phys. Rev. **137**, B1022 (1965); **139**, B1638 (1965).

served axial-vector current (PCAC), which demands $m_\pi \neq 0$. It is tempting to regard this condition as the remnant of a guage condition imposed upon the theory at an earlier stage (before symmetry breaking) when $m_\pi = 0$. We cannot make this connection precise without a dynamical framework which prescribes the dependence of the amplitudes on the pion mass. However, that the Adler consistency condition formulated for *zero-mass pion theories* is connected with pion gauge invariance is already apparent in the early work of Nishijima,[3] where the vanishing of amplitudes at a pion threshold was essentially derived from the pion-translation invariance of a massless field. Recently both Hamilton and Martinis[4] have studied low-energy theorems from the pion-gauge point of view and come to the same conclusion, i.e., that any transition matrix element involving connected zero-mass pions must vanish in the limit of zero pion four-momentum if the S matrix is pion-gauge-invariant. Thus, we are motivated to study the underlying nature of the invariance, which, unlike most symmetries (which are homogeneous linear transformations) does not leave the vacuum invariant and is not implemented by a unitary Fock-space operator. By considering the commutator of the S matrix with the generator χ of the pion gauge transformation and by expanding the S matrix in improper ("chiral") eigenstates of χ, we are led to two proofs of the gauge condition which we believe are more transparent than existing derivations and which exhaust the implications of the pion gauge invariance.

The chiral states which emerge in this approach are interesting in their own light and are related to the coherent states[5] first applied in the context of quantum optics. Our investigation of the displaced vacuums (which are coherent states generated from the Fock vacuum by the action of the operator $e^{i\chi}$ which implements the gauge translation) has been made possible by recent advances in the same context.[6] To make the pion gauge condition more plausible we apply the same methods to the derivation of the gauge condition in quantum electrodynamics.

We do not consider here the second main result of the current algebra, the Adler-Weisberger consistency condition, which is a statement of the universality of the constant multiplying the leading power of ν in an expansion of the amplitude for $\pi X \to \pi X$. This universality is certainly not a consequence of pion guage invariance alone. In fact, nowhere do we assume that the amplitude for zero-mass pion emission or absorption has an expansion about $\mathbf{k}=0$. In a true zero-mass pion theory this point is above a real or virtual threshold and such an expansion probably does not exist. In this respect, we find the results of Hamilton[4] unsatisfactory because they depend on an artificial distinction between zero-mass "external" pions and massive "internal" pions. Some of the extrapolation procedures of current algebra calculations may suffer similar defects although they are more adroitly concealed. We can say nothing from gauge invariance alone about the functional nature of the approach to zero of soft-pion amplitudes; this is a detailed result of the behavior of the current for vanishing four-momentum. In this context we remark that the gauge invariance does place a limit on the strength of any singularities in a model for the effective current; without this restriction, for example, the interaction term $\partial^\mu \phi j_\mu(x)$, where $j_\mu(x)$ is a prescribed c-number current is only formally gauge-invariant. One must be careful, therefore, when applying the gauge conditions to effective Lagrangians in which the current is essentially approximated by a perturbation calculation which may not satisfy this restriction. It is, of course, impossible to say, in the framework of these Lagrangians, if the "exact" current obeys the restriction since this requires a knowledge of the current in terms of the asymptotic fields which is tantamount to a complete solution to the problem (in which case gauge conditions would be unnecessary).

Finally, we remark that many zero-mass pion theories are not invariant against gauge translations of the pion interpolating field alone but require simultaneous transformations of the other interpolating fields in the theory (unlike the gradient coupling models of pion interactions, in which, for example, the nucleon fields are not transformed). The invariance in these theories is analogous to electromagnetic gauge invariance of the second kind in which the vector potential undergoes the inhomogeneous transformation and the charged fields the homogeneous transformations. However, for some of these zero-mass pion theories, the set of transformations reduce, when expressed in terms of the asymptotic fields, to a simple translation of the asymptotic pion field, all other fields going into themselves, that is, the generator of the transformation when evaluated in terms of the asymptotic fields contains only a pion part. These theories also obey the suppositions of our proof and hence obey the low-energy theorem for zero-mass pion processes.

2. PROPERTIES OF GAUGE TRANSLATIONS OF THE FIRST KIND

We consider theories invariant against arbitrary independent c-number translations of a set of pseudoscalar fields ϕ_α:

$$\phi_\alpha \to \phi_\alpha + c_\alpha = \phi_\alpha{}' . \tag{2.1}$$

Examples are the field theory of free zero-mass pions and the gradient coupling model of pion-nucleon scattering [in which the ϕ_α ($\alpha = 1, 2, 3$) are the Hermitian

[3] K. Nishijima, Nuovo Cimento **11**, 698 (1959).

[4] J. Hamilton, Nucl. Phys. **B1**, 449 (1967); M. Martinis, Nuovo Cimento **56A**, 935 (1968).

[5] J. R. Klauder and E. C. G. Sudarshan, *Fundamentals of Quantum Optics* (W. A. Benjamin, Inc., New York, 1968), Chap. 7 and the references quoted in this book.

[6] J. R. Klauder and J. McKenna, J. Math. Phys. **6**, 68 (1965); J. R. Klauder, J. McKenna, and E. J. Woods, *ibid.* **7**, 822 (1966); T. W. B. Kibble, *ibid.* **9**, 315 (1968).

pion fields with zero mass] and the interaction of zero-mass pions with a c-number four-vector current. Counter examples are the gradient coupling model with pion mass terms, the γ_5-coupling model of pions and nucleons, with or without mass terms, and the usual electromagnetic interaction of charged pions, which is invariant against electromagnetic gauge transformations of the second kind but not against pion gauge transformations of the charged pions whether or not they have zero mass. This is not to say that pion gauge (translation) invariance is in conflict with charge conservation or isospin invariance; these depend only on the invariance of the theory against charge and isospin transformations of the first kind. For example, the charge- and isospin-conserving-gradient coupling model has pion gauge translation invariance, for the charged as well as the neutral pions, as long as they have zero mass.

For definiteness we concentrate on pion gauge transformations. Then the translation (2.1), restricted to asymptotic pion fields, is implemented by the formally unitary, linear but inhomogeneous transformation

$$U^\dagger(c_\alpha)\phi_\alpha(x)U(c_\alpha)=\phi_\alpha(x)+c_\alpha\,,$$
$$U(c_\alpha)=\exp\left(-ic_\alpha\int d^3x\ \partial_0\phi_\alpha(x)\right)\equiv e^{-ic_\alpha\chi_\alpha}\,, \quad (2.2)$$

where the generator χ_α is the space integral of the current $\partial_\mu\phi_\alpha(x)$. Since

$$\langle 0|U^\dagger(c_\alpha)\phi_\alpha(x)U(c_\alpha)|0\rangle=\langle 0|\phi_\alpha(x)+c_\alpha|0\rangle$$
$$=c_\alpha\neq\langle 0|\phi_\alpha(x)|0\rangle\,, \quad (2.3)$$

$U(c_\alpha)$ does not leave the Fock vacuum invariant. The translated vacuums $|c_\alpha\rangle\equiv U(c_\alpha)|0\rangle$ are "orthogonal" (in a sense to be made precise in the next section) to the Fock vacuum and lie in a continuum of spaces in each of which there is a representation of the commutation relations unitarily inequivalent to the Fock representation. [There is, however, a larger space (than that which carries the usual Fock representation) on which $U(c_\alpha)$ is unitary, namely, the nonseparable infinite tensor product space of von Neumann.[7]] These assertions will be justified in Sec. III. It is clear by now, however, that the gauge translation invariance is not a symmetry in the usual sense.[8] (In the usual terminology it is "spontaneously" broken, although this language seems to us misleading.) Nonetheless, the invariance, an automorphism of the field algebra, has physical consequences that can be extracted much the same way as results of the usual symmetries (which are unitarily implemented and leave the vacuum invariant) are extracted, that is, by expanding the external states in eigenstates of the conserved operators and demanding that the S matrix be diagonal in this representation. To this end we shall in Sec. 3 diagonalize the generators χ_α and establish an expansion of Fock-space states in terms of their eigenvectors. First, however, we recall the following well-known properties of the χ_α:

$$[\chi_\alpha,\chi_\beta]=0\,, \quad (2.4)$$
$$[T_\alpha,\chi_\beta]=i\epsilon_{\alpha\beta\gamma}\chi_\gamma\,, \quad (2.5)$$
$$\{\mathcal{P},\chi_\alpha\}_+=0\,, \quad (2.6)$$

where T_α is the generator of isospin rotations and $\mathcal{P}$ is the parity operator. We shall call χ_α the pion chirality and refer to its eigenstates as chiral states. (Since chirality is often associated with the commutation relations of "chiral" $SU(2)$, $[\chi_\alpha,\chi_\beta]=i\epsilon_{\alpha\beta\gamma}\chi_\gamma$, we repeat that we are for the moment treating only pion chirality and theories invariant against c-number displacements of the pion field; for such theories the χ_α commute and may be simultaneously diagonalized.) It is clear from these commutation relations that the transformed vacuums $U(c_\alpha)|0\rangle$ are not charge states (except when $\alpha=3$) and do not have definite parity. As we shall see, they are degenerate ground states composed of coherent combinations of zero- four-momentum pions.

In what follows we shall use covariant normalization for the annihilation-creation operators, that is,

$$\phi(x)=\frac{1}{[2(2\pi)^3]^{1/2}}\int\frac{d^3k}{k_0}[a(\mathbf{k})e^{-ik\cdot x}+a^\dagger(\mathbf{k})e^{ik\cdot x}]\,, \quad (2.7)$$
$$[a(\mathbf{k}),a^\dagger(\mathbf{k}')]=k_0\delta^3(\mathbf{k}-\mathbf{k}')\,, \quad (2.8)$$

where we drop isospin indices for the time being. Since the gauge translation (2.1) induces on the annihilation operators the singular transformation

$$a(\mathbf{k})\to a(\mathbf{k})+\tfrac{1}{2}c[2(2\pi)^3]^{1/2}k_0\delta^3(\mathbf{k})\,, \quad (2.9)$$

it is a mathematical convenience to use, instead of sharp momentum states, a complete set of orthonormal solutions of the Klein-Gordon equation

$$\Box f_i(x)=0\,,$$
$$(f_i,f_j)=i\int d^3x\ f_i^*(x)\overleftrightarrow{\partial_0}f_j(x)=\delta_{ij} \quad (2.10)$$

with Fourier transform

$$f_i(x)=\frac{2}{[2(2\pi)^3]^{1/2}}\int d^4k\ \theta(k)\delta(k^2)e^{-ik\cdot x}\tilde{f}_i(\mathbf{k}) \quad (2.11)$$

and momentum-space orthonormality and completeness relations

$$\int\frac{d^3k}{k_0}\tilde{f}_i^*(\mathbf{k})\tilde{f}_j(\mathbf{k})=\delta_{ij}\,,$$
$$\sum_i\tilde{f}_i^*(\mathbf{k})\tilde{f}_i(\mathbf{k}')=k_0\delta^3(\mathbf{k}-\mathbf{k}')\,. \quad (2.12)$$

[7] J. von Neumann, Compositio Math. 6, 1 (1938).

[8] R. F. Streater, Proc. Roy. Soc. (London) A287, 510 (1967), and references therein.

Then
$$\phi(x)=\sum_i [f_i(x)a_i+f_i^*(x)a_i^\dagger], \tag{2.13}$$
where
$$\sum_i \tilde{f}_i(\mathbf{k})a_i=a(\mathbf{k}),$$
$$[a_i,a_j^\dagger]=\delta_{ij}. \tag{2.14}$$

Then the transformation corresponding to (2.9) is
$$a_i\rightarrow a_i+\tfrac{1}{2}c[2(2\pi)^3]^{1/2}\tilde{f}_i(0), \tag{2.15}$$
where from now on we assume (without loss of generality) that the $\tilde{f}_i(\mathbf{k})$ may be chosen to be real. We thus have the following representations of the chirality operators
$$\chi=\int d^3x\,\partial_0\phi(x)=(2\pi)^{3/2}i[a^\dagger(0)-a(0)]/\sqrt{2}, \tag{2.16a}$$
$$\begin{aligned}\chi&=(2\pi)^{3/2}\sum_i i\tilde{f}_i(0)(a_i^\dagger-a_i)/\sqrt{2}\\ &\equiv(2\pi)^{3/2}\sum_i \tilde{f}_i(0)\chi_i,\end{aligned} \tag{2.16b}$$
where
$$\chi_i=i(a_i^\dagger-a_i)/\sqrt{2}. \tag{2.17}$$

Now we consider a theory in which the T matrix, $T=i(1-S)$, is invariant against the transformation $U(c)=e^{-ic\cdot\chi}$. Then, if the states $\langle f|$ and $|i\rangle$ contain no zero-four-momentum pions, we have
$$\begin{aligned}\langle f|T|i\rangle=\langle f|U^\dagger(c)TU(c)|i\rangle&\equiv{}_c\langle f|T|i\rangle_c\\ &\equiv\langle f,\pi_c|T|i,\pi_c\rangle,\end{aligned} \tag{2.18}$$
where $|i\rangle_c\equiv U(c)|i\rangle=\{i\}U(c)|0\rangle=\{i\}|c\rangle$, $|c\rangle=\pi_c|0\rangle=|\pi_c\rangle$, $\{i\}$ represents the product of operators which create the state $|i\rangle$ out of the vacuum $|0\rangle$, and $|\pi_c\rangle$ is a coherent state of zero-four-momentum pions to be defined in Sec 3. By hypothesis $\{i\}$ contains no zero-four-momentum pions and hence is invariant under $U(c)$. Therefore, when calculating T-matrix elements between states with no zero-four-momentum pions, it is irrelevant which of the transformed vacuums we use; or, equivalently, a coherent state of zero-four-momentum pions π_c may be added to the initial and final state built on the untransformed vacuums without altering the matrix element.

If the initial or final state contain a zero-four-momentum pion (in the limit), then the equivalent result is
$$\begin{aligned}\langle f|T|i,a^\dagger(\mathbf{k})\rangle&={}_c\langle f|T|i,a^\dagger(\mathbf{k})\rangle_c-\tfrac{1}{2}c[2(2\pi)^3]^{1/2}k_0\delta^3(\mathbf{k})\\&\qquad\times{}_c\langle f|T|i\rangle_c\\&={}_c\langle f|T|i,a^\dagger(\mathbf{k})\rangle_c-\tfrac{1}{2}c[2(2\pi)^3]^{1/2}k_0\delta^3(\mathbf{k})\\&\qquad\times\langle f|T|i\rangle,\end{aligned} \tag{2.19}$$
where we have used the result of (2.18). To investigate (2.19) we construct a normalized wave packet of pion states
$$g_\pi|0\rangle=|g_\pi\rangle=\int d^3k\,\frac{g(\mathbf{k})}{\sqrt{k_0}}a(\mathbf{k})|0\rangle,$$
$$\int d^3k|g(\mathbf{k})|^2=1\Rightarrow\langle g_\pi|g_\pi\rangle=1. \tag{2.20}$$

Then, we have
$$\begin{aligned}\langle f|T|i,g_\pi\rangle={}_c\langle f|T|i,g_\pi\rangle_c-\tfrac{1}{2}c[2(2\pi)^3]^{1/2}\langle f|T|i\rangle\\ \times\lim_{\mathbf{k}\to 0}(\sqrt{k_0})g(\mathbf{k}).\end{aligned} \tag{2.21}$$

Hence, the absorption matrix element for a pion state with wave function $g(\mathbf{k})/\sqrt{k_0}$ is independent of the vacuum used to build the external states, provided $\lim_{\mathbf{k}\to 0}(\sqrt{k_0})g(\mathbf{k})=0$. Such states are not gauge-invariant but are the result of gauge-invariant operators acting on the noninvariant vacuum. If we restrict the space of allowable pion wave functions to those which have the property $\lim_{\mathbf{k}\to 0}(\sqrt{k_0})g(\mathbf{k})=0$, then in this subspace (which specifically excludes pions with momenta sharp about $\mathbf{k}=0$) the results of pion gauge invariance are trivial: Matrix elements are independent of the vacuums $|0\rangle_c$; the mathematically inequivalent vacuums are physically equivalent. The invariance is therefore interesting only for pions in the limit of vanishing four-momenta. We show below that the pion gauge invariance implies that all transition matrix elements vanish when such zero-four-momentum pions are involved as external states. We shall do this in two ways, first by an expansion in momentum space of the S operator in terms of n-point functions of the pion fields and second by an expansion in terms of chiral states of the initial and final pions. For this purpose we devote Sec. 3 to the diagonalization of χ.

3. CHIRAL EIGENSTATES

We initially restrict ourselves to a single mode $\chi_i=i(a_i^\dagger-a_i)/\sqrt{2}$ and diagonalize χ_i in the subspace H_i of pions with momentum distribution $\tilde{f}_i(\mathbf{k})$. Since the χ_i mutually commute, the finite tensor product $|\lambda_1\rangle|\lambda_2\rangle\cdots|\lambda_N\rangle$ is an eigenvector of
$$\sum_i^N c_i\chi_i$$
in the space
$$H\otimes^N=\prod_i^N H_i\otimes$$
with eigenvalue
$$\sum_i^N c_i\lambda_i$$
if $\chi_i|\lambda_i\rangle=\lambda_i|\lambda_i\rangle$. The nontrivial extension to a system with infinitely many degrees of freedom (for example,

a field theory) will be considered at the end. For the moment, then, we drop the mode index (i). Most of the results of this section are, in one context or another, well known.

For generality we consider the two combinations

$$\chi_q=(a^\dagger+a)/\sqrt{2}\,,\quad \chi_p=i(a^\dagger-a)/\sqrt{2}\,. \tag{3.1}$$

Then we have

$$\begin{aligned}&[\chi_q,\chi_p]=i\,, \qquad N=a^\dagger a=\tfrac{1}{2}(\chi_p{}^2+\chi_q{}^2)\,,\\ &[N,\chi_q]=-i\chi_p\,,\quad [N,\chi_p]=i\chi_q\,,\end{aligned} \tag{3.2}$$

where the purpose of the notation is to suggest the result, namely, the eigenvectors $|\lambda_q\rangle$ and $|\lambda_p\rangle$, with

$$\begin{aligned}\chi_q|\lambda_q\rangle=\lambda_q|\lambda_q\rangle\,,\\ \chi_p|\lambda_p\rangle=\lambda_p|\lambda_p\rangle\,,\end{aligned} \tag{3.3}$$

are simply the coordinate and momentum eigenstates of a harmonic oscillator with a single mode. In terms of the number state $|n\rangle=(a^\dagger)^n|0\rangle/\sqrt{(n!)}$, we have

$$|\lambda_q\rangle=\sum_n|n\rangle\langle n|\lambda_q\rangle=\sum_n\frac{e^{-\lambda_q{}^2/2}H_n(\lambda_q)}{(2^n n!\sqrt{\pi})^{1/2}}|n\rangle\,, \tag{3.4a}$$

$$|\lambda_p\rangle=\sum_n|n\rangle\langle n|\lambda_p\rangle=\sum_n(i)^n\frac{e^{-\lambda_p{}^2/2}H_n(\lambda_p)}{(2^n n!\sqrt{\pi})^{1/2}}|n\rangle\,, \tag{3.4b}$$

where $H_n(\lambda)$ are the Hermite polynomials with normalization $H_0(\lambda)=1$. From the classical theory we know we have orthonormality and completeness in the improper sense

$$\begin{aligned}\langle\lambda|\lambda'\rangle=\delta(\lambda-\lambda')\,,\\ \int d\lambda|\lambda\rangle\langle\lambda|=1\,,\end{aligned} \tag{3.5}$$

where, when statements apply to both the $|\lambda_p\rangle$ and $|\lambda_q\rangle$, we drop the subscripts. The states $|\lambda\rangle$ form an improper basis in the Hilbert space of square integrable functions defined on the real line with scalar product

$$\langle f|g\rangle=\int d\lambda\, f^*(\lambda)g(\lambda)\,. \tag{3.6}$$

In this space the expansion coefficients $\langle n|\lambda\rangle\equiv f_n(\lambda)$ form a (proper) orthonormal basis. The sets $\langle f|\lambda_p\rangle$ and $\langle f|\lambda_q\rangle$ are Fourier transforms of each other. In the space of $f(\lambda_q)$, χ_q is represented by multiplication by λ_q and χ_p by the operation $-i\partial/\partial\lambda_q$. In short, the representation space $f(\lambda)$ in which the chiral eigenstates form an improper basis is simply the usual Schrödinger representation space of the canonical commutation relations of a single mode.[9]

To study the displaced vacuums, which we have asserted are coherent states of zero-four-momentum pions, it is useful to define a general Hermitian linear combination of the annihilation and creation operators

$$\chi_\theta=(e^{i\theta}a^\dagger+e^{-i\theta}a)/\sqrt{2}=\cos\theta\,\chi_q+\sin\theta\,\chi_p\,. \tag{3.7}$$

An elementary calculation yields the eigenvector $|\lambda_\theta\rangle$ of χ_θ,

$$|\lambda_\theta\rangle=\sum_n\frac{1}{(2^n n!\sqrt{\pi})^{1/2}}e^{in\theta}H_n(\lambda_\theta)e^{-\lambda_\theta{}^2/2}|n\rangle\,, \tag{3.8}$$

which reduces to $|\lambda_q\rangle$ and $|\lambda_p\rangle$ when $\theta=0$ and $\frac{1}{2}\pi$, respectively. There is a close connection between the general chiral eigenstates $|\lambda_\theta\rangle$ and the coherent states $|\alpha\rangle\equiv|\rho e^{i\theta}\rangle\equiv|x,y\rangle$ defined by

$$a|\alpha\rangle=\alpha|\alpha\rangle\,, \tag{3.9}$$

$$|\alpha\rangle=\sum_n\frac{\alpha^n e^{-|\alpha|^2/2}|n\rangle}{\sqrt{(n!)}}\,. \tag{3.10}$$

The coherent states are generated by the action of the unitary displacement operator

$$D(\alpha)=\exp(\alpha a^\dagger-\alpha^* a)=\exp(-i\rho\sqrt{2}\chi_{\theta+\pi/2})\,, \tag{3.11}$$

where

$$D^{-1}(\alpha)aD(\alpha)=a+\alpha \tag{3.12}$$

and

$$|\alpha\rangle=D(\alpha)|0\rangle\,. \tag{3.13}$$

Hence the chiral states $|\lambda_{\theta+\pi/2}\rangle$ are eigenstates of the displacement operator $D(\rho,\theta)$ with eigenvalue $\exp(-i\rho\sqrt{2}\lambda_{\theta+\pi/2}$. The χ_q and χ_p generate imaginary and real displacements, respectively,

$$\begin{aligned}e^{-i\rho\sqrt{2}\chi_q}|0\rangle=|0,-\rho\rangle\,,\\ e^{-i\rho\sqrt{2}\chi_q}|0\rangle=|\rho,0\rangle\end{aligned} \tag{3.14}$$

and χ_θ is invariant against displacements along rays with phase $\theta+\frac{1}{2}\pi$, e.g., $[D(x,0),\chi_p]=0$. An elementary calculation yields the expansion of the $|\lambda\rangle$ states in the coherent-state representation, e.g., for $|\lambda_p\rangle$ we have

$$|\lambda_p\rangle=\frac{1}{\pi}\int d^2\alpha|\alpha\rangle\langle\alpha|\lambda_p\rangle$$

and

$$\langle\alpha|\lambda_p\rangle=\frac{1}{\sqrt[4]{\pi}}\exp(i\lambda_p\sqrt{2}\alpha^*-\tfrac{1}{2}|\alpha|^2-\tfrac{1}{2}\alpha^{*2}-\tfrac{1}{2}\lambda_p{}^2)\,. \tag{3.15}$$

Finally, we note that the chiral states can be represented by integrals over the real and imaginary parts of the coherent states:

$$\begin{aligned}|\lambda_q\rangle=\frac{1}{\sqrt{2}}\pi^{-3/4}e^{-\lambda_q{}^2/2}\int dy\left|\frac{\lambda_q}{\sqrt{2}},y\right\rangle,\\ |\lambda_p\rangle=\frac{1}{\sqrt{2}}\pi^{-3/4}e^{-\lambda_p{}^2/2}\int dx\left|x,\frac{\lambda_p}{\sqrt{2}}\right\rangle.\end{aligned} \tag{3.16}$$

[9] J. von Neumann, *Mathematische Grundlagen der Quantenmechanik* (Dover Publications, Inc., New York, 1943).

Thus, within the Hilbert space H_i of a single mode, the gauge transformation has familiar and elementary properties. A vector $|f\rangle \in H_i$ can be represented alternatively by a square summable function $\langle n|f\rangle$ defined on the positive integers, a square integrable function $\langle\lambda|f\rangle$ defined on the real line, or a square integrable function $\langle\alpha|f\rangle$ defined on the complex plane. Within the subspace H_i the gauge transformation induces the translation (2.15), which is unitarily implemented by the displacement

$$U_i(\tfrac{1}{2}c[2(2\pi)^3]^{1/2}\tilde{f}_i(0))=\exp[-ic(2\pi)^3\tilde{f}_i(0)\chi_i], \quad (3.17)$$

whose action on the subvacuum $|0_i\rangle$ (belonging to H_i) is to create a coherent state $|\alpha_i\rangle \in H_i$ with $\alpha_i = c[\frac{1}{2}(2\pi)^3]^{1/2}\tilde{f}_i(0)$. In H_i, the $|\lambda_i\rangle$ are improper basis vectors which are eigenstates of the translation and its generator. The extension to a finite tensor-product space

$$H^N=\prod_i^N H_i\otimes$$

proceeds (as sketched in the beginning of this section) without incident. Thus, we have

$$|\{\lambda\}\rangle=|\lambda_1\rangle|\lambda_2\rangle|\lambda_3\rangle\cdots|\lambda_N\rangle,$$

$$\langle\{\lambda\}|\{\lambda'\}\rangle=\prod_i^N\delta(\lambda_i-\lambda_i'),$$

$$\int\prod_i^N[d\lambda_i]|\{\lambda\}\rangle\langle\{\lambda\}|=1 \text{ in } H^N, \quad (3.18)$$

$$\prod_i^N U_i(\tfrac{1}{2}c[2(2\pi)^3]^{1/2}\tilde{f}_i(0))|0\rangle$$

$$=\exp[-ic(2\pi)^{3/2}\sum_i\tilde{f}_i(0)\chi_i]|0\rangle=\prod_i^N|\alpha_i\rangle,$$

where

$$\prod_i^N U_i$$

is a unitary operator in H^N.

For $N\to\infty$ it is well known that complications arise because of the possibility of divergent products and sums.[6,7] These problems have been discussed at length in the literature.[5–7] In the following we shall largely follow the exposition of Kibble. In

$$H_\otimes=\prod_i^\infty\otimes H_i$$

a general product vector $|\psi\rangle$ is defined by a sequence $\{|\psi_i\rangle\}$ with $|\psi\rangle=\pi_i^\infty|\psi_i\rangle$, where $|\psi_i\rangle\in H_i$. Two vectors $|\phi\rangle$ and $|\psi\rangle$ are defined to be equivalent (denoted by $|\phi\rangle\sim|\psi\rangle$), if

$$\sum_i|1-\langle\phi_i|\psi_i\rangle|<\infty, \quad (3.19)$$

in which case their scalar product is defined by

$$\langle\phi|\psi\rangle=\prod_i^\infty\langle\phi_i|\psi_i\rangle. \quad (3.20)$$

If formally divergent products are assigned the value zero, then inequivalent vectors are orthogonal. $H_\otimes$ with scalar product so defined is the nonseparable infinite tensor-product space of von Neumann.[7] On $H_\otimes$, we have

$$U_\otimes=\exp[-ic(2\pi)^{3/2}\sum_i\tilde{f}_i(0)\chi_i], \quad (3.21)$$

which is unitary, i.e., the gauge transformation is unitarily implemented; the Fock vacuum $|0\rangle=\prod_i^\infty|0_i\rangle$ is transformed into the coherent tensor-product state $\prod_i^\infty|\alpha_i\rangle$ with norm

$$\prod_i^\infty\langle\alpha_i|\alpha_i\rangle=1. \quad (3.22)$$

However, $U_\otimes$ is not unitary on the Fock space, $H_\otimes(0)$, which is a subspace of $H_\otimes$ consisting of all vectors $|\psi\rangle$ equivalent to the vacuum $|0\rangle=\prod_i^\infty|0_i\rangle$, that is, all vectors $\prod_i^\infty|\psi_i\rangle$ for which

$$\sum_i^\infty|1-\langle 0_i|\psi_i\rangle|<\infty. \quad (3.23)$$

$U_\otimes$ in general takes a space $H_\otimes(\phi)$ of vectors equivalent to a given vector ϕ into a space $H_\otimes(\phi')$ with $|\phi'\rangle = U_\otimes|\phi\rangle$. Only in the special case when $U_\otimes|\phi\rangle\sim|\phi'\rangle$ is $U_\otimes$ a unitary operator in $H_\otimes(\phi)$. In the case of the gauge transformation, however, $\sum_i|1-\langle 0_i|U_i|0_i\rangle|$ is divergent, the transformed vacuum $U_\otimes|0\rangle$ is orthogonal in the above sense to $|0\rangle$, it does not lie in $H_\otimes(0)$, and $U_\otimes$ is not a unitary operator on $H_\otimes(0)$. Nonetheless, the transformation is an isometry; the transformed vacuums are normalized to unity, even though they do not lie in the Fock space $H_\otimes(0)$.

The $|\{\lambda\}\rangle$ form an improper basis in $H_\otimes$ and hence in $H_\otimes(0)$, that is we have the expansion

$$|n_1n_2n_3\cdots\rangle\equiv|n_1\rangle|n_2\rangle|n_3\rangle\cdots$$

$$=\int d\lambda_1|\lambda_1\rangle\langle\lambda_1|n_1\rangle\int d\lambda_2|\lambda_2\rangle\langle\lambda_2|n_2\rangle\cdots$$

$$\equiv\int\prod_i[d\lambda_i]|\{\lambda\}\rangle\langle\{\lambda\}|n_1n_2n_3\cdots\rangle, \quad (3.24)$$

where $\{n_i\}$ is a sequence of occupation numbers for pions in mode (i). $|\{n_i\}\rangle$ is a Fock-space state provided $\sum_i|1-\langle 0_i|n_i\rangle|<\infty$, which is the case if all but a finite number of modes are unoccupied, but the expansion is valid in any case, simply from the property

$$|n_i\rangle=\int d\lambda_i|\lambda_i\rangle\langle\lambda_i|n_i\rangle.$$

We shall use this expansion in deriving consequences of the pion gauge invariance. Specifically, we shall use the fact that $\langle\{\lambda\}|T|\{\lambda'\}\rangle$ vanishes unless the total chirality of the initial and final states ($\chi_{\text{init}}\propto\sum_i f_i(0)\lambda_i'$, $\chi_{\text{fin}}\propto\sum_i f_i(0)\lambda_i$) is equal, that is,

$$\langle\{\lambda\}|T|\{\lambda'\}\rangle=\delta(\sum_i f_i(0)(\lambda_i-\lambda_i'))\langle\{\lambda\}|T|\{\lambda'\}\rangle. \quad (3.25)$$

4. GAUGE CONDITIONS FOR SCATTERING AMPLITUDES

In this section we derive the conditions which an arbitrary scattering matrix element (involving connected pions) has to obey to be invariant against the pion gauge transformation (2.2) introduced in the previous sections. We shall derive these conditions employing two different methods. In the first method,[10] we assume that the general expansion of the S operator in terms of in-fields of the pion exists and calculate the commutator of this expansion of S with the chirality operator χ_α. In order that the S operator be gauge-invariant this commutator must vanish, which yields conditions on individual S-matrix elements. In the second derivation, we start from chiral eigenstates and demand that the S-matrix elements for arbitrary states be diagonal in the chirality quantum number. We then arrive at the same result as with the first method if in addition we assume the usual crossing symmetry.

We shall also apply the first method to quantum electrodynamics to show that this derivation of gauge conditions is quite natural and leads to the familiar result in this case also.

As already mentioned, in the first method we assume that the scattering operator S has an expansion in normal ordered products of pion in-fields $\phi_{\text{in},\alpha}(x)$:

$$S=\sum_{n=0}\frac{1}{n!}\int d^4x_1\int d^4x_2 \cdots\int d^4x_n \sum_{\alpha_1,\alpha_2\cdots\alpha_n} S_{\alpha_1,\alpha_2\cdots\alpha_n}{}^{(n)}(x_1,x_2,\cdots,x_n) \times{:}\phi_{\text{in},\alpha_1}(x_1)\phi_{\text{in},\alpha_2}(x_2)\cdots\phi_{\text{in},\alpha_n}(x_n){:}. \quad (4.1)$$

The expansion coefficients $S^{(n)}$ are operators which may be expanded in fields corresponding to all the other particles stable under strong interactions. We note that $S^{(n)}$ yields only the connected part concerning the pion lines in a scattering process involving n pions.

The commutator C_β between the chirality operator $\chi_\beta=\int d^3x\,\partial_0\phi_\beta(x)$ and S,

$$C_\beta=[\chi_\beta,S], \quad (4.2)$$

is calculated using the following formula[11]:

$$[\phi_{\text{in},\beta}(x),S]=i\int d^4x'\Delta(x-x')\frac{\delta}{\delta\phi_{\text{in},\beta}(x')}S, \quad (4.3)$$

where, for example,

$$\delta\phi_\alpha(y)/\delta\phi_\beta(x)=\delta_{\alpha\beta}\delta^4(x-y). \quad (4.4)$$

The result for C_β is

$$C_\beta=\sum_n\frac{1}{n!}\int d^3x\int d^4x'\int d^4x_1 \cdots\int d^4x_n S_{\alpha_1\cdots\alpha_n\beta}{}^{(n+1)}(x_1,\cdots,x_n,x') \times{:}\phi_{\text{in},\alpha_1}(x_1)\cdots\phi_{\text{in},\alpha_n}(x_n){:}. \quad (4.5)$$

To obtain the final result it is more appropriate to express C_β by the Fourier transforms of the $S^{(n)}$, defined by

$$S_{\alpha_1\cdots\alpha_n}{}^{(n)}(x_1,\cdots,x_n)=\frac{1}{(2\pi)^{5n/2}}\int d^4k_1 \cdots\int d^4k_n e^{i(k_1\cdot x_1+\cdots+k_n\cdot x_n)}\tilde{S}_{\alpha_1\cdots\alpha_n}(k_1,\cdots,k_n). \quad (4.6)$$

We have then

$$C_\beta=\sum_n\frac{1}{n!}\int d^4k_1\cdots\int d^4k_n\tilde{\sigma}_{\alpha_1\cdots\alpha_n,\beta}{}^{(n)}(k_i,\cdots,k_n) \times{:}\tilde{\phi}_{\text{in},\alpha_1}(k_1)\cdots\tilde{\phi}_{\text{in},\alpha_n}(k_n){:}\,\delta(k_1^2)\cdots\delta(k_n^2), \quad (4.7)$$

with

$$\tilde{\sigma}_{\alpha_1\cdots\alpha_n,\beta}{}^{(n)}(k_1,\cdots,k_n) =(-i)(2\pi)^{3/2}\int d^4k\, k_0\epsilon(k)\delta(k^2)\delta^3(\mathbf{k}) \times\tilde{S}_{\alpha_1\cdots\alpha_n,\beta}{}^{(n+1)}(k_1,\cdots,k_n,k). \quad (4.8)$$

By taking repeated commutators of C_β with $\phi_{\text{in},\alpha_1}(z_1)$, $\cdots$, $\phi_{\text{in},\alpha_n}(z_n)$ and taking expectation values in a state without pions, we can express all $\tilde{\sigma}^{(n)}(k_1,\cdots,k_n)$ by these repeated commutators with C_β.

Because of pion gauge invariance $C_\beta=0$ and therefore

$$\tilde{\sigma}_{\alpha_1,\alpha_2\cdots\alpha_n,\beta}{}^{(n)}(k_1\cdots k_n)=0. \quad (4.9)$$

After evaluating the integral over k in (4.8), we see that (4.9) is equivalent to

$$\lim_{|\mathbf{k}|\to0}(\tilde{S}_{\alpha_1\cdots\alpha_n,\beta}{}^{(n+1)}(k_1,\cdots k_n,\mathbf{k},\sqrt{\mathbf{k}^2}) +\tilde{S}_{\alpha_1\cdots\alpha_n,\beta}{}^{(n+1)}(k_1,\cdots,k_n,-\mathbf{k},-\sqrt{\mathbf{k}^2}))=0. \quad (4.10)$$

[10] This is basically the method suggested by Nishijima (Ref. 3) and recently elaborated upon by Hamilton and by Martinis (Ref. 4).

[11] S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Harper and Row Publications, Inc., New York, 1964). We use the metric and γ-matrix convention of this text.

In (4.10) the matrix element $\tilde{S}^{(n+1)}(\cdots,\mathbf{k},\sqrt{\mathbf{k}^2})$ is the S-matrix element for an ingoing pion with momentum k, whereas in $\tilde{S}^{(n+1)}(\cdots -\mathbf{k},\sqrt{\mathbf{k}^2})$ the pion is outgoing with momentum k. For $\mathbf{k}\to 0$ both functions approach the same limit and the condition finally is

$$\lim_{|\mathbf{k}|\to 0}\tilde{S}_{\alpha_1\cdots\alpha_n,\beta}{}^{(n+1)}(k,\cdots k_n,\mathbf{k},\sqrt{\mathbf{k}^2})=0. \quad (4.11)$$

Therefore, we have the theorem: Any scattering matrix element involving pions is gauge-invariant only if its connected part vanishes for vanishing four-momentum k of any pion selected.

The rather complicated derivation above is equivalent to iteration of the following rather simple exercise. Let $|f\rangle$ and $|i\rangle$ be states containing no zero-mass pions. Then the matrix element of the commutator C_β between these states is

$$\langle f|C_\beta|i\rangle=\langle f|[\chi_\beta,S]|i\rangle$$
$$=(2\pi)^{3/2}\frac{i}{\sqrt{2}}\langle f|[(a^\dagger(0)-a(0)),S]|i\rangle \quad (4.12)$$

and from $C_\beta=0$ we have

$$\langle fa(0)|S|i\rangle+\langle f|S|a^\dagger(0)i\rangle=0, \quad (4.13)$$

which is equivalent to (4.10).

As an illustration of the method outlined above we derive the well-known condition for photon gauge invariance. Gauge invariance for photo processes means that the S operator is invariant against the transformation

$$A_{\mu,\mathrm{in}}(x)\to A_{\mu,\mathrm{in}}(x)+\partial_\mu\Lambda(x), \quad (4.14)$$

where $A_{\mu,\mathrm{in}}(x)$ is the ingoing photon field and $\Lambda(x)$ is a c-number solution of the Klein-Gordon equation. That S be invariant against the transformation (4.14) requires

$$C_\Lambda=[\chi_\Lambda,S]=0, \quad (4.15)$$

where

$$\chi_\Lambda=-\int d^3x[\partial_0\Lambda\partial^\mu A_{\mu,\mathrm{in}}(x)-\Lambda(x)\partial_0\partial^\mu A_{\mu,\mathrm{in}}(x)] \quad (4.16)$$

is the generator of the gauge transformation

$$e^{i\cdot\chi_\Lambda}A_{\mu,\mathrm{in}}(x)e^{-i\cdot\chi_\Lambda}=A_{\mu,\mathrm{in}}(x)+\partial_\mu\Lambda(x). \quad (4.17)$$

It should be noted at this point that the operator χ_Λ has the same peculiar mathematical properties as the pion chirality operator χ_α, as discussed in Sec. 2.

In close similarity to the pion case we assume that the scattering operator S can be expanded in normal ordered products of the ingoing photon field:

$$S=\sum_{n=0}^{\infty}\int d^4x_1\cdots\int d^4x_n S^{(n)\mu_1\cdots\mu_n}(x_1,\cdots,x_n)$$
$$\times:A_{\mu_1,\mathrm{in}}(x_1)\cdots A_{\mu_n,\mathrm{in}}(x_n):. \quad (4.18)$$

Then the commutator is calculated with the same methods as in the pion case. The result is

$$C_\Lambda=\sum_n\frac{1}{n!}\int d^4x_1\cdots\int d^4x_n\int d^3x\int d^4x'(-i)$$
$$\times[\Lambda(x)\partial_0{}^x\partial_\mu{}^x\Delta(x-x')-\partial_0{}^x\Lambda(x)\partial_\mu{}^x\Delta(x-x')]$$
$$\times S^{(n+1)\mu_1\cdots\mu_n\mu}(x_1,\cdots,x_n,x')$$
$$\times:A_{\mu_1,\mathrm{in}}(x_1)\cdots A_{\mu_n,\mathrm{in}}(x_n):, \quad (4.19)$$

which we express by the Fourier transform of $\Lambda(x)$ and $S^{(n)}$ defined by

$$\Lambda(x)=\frac{1}{(2\pi)^{3/2}}\int d^4k\,\delta(k^2)e^{-ik\cdot x}\tilde{\Lambda}(k), \quad (4.20)$$

$$S^{(n)\mu_1\cdots\mu_n}(x_1\cdots x_n)=\frac{1}{(2\pi)^{5n/2}}\int d^4k_1$$
$$\cdots\int d^4k_n\tilde{S}^{(n)\mu_1\cdots\mu_n}(k_1,\cdots,k_n). \quad (4.21)$$

Then we obtain

$$C_\Lambda=\sum_n\frac{1}{n!}\int d^4k_1\cdots\int d^4k_n\tilde{\sigma}^{(n)\mu_1\cdots\mu_n}(k_1,\cdots,k_n)$$
$$\times:\tilde{A}_{\mu_1,\mathrm{in}}(k_1)\cdots A_{\mu_n,\mathrm{in}}(k_n):\delta(k_1{}^2)\cdots\delta(k_n{}^2), \quad (4.22)$$

with

$$\tilde{\sigma}^{(n)\mu_1\cdots\mu_n}(k_1,\cdots,k_n)=\int d^4k\int d^4p\,\delta(k^2)\epsilon(k)\delta(p^2)\tilde{\Lambda}(p)$$
$$\times k_\mu\tilde{S}^{(n+1)\mu_1\cdots\mu_n,\mu}(k_1,\cdots,k_n,k)(p_0-k_0)$$
$$\times e^{i(p_0-k_0)x_0}\delta^3(\mathbf{p}-\mathbf{k}). \quad (4.23)$$

Since $C_\Lambda=0$, we have

$$\tilde{\sigma}^{\mu_1\cdots\mu_n,\mu}(k_1,\cdots,k_n)=0. \quad (4.24)$$

By evaluating the integral in (4.23), we get from this condition

$$\int\frac{d^3k}{2\omega_k}\{e^{2i\omega_kx_0}\tilde{\Lambda}(\mathbf{k})k_\mu\tilde{S}^{(n+1)\mu_1\cdots\mu_n,\mu}(k_1,\cdots k_n,k)|_{k_0=-\omega_k}$$
$$+e^{-2i\omega_kx_0}\tilde{\Lambda}^*(-\mathbf{k})k_\mu\tilde{S}^{(n+1)\mu_1\cdots\mu_n,\mu}$$
$$\times(k_1,\cdots,k_n,k)|_{k_0=+\omega_k}\}=0, \quad (4.25)$$

where $\omega_k=\sqrt{\mathbf{k}^2}$ and $\tilde{\Lambda}(\mathbf{k})=\tilde{\Lambda}(k)$ for $k_0>0$, $\tilde{\Lambda}^*(-\mathbf{k})=\tilde{\Lambda}(k)$ for $k_0<0$. This last condition (4.25) must be fulfilled for any complex function $\tilde{\Lambda}(\mathbf{k})$, which is possible only if

$$k_\mu\tilde{S}^{(n+1)\mu_1\cdots\mu_n,\mu}(k_1,\cdots,k_n,k)=0, \quad (4.26)$$

where $k_0=\pm\omega_k$. Equation (4.26) is the familiar gauge condition in quantum electrodynamics. It is well known that this condition ensures the right Lorentz transformation properties of S for massless vector mesons.[1] A simplified version of this derivation for particular processes along the lines of the method (4.12) and (4.13) has been carried out by Rollnik.[12]

In the following paragraph we shall prove the pion gauge condition using chiral eigenstates. In this method the assumption that the scattering operator has an expansion in normal products of asymptotic states is not needed. Only crossing symmetry and properties of the chiral eigenstates constructed in Sec. 3 are employed. To make our presentation transparent we shall not derive explicitly the result for the most general case of a finite number of zero momentum pions in the initial and final state. Instead we derive the theorem for three examples with one or two zero-momentum pions in the initial or final state. That the most general case can be derived with the same methods will then become obvious. We denote by $|i\rangle$, $|f\rangle$ states with no pions; i and f stand for operators creating all the other particles. A state with one zero-momentum pion is, in terms of the wave packet expansions

$$|i,a^\dagger(0)\rangle=\sum_j \tilde{f}_j(0)|i,\cdots 1_j,\cdots\rangle. \qquad (4.27)$$

The wave packet functions were introduced in Sec. 2, and $|i,\cdots,1_j,\cdots\rangle$ stands for the following Fock-space vector: $|i,0_1,0_2,\cdots,0_{j-1},1_j,0_{j+1},\cdots\rangle$. As outlined in Sec. 3, we can expand such vectors in eigenstates of the operator χ_j. The general formula is

$$|i,n_1n_2,\cdots\rangle=\int\prod_k^\infty[d\lambda_k\langle\lambda_k|n_k\rangle]|i,\lambda_1,\lambda_2,\cdots\rangle$$

$$\equiv\int\prod_k^\infty[d\lambda_k\langle\lambda_k|n_k\rangle]|i,\{\lambda\}\rangle. \qquad (4.28)$$

This formula is used to decompose the matrix element $\langle f,\cdots,1_j,\cdots|T|i\rangle$ of the T operator $T=i(1-S)$ with one pion in state j in the final state:

$$\langle f,\cdots,1_j,\cdots|T|i\rangle$$

$$=\int\int\prod_{k\neq j}[d\lambda_k\langle 0_k|\lambda_k\rangle]d\lambda_j\langle 1_j|\lambda_j\rangle$$

$$\times\prod_l[d\lambda_l'\langle\lambda_l'|0_l\rangle]\langle f,\{\lambda\}|T|i,\{\lambda'\}\rangle$$

$$=i\sqrt{2}\int\prod_k[d\lambda_k\langle 0_k|\lambda_k\rangle]\prod_l[d\lambda_l'\langle\lambda_l'|0_l\rangle]$$

$$\times\lambda_j\langle f,\{\lambda\}|T|i,\{\lambda'\}\rangle. \qquad (4.29)$$

In this last step, we made use of the relation $\langle 1_j|\lambda_j\rangle=i\sqrt{2}\langle 0_j|\lambda_j\rangle$ following from (3.4b). Similarly, but with an essential sign change stemming from the fact that $\langle 1_j|\lambda_j\rangle=-\langle\lambda_j|1_j\rangle$, we have

$$\langle f|T|i,\cdots 1_j\cdots\rangle$$

$$=-i\sqrt{2}\int\prod_k[d\lambda_k\langle 0_k|\lambda_k\rangle]\prod_l[d\lambda_l'\langle\lambda_l'|0_l\rangle]\lambda_j'$$

$$\times\langle f,\{\lambda\}|T|i,\{\lambda'\}\rangle. \qquad (4.30)$$

We now form the sum:

$$\langle f,a(0)|T|i\rangle+\langle f|T|i,a^\dagger(0)\rangle$$

$$=\sum_j\tilde{f}_j(0)[\langle f,\cdots 1_j\cdots|T|i\rangle+\langle f|T|i,\cdots 1_j\cdots\rangle]$$

$$=i\sqrt{2}\int\prod_k[d\lambda_k\langle 0_k|\lambda_k\rangle]\prod_l[d\lambda_l'\langle\lambda_l'|0_l\rangle]$$

$$\times[\sum_j\tilde{f}_j(0)(\lambda_j-\lambda_j')]\langle f,\{\lambda\}|T|i,\{\lambda'\}\rangle. \qquad (4.31)$$

However, since the total pion chirality $\chi\propto\sum_j\tilde{f}_j(0)\chi_j$ is conserved, $\langle f,\{\lambda\}|T|i,\{\lambda'\}\rangle$ is proportional to $\delta(\sum_j\tilde{f}_j(0)(\lambda_j-\lambda_j'))$ and the right-hand side of (4.31) vanishes:

$$\langle f\,a(0)|T|i\rangle+\langle f|T|i,a^\dagger(0)\rangle\equiv T_{10}{}^{fi}+T_{01}{}^{fi}=0. \qquad (4.32)$$

Because of crossing symmetry we have for zero-momentum pions $T_{10}{}^{fi}=T_{01}{}^{fi}$ and hence the desired result, $T_{10}{}^{fi}=T_{01}{}^{fi}=0$. We remark that the crossing property for $k=0$ is an additional assumption in this method of derivation. In the first method it came with the assumption that the S operator can be expanded into asymptotic fields, that is, the field-operator expansion is manifestly crossing symmetric.

This method of using chiral eigenstates can be applied to general connected amplitudes in which there are a finite number of zero-momentum pions in the initial or final state or amplitudes in which there are a finite number of pions at least one of which has zero four-momentum. We illustrate how the proof goes by the following two concrete examples. First we consider the product state

$$|\cdots,1_i,\cdots\rangle|\cdots,1_j,\cdots\rangle$$

$$=\sqrt{2}\delta_{ij}|\cdots,2_j,\cdots\rangle+(1-\delta_{ij})|\cdots,1_i,\cdots,1_j,\cdots\rangle$$

$$=\sqrt{2}\delta_{ij}\int\prod_{k\neq j}[d\lambda_k\langle\lambda_k|0_k\rangle]d\lambda_j\langle\lambda_j|2_j\rangle|\{\lambda\}\rangle$$

$$+(1-\delta_{ij})\int\prod_{k\neq i,j}[d\lambda_k\langle\lambda_k|0_k\rangle]d\lambda_i\langle\lambda_i|1_i\rangle$$

$$\times d\lambda_j\langle\lambda_j|1_j\rangle|\{\lambda\}\rangle$$

$$=\int\prod_k[d\lambda_k\langle\lambda_k|0_k\rangle]\delta_{ij}(1-2\lambda_i\lambda_j)|\{\lambda\}\rangle. \qquad (4.33)$$

To derive this representation we used the following properties of the coefficients $\langle\lambda_i|n_i\rangle$, which follow from

[12] H. Rollnik, Z. Physik **161**, 370 (1961).

(3.4b):

$$\langle\lambda_i|1_i\rangle=-i\sqrt{2}\lambda_i\langle\lambda_i|0_i\rangle,\quad \langle\lambda_i|2_i\rangle=2^{-1/2}(1-2\lambda_i^2)\langle\lambda_i|0_i\rangle. \tag{4.34}$$

Therefore, we have

$$\langle f|T|i,a(\mathbf{p})a^\dagger(0)\rangle =\sum_{i,j}\tilde{f}_i(\mathbf{p})\tilde{f}_j(0)\langle f|T(|i,\cdots,1_i,\cdots\rangle|\cdots,1_j,\cdots\rangle) =\sum_{i,j}\tilde{f}_i(\mathbf{p})f_i(0)\int\prod_l[d\lambda_l'\langle 0|\lambda_l'\rangle]\int\prod_k[d\lambda_k\langle\lambda_k|0_k\rangle] \times(\delta_{ij}-2\lambda_i\lambda_j)\langle f,\{\lambda'\}|T|i,\{\lambda\}\rangle \tag{4.35}$$

and, similarly,

$$\langle f,a(\mathbf{p})|T|i,a^\dagger(0)\rangle =\sum_{i,j}f_i(\mathbf{p})f_j(0)\int\prod_l[d\lambda_l'\langle 0_l|\lambda_l\rangle]\prod_k[d\lambda_k\langle\lambda_k|0_k\rangle] \times 2\lambda_i'\lambda_j\langle f,\{\lambda'\}|T|i,\{\lambda\}\rangle. \tag{4.36}$$

Forming the sum and using the completeness relations for $\tilde{f}(\mathbf{p})$, we obtain

$$\langle f,a(\mathbf{p})|T|i,a^\dagger(0)\rangle+\langle f|T|i,a^\dagger(\mathbf{p})a^\dagger(0)\rangle =\sum_j f_j(\mathbf{p})\lambda_j\Big[\sum_i(\lambda_i'-\lambda_i)f_i(0)\Big]\int\prod_l[d\lambda_l'\langle 0_l|\lambda_l'\rangle] \times\prod_k[d\lambda_k\langle\lambda_k|0_k\rangle]+\sum_i f_i(\mathbf{p})f_i(0)\int\prod_l[d\lambda_l'\langle 0_l|\lambda_l'\rangle] \times\prod_k[d\lambda_k\langle\lambda_k|0_k\rangle]\langle f,\{\lambda'\}|T|i,\{\lambda\}\rangle =p_0\delta^3(\mathbf{p})\langle f|T|i\rangle, \tag{4.37}$$

where we used again the conservation of the total pion chirality. To interpret this last result we recall that $|f\rangle$ and $|i\rangle$ are arbitrary states containing no pions (of any momenta). When, in addition, there is at least one pion in the final and one in the initial state, then disconnected amplitudes are possible in which one or more pions go straight through without interacting,

$$\langle f\,a(\mathbf{p})|T|i,a^\dagger(0)\rangle=\langle f,a(\mathbf{p})|T^{\text{con}}|i,a^\dagger(0)\rangle +p_0\delta^3(\mathbf{p})\langle f|T|i\rangle. \tag{4.38}$$

As we already observed earlier, the gauge conditions constrain only the connected part of the amplitudes. Generally in this context it is convenient to decompose T into its connected (T^{con}) and disconnected (T^{dis}) parts (with respect to pions only). When disconnected pions are not possible, we define $T^{\text{con}}\equiv T$ and $T^{\text{dis}}=0$. Then (4.37) reads

$$\langle f,a(\mathbf{p})|T^{\text{con}}|i,a^\dagger(0)\rangle+\langle f|T^{\text{con}}|i,a^\dagger(\mathbf{p})a^\dagger(0)\rangle=0. \tag{4.39}$$

We may now take the limit $\mathbf{p}=0$, with the result

$$T_{11}^{\text{con}}=-T_{02}^{\text{con}}. \tag{4.40}$$

From the crossing property we have $T_{11}^{\text{con}}=T_{02}^{\text{con}}$, whence

$$T_{11}^{\text{con}}=T_{02}^{\text{con}}=0. \tag{4.41}$$

Quite analogously we derive

$$\langle f,a(0)|T^{\text{con}}|i,a^\dagger(\mathbf{p})\rangle+\langle f|T^{\text{con}}|i,a^\dagger(\mathbf{p})a^\dagger(0)\rangle=0. \tag{4.42}$$

Again, using the crossing property for zero four-momentum pions, this implies that both amplitudes vanish separately, even when only one of the two pions has vanishing four-momentum, another special case of the general theorem. In the above derivations (using chiral states) we have, in the interest of simplicity, dropped the isospin indices, proceeding as if only one species of pion appeared in the initial and final states. Since the χ_α are separately conserved, it is clear that the method can be generalized to include arbitrary pion isospin configurations in the initial and final state.

Presumably, this method of diagonalizing the chirality operator can also be pursued to derive the gauge conditions in quantum electrodynamics by diagonalizing the operator χ_Λ. But we shall not do so in this paper which is primarily concerned with pion guage invariance. Finally, we remark that despite the fact that the chirality χ connects the Fock space with spaces orthogonal to it and therefore cannot be diagonalized in the Fock space, it nonetheless can be diagonalized in a larger space which has the Fock space as a subspace. Then as the derivation outlined above shows these eigenstates can be used to infer physical consequences of the symmetry. Thus, it does not appear to us necessary to invoke such notions as weak conservation of chirality as has been advoctated by Nambu and Lurié.[13]

5. APPLICATIONS

In the previous sections we showed that the invariance of the S matrix against the transformation $\phi^{\text{in}}\to\phi^{\text{in}}+c$ generated by $\chi=\int d^3x\,\partial_0\phi^{\text{out}}$, implies the low-energy condition for zero mass-pion theories. The framework so far has been the usual general assumption of an S operator defined on a Hilbert space of asymptotic free-particle states, sometimes augmented by the assumption that S has an expansion in terms of the asymptotic fields. In this section we outline the conditions which the gauge invariance imposes on the πN scattering amplitudes and total cross sections. Then we look for Lagrangian models in which the invariance is realized and check the vanishing of scattering amplitudes for soft-pion emission or absorption processes.

Let us consider $\pi_\beta(q)+N(p)\to\pi_\alpha(q')+N(p')$ scattering with the momenta and isospin indices of the ingoing and outgoing particles as indicated. Then in Sec. 4 it

[13] Y. Nambu and D. Lurié, Phys. Rev. **125**, 1429 (1962).

was stated: The scattering amplitude $T_{\alpha\beta}(p',r',q';p,r,q)$ for this process is gauge-invariant if

$$T_{\alpha\beta}(p',r',q';p,r,q)=0 \tag{5.1}$$

for $q=0$ or $q'=0$ and for any choice of isospin indices α and β and nucleon spins r and r'. The condition (5.1) can be exploited in different ways. One possibility is to consider the forward scattering amplitude split into the isospin symmetric and antisymmetric part, defined by

$$T_{\alpha\beta}=T^{(+)}\delta_{\alpha\beta}+T^{(-)}\tfrac{1}{2}[\tau_\alpha,\tau_\beta]. \tag{5.2}$$

$T^{(+)}$ as a function of $\nu=p\cdot q=\frac{1}{4}(s-u)$ for $t=0$ is even in ν, whereas $T^{(-)}$ is odd and the gauge principle requires that $T^{(\pm)}$ has no constant terms. In the literature this result is usually stated in the form that the power series expansion around $\nu=0$ of the two amplitudes starts with ν^2 in the case of $T^{(+)}$ and with ν in the case of $T^{(-)}$, using crossing symmetry. But $\nu=0$ is a complicated branch point where for zero-mass pions the unitary cuts of all possible intermediate states coincide. Thus, we may also have an appreciable imaginary part near $\nu=0$. Because of crossing symmetry we have

$$\mathrm{Im}T^{(\pm)}(\nu)=\mp\mathrm{Im}T^{(\pm)}(-\nu)$$

and

$$\mathrm{Re}T^{(\pm)}(\nu)=\pm\mathrm{Re}T^{(\pm)}(-\nu).$$

For $\nu\geqslant 0$ the imaginary part is related to the total cross section $\sigma^{(\pm)}(\nu)=\frac{1}{2}[\sigma_{\pi^-}(\nu)\pm\sigma_{\pi^+}(\nu)]$ by

$$\mathrm{Im}T^{(\pm)}(\nu)=2\nu\sigma^{(\pm)}(\nu). \tag{5.3}$$

The gauge principle requires that $\mathrm{Re}T^{(\pm)}(\nu)$ and $\mathrm{Im}T^{(\pm)}(\nu)$ vanish for $\nu=0$. This means that the total cross sections $\sigma^{(\pm)}(\nu)$ are less singular than $1/\nu$ as ν approach zero. Of course, in a theory with zero-mass pions, total cross sections may be singular for $\nu\to 0$ since in this case an infinite number of pions can be produced. Using dispersion relations, the conditions on the real parts can be transformed into integral relations for the total cross sections. From $\mathrm{Re}T^{(-)}(\nu=0)=0$ we infer (assuming an unsubtracted dispersion relation)

$$\lim_{\nu\to 0}\frac{\nu}{\pi}P\int_0^\infty d\nu'\,\frac{\nu'\sigma^{(-)}(\nu')}{\nu'^2-\nu^2}=0. \tag{5.4}$$

In (5.4) the integral over ν includes also possible discrete singularities like one-nucleon pole contributions. We remark that in theories with a finite pion mass, both conditions on $\mathrm{Im}T^{(-)}(\nu)$ and (5.4) are fulfilled automatically since total cross sections in such theories approach finite values and the extra factor ν in (5.4) is necessary for crossing symmetry. The conclusions from $\mathrm{Re}T^{(+)}(\nu)=0$ are more interesting. If the dispersion relation for $T^{(+)}(\nu)$ is also unsubtracted, we have

$$\lim_{\nu\to 0}\frac{P}{\pi}\int_0^\infty d\nu'\,\frac{\nu'^2\sigma^{(+)}(\nu')}{\nu'^2-\nu^2}=0, \tag{5.5}$$

which is possible only for $\sigma^{(+)}(\nu')\equiv 0$. From this we conclude that the dispersion relation for $T^{(+)}(\nu)$ must be subtracted which agrees with the empirical finding that total cross sections $\sigma^{(+)}(\nu)$ do not vanish for large ν. Sometimes it is inferred from $\mathrm{Re}T^{(+)}(\nu=0)=0$ that the s-wave scattering length $a^{(+)}$ vanishes.[4] But this inference can be made only if it is known that $\mathrm{Re}T^{(+)}(\nu)$ for $\nu=0$ is dominated by the s-wave contribution. In the forward scattering amplitude $T(\nu)$ all partial waves contribute at threshold if the pion mass is zero. Of course, in the realistic case of nonvanishing pion mass all partial waves except the s wave vanish at threshold $q=0$. Only if this relation between the s wave and the higher partial waves is maintained for $m_\pi=0$ can we say that $a^{(+)}=0$.

We now consider simple models which are pion gauge-invariant. First, we shall consider gradient-coupling models, that is, those in which the interaction has the form $j_\mu\partial^\mu\phi$, where j_μ is a c number or an operator functional of fields other than the pion field. Derivative coupling by itself, however, is not sufficient to guarantee gauge invariance of the Lagrangian density; one must impose boundedness conditions on the current $j_\mu(x)$. Under $\phi(x)\to\phi(x)+c$, we have $\tilde{\phi}(k)\to\tilde{\phi}(k)+c\delta^4(k)$ and therefore, in momentum space,

$$\frac{1}{(2\pi)^4}\int d^4x\, e^{ik\cdot x}j_\mu(x)\partial^\mu\phi(x)$$

$$=\int d^4k'\,\tilde{\jmath}_\mu(k-k')k'^\mu\tilde{\phi}(k')\to\int d^4k'\,\tilde{\jmath}_\mu(k-k')k'^\mu\tilde{\phi}(k')$$

$$+c\lim_{k'\to 0}\tilde{\jmath}_\mu(k-k')k'^\mu. \tag{5.6}$$

Therefore, if the interaction term is to be gauge-invariant, $\lim_{k'\to 0}k'^\mu\tilde{\jmath}_\mu(k-k')$ must vanish for any four-vector k. This does not prevent $j_\mu(k)$ from being singular, that is, one cannot infer that $k'^\mu\tilde{\jmath}_\mu(k-k')$ is proportional to k'^μ at $k'^\mu\to 0$. This is precisely the case, as we shall presently see, in the gradient-coupling model for πN scattering, where the elastic amplitude in second-order perturbation theory has a leading term proportional to ν whereas if $j_\mu(k)$ were nonsingular, the leading term would be ν^2; thus, to determine the exact behavior of the amplitude near $k_\mu=0$ one must make detailed dynamical assumptions independent of the gauge invariance of the theory. The simplest gauge-invariant model of this kind is a gradient coupling with c-number source. The Lagrangian is

$$\mathfrak{L}=+\tfrac{1}{2}(\partial\phi)^2-j_{\mu,\alpha}(x)\partial^\mu\phi_\alpha(x), \tag{5.7}$$

with a given c-number axial-vector function $j_{\mu,\alpha}(x)$ which depends also on the momenta p and p' and the spins of σ and σ' of the ingoing and outgoing nucleons producing this source. We assume explicitly that

$$\lim_{k\to 0} k^\mu j_{\mu,\alpha}(k)=0\,, \tag{5.8}$$

where $j_{\mu,\alpha}(k)$ is the Fourier transform of $j_{\mu,\alpha}(x)$, defined below. This problem is similar to the problem of the quantized radiation field interacting with a classical current density and can be solved exactly. The S operator can easily be expressed by the Fourier transform of the current density, defined by

$$j_{\mu,\alpha}(x)=\int \frac{d^4k\sqrt{2}\theta(k)}{(2\pi)^{5/2}}(j_{\mu,\alpha}(k)e^{-ik\cdot x}+j_{\mu,\alpha}{}^*(k)e^{ik\cdot x})\,. \tag{5.9}$$

It has the following form:

$$S=\exp\left\{\int \frac{d^3k}{2k_0}[k^\mu j_{\mu,\alpha}(k)a_{\alpha,\text{in}}{}^\dagger(\mathbf{k})-k^\mu j_{\mu,\alpha}{}^*(k)a_{\alpha,\text{in}}(\mathbf{k})]\right\}, \tag{5.10}$$

which can be transformed into

$$S=\exp\left\{\int \frac{d^3k}{k_0}k^\mu j_{\mu,\alpha}(k)a_{\alpha,\text{in}}{}^\dagger(\mathbf{k})\right\}\exp\left\{-\int \frac{d^3k}{k_0}k^\mu j_{\mu,\alpha}{}^*(k)a_{\alpha,\text{in}}(\mathbf{k})\right\}$$
$$\times\exp\left\{-\frac{1}{2}\int \frac{d^3k}{k_0}(k^\mu j_{\mu,\alpha}(k))(k^\nu j_{\nu,\alpha}(k))\right\}. \tag{5.11}$$

We assume that $j_\mu(k)$ is chosen such that all integrals exist. We have explicitly $[X,S]=0$ because of the assumption (5.8). It is clear that the S matrix in this model is itself a displacement operator which induces the c-number translation

$$a_{\alpha,\text{out}}(\mathbf{k})=S^{-1}a_{\alpha,\text{in}}(\mathbf{k})S=a_{\alpha,\text{in}}(\mathbf{k})+k^\mu j_\mu{}^*(k)\,. \tag{5.12}$$

Thus, the c-number source radiates coherently, that is, pions are produced in coherent states. S is unitary in Fock space if

$$\int \frac{d^3k}{k_0}|k^\mu j_{\mu,\alpha}(k)|^2<\infty\,. \tag{5.13}$$

From the general formula (5.11), it is easy to calculate the matrix elements for particular pion scattering and production processes. For example, we obtain for the production of n pions with momenta $k_1, k_2, \cdots, k_n$ and isotopic spin quantum numbers $\alpha_1, \alpha_2, \cdots, \alpha_n$:

$$\begin{aligned}\langle k_1,\alpha_1; k_2\alpha_2;\cdots k_n,\alpha_n\ \text{out}|0\ \text{in}\rangle &=\langle 0_{\text{out}}|0_{\text{in}}\rangle\{k_1{}^\mu j_{\mu,\alpha_1}(k_1)k_2{}^\mu j_{\mu,\alpha_2}(k_2)\\ &\qquad\cdots k_n{}^\mu j_{\mu,\alpha_n}(k_n)\}\end{aligned} \tag{5.14}$$

or for the production of n pions from one incoming pion (connected amplitudes only)

$$\begin{aligned}\langle k_1,\alpha_1;\cdots k_n,\alpha_n\ \text{out}|q,\beta\ \text{in}\rangle^{\text{con}} &=\langle 0_{\text{out}}|0_{\text{in}}\rangle k_1{}^\mu j_{\mu,\alpha_1}(k_1)k_2{}^\mu j_{\mu,\alpha_2}(k_2)\\ &\qquad\cdots k_n{}^\mu j_{\mu,\alpha_n}(k_n)(-q^\mu j_{\mu,\beta}{}^*(q))\,,\end{aligned} \tag{5.15}$$

with

$$\langle 0_{\text{out}}|0_{\text{in}}\rangle=\exp\left\{-\frac{1}{2}\int \frac{d^3k}{k_0}k^\mu j_{\mu,\alpha}(k)k^\nu j_{\nu,\alpha}{}^*(k)\right\}. \tag{5.16}$$

$\langle 0_{\text{out}}|0_{\text{in}}\rangle$ is the transition matrix element without absorption or emission of pions which stands in a realistic theory for the elastic scattering of say protons caused by virtual intermediate pions. Although we admit that this simple source theory has many unrealistic features, there are some properties interesting enough to be pointed out. As to be expected, the final results (5.14)–(5.16) are pion gauge-invariant. The matrix elements vanish if any of the pion four-momenta go to zero. Second, the formulas (5.14) and (5.15) can be used to say something about threshold behavior of the matrix elements in a theory with nonzero mass pions. We add a mass term $-\frac{1}{2}m_\pi{}^2(\phi_\alpha)^2$ to the Lagrangian in (5.7). Then it follows from (5.14) that the matrix element for production of n pions is proportional to $(m_\pi)^n$ if all the pion momenta $\mathbf{k}_i$ ($i=1,\cdots,n$) are zero.

We now consider simple models which have been studied in the past in connection with PCAC and more recently as substitutes for $SU(2)\times SU(2)$ current-algebra schemes. We shall show that all these models are pion gauge-invariant although the transformations of the interpolating fields differ in the various models. The simplest model is the gradient-coupling model of pions and nucleons defined by the Lagrangian

$$\mathfrak{L}=\tfrac{1}{2}(\partial\tilde{\phi})^2-\bar{\psi}(-i\gamma_\mu\partial^\mu+m+if\gamma_\mu\gamma_5\boldsymbol{\tau}\cdot\partial^\mu\tilde{\phi})\psi\,. \tag{5.17}$$

This Lagrangian is formally invariant against the transformation

$$\phi_\alpha\to\phi_\alpha+c\,. \tag{5.18}$$

The current associated with this symmetry has the form

$$A_{\mu,\alpha}=-j_{\mu,\alpha}+\partial_\mu\phi_\alpha\,, \tag{5.19a}$$

where

$$j_{\mu,\alpha}=if\bar{\psi}\gamma_\mu\gamma_5\tau_\alpha\psi\,. \tag{5.19b}$$

The axial-vector current $A_{\mu,\alpha}(x)$ is conserved as can be seen explicitly from the equation of motion for the pion field

$$\partial^\mu \partial_\mu \phi_\alpha = \partial^\mu j_{\mu,\alpha}. \qquad (5.20)$$

The operator

$$\chi_\alpha = \int d^3x\, A_{0,\alpha} = \int d^3\chi\, (\partial_0 \phi_\alpha - j_{0,\alpha}) \qquad (5.21)$$

generates the transformation (5.18) by virtue of the canonical commutation relations. In the following we shall assume that, although the gradient-coupling model has such undesirable properties as unrenormalizability, asymptotic fields in the usual sense exist for this model. Then the asymptotic pion field $\phi_{\alpha,\mathrm{in}}(x)$ undergoes the same transformation,

$$\phi_{\alpha,\mathrm{in}} \to \phi_{\alpha,\mathrm{in}} + c\,, \qquad (5.22)$$

as the interpolating field ϕ_α. The generator of the translation of $\phi_{\alpha,\mathrm{in}}$ is

$$\chi_{\alpha,\mathrm{in}} = \int d^3\chi\, \partial_0 \phi_{\alpha,\mathrm{in}}\,. \qquad (5.23)$$

Since χ_α is independent of time, χ_α should be equal to $\chi_{\alpha,\mathrm{in}}$. This is the case if the space integral

$$\int d^3x\, j_{0,\alpha}(x) \qquad (5.24)$$

converges to zero as a weak limit for $t \to \pm\infty$. That this may happen is already plausible from the old "adiabatic switching" hypothesis. In the context of electrodynamics Källén[14] has imposed this property by the formal device of a damping factor $e^{-\alpha|x_0|}$, a procedure which is applicable in the pion theories also. Instead we give the following formal derivation based on the Yang-Feldman equation for the pion field in the usual formulation[15]:

$$\phi_\alpha(x) = \phi_{\alpha,\mathrm{in}}(x) + \int d^4y\, \Delta_R(x-y) \partial^\mu j_{\mu,\alpha}(y)\,. \qquad (5.25)$$

Using this equation, we substitute for $\partial_0 \phi_\alpha$ in (5.21) the time derivative of (5.25). Then we have

$$\chi_\alpha = \chi_{\alpha,\mathrm{in}} + \int d^3x \int d^4y\, \partial_0{}^x \Delta_R(x-y) \partial^\mu j_{\alpha,\mathrm{in}}(y) - \int d^3x\, j_{0,\alpha}(x)\,. \qquad (5.26)$$

We evaluate the second term in (5.26) in momentum space, disregarding all complications which evolve from interchanges of orders of integration:

$$\int d^3x \int d^4y\, \partial_0{}^x \Delta_R(x-y) \partial^\mu j_{\mu,\alpha}(y) = \int d^3x \int d^4k\, \frac{e^{-ik\cdot x} k_0 k^\mu \tilde{j}_{\mu,\alpha}(k)}{(k_0+i\epsilon)^2 - \mathbf{k}^2} = (2\pi)^3 \int dk_0 e^{-ik_0x_0} \tilde{j}_{0,\alpha}(k_0,0) = \int d^3x\, j_{0,\alpha}(x)\,. \qquad (5.27)$$

Thus the second term in (5.26) is equal to the last term and we have $\chi_\alpha = \chi_{\alpha,\mathrm{in}}$.

Then solutions of the gradient-coupling model, under the condition that they exist, are obviously pion gauge-invariant in the sense discussed in the previous sections. Therefore in this model all elastic and inelastic π-N scattering amplitudes should vanish if the four-momentum q of one of the pions goes to zero. It might be interesting to check the specific behavior of these amplitudes for $q \to 0$ in one simple example. We shall do this for elastic $\pi(q) + N(p) \to \pi(q') + N(p')$ scattering, where q, p, q', and p' are the momenta of the participating particles as indicated. Unfortunately we can do this only in lowest-order perturbation theory. The matrix elements $T^{(\pm)}$, where $(\pm)$ stands for symmetric and antisymmetric isospin combinations, are considered as functions of $t = (q-q')^2$ and $s = (p+q)^2$.

For q or $q' \to 0$ we have $t \to 0$ and s, $u \to m^2$. Therefore, part of the information about the behavior in the limit q or $q' \to 0$ is apparent from the form of the forward scattering amplitude $T_0(\nu) = T^{(\pm)}(\nu, t=0)$ with $\nu = p\cdot q$. In lowest order of f the amplitudes $T_0{}^{(\pm)}(\nu)$ have the following form:

$$T_0{}^{(+)}(\nu) = 0\,, \quad T_0{}^{(-)}(\nu) = 2f^2\nu/m\,. \qquad (5.28)$$

Thus both amplitudes vanish for $q = q' \to 0$ as expected.

It is difficult to calculate pion-production amplitudes in this model. A first step in this direction has been taken by Perrin.[16] But as far as we can see, his result is not pion gauge-invariant, presumably because of the approximations necessary to derive his final result.

It appears that in the gradient-coupling model, the asymptotic pion gauge invariance is realized in the most simple way. The free and the total Lagrangian have the same symmetry $\phi_\alpha \to \phi_\alpha + c_\alpha$. Therefore, the axial-vector charges $\chi_\alpha(t) = \int d^3x\, A_{0,\alpha}(x)$ are translation operators obeying $[\chi_\alpha(t), \chi_\beta(t)] = 0$ for all t and not just for $t \to \pm\infty$. The situation is different in some of the models considered recently in connection with $SU(2) \times SU(2)$ chiral symmetry. An old example of such models is the familiar "linear" σ model studied by Gell-Mann and Lévy.[17] For our purpose we are interested only in a limiting case of this model, namely for zero-

[14] G. Källén, in *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1958), Vol. V, Pt. 1.

[15] See, for example, S. Schweber, Ref. 11.

[16] R. Perrin, Phys. Rev. **162**, 1343 (1967).

[17] M. Gell-Mann and M. Lévy, Nuovo Cimento **16**, 705 (1960), and older papers quoted therein.

mass pions without symmetry breaking. Then the Lagrangian of the σ model is

$$\mathcal{L}=\tfrac{1}{2}(\partial\boldsymbol{\phi})^2+\tfrac{1}{2}(\partial\sigma)^2-\tfrac{1}{2}m_\sigma^2\sigma^2-\bar{\psi}(-i\gamma^\mu\partial_\mu+m)\psi \\ +f\bar{\psi}(\sigma+\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\phi})\psi-\tfrac{1}{2}(m_\sigma/2m)^2f^2(\sigma^2+\boldsymbol{\phi}^2)^2 \\ +(m_\sigma^2/2m)f\sigma(\sigma^2+\boldsymbol{\phi}^2)\,, \quad (5.29)$$

where σ stands for the scalar field with $I=0$. This Lagrangian is invariant under the following gauge transformation which we write in infinitesimal form[18]

$$\psi\to[1+(f/2m)\gamma_5\boldsymbol{\tau}\cdot\mathbf{c}]\psi\,, \\ \sigma\to\sigma+(f/m)\boldsymbol{\phi}\cdot\mathbf{c}\,, \quad (5.30) \\ \boldsymbol{\phi}\to\boldsymbol{\phi}+\mathbf{c}-(f/m)\sigma\mathbf{c}.$$

The current generating this transformation has the following form:

$$A_{\mu,\alpha}=\partial_\mu\phi_\alpha+(f/m)\bar{\psi}i\gamma_\mu\gamma_5(\tfrac{1}{2}\tau_\alpha)\psi \\ +(f/m)(\phi_\alpha\partial_\mu\sigma-\sigma\partial_\mu\phi_\alpha)\,. \quad (5.31)$$

Equation (5.31) specifies the transformation for the interpolating fields ψ, ϕ_α, and σ at finite t. Again it is plausible from the adiabatic hypothesis that the transformations of the corresponding asymptotic fields $\psi_{\rm in}$, $\phi_{\alpha,\rm in}$, and $\sigma_{\rm in}$ are the following:

$$\psi_{\rm in}\to\psi_{\rm in}\,, \\ \sigma_{\rm in}\to\sigma_{\rm in}\,, \quad (5.32) \\ \phi_{\alpha,\rm in}\to\phi_{\alpha,\rm in}+c_\alpha\,,$$

and similarly for the outgoing fields. This means the σ model above has the pion gauge invariance of the asymptotic pion field considered in the previous sections. To support this statement we show (following the procedure in the gradient coupling model) that the generator of the transformations (5.30), namely, $\chi_\alpha=\int d^3x\,A_{0,\alpha}$ with $A_{0,\alpha}$ given by (5.31), is equal to $\chi_{\alpha,\rm in}$, the generator of the transformations (5.32). For this purpose we write the axial-vector current (5.31) in the form

$$A_{\mu,\alpha}=\partial_\mu\phi_\alpha-j_{\mu,\alpha}\,, \quad (5.33a)$$

where

$$j_{\mu,\alpha}=-(f/m)\bar{\psi}i\gamma_\mu\gamma_5(\tfrac{1}{2}\tau_\alpha)\psi \\ -(f/m)(\phi_\alpha\partial_\mu\sigma-\sigma\partial_\mu\phi_\alpha)\,. \quad (5.33b)$$

Because the axial-vector current is conserved, $\partial^\mu A_{\mu,\alpha}=0$ or equivalently, from the equation of motion for the pion field, we have

$$\partial^\mu\partial_\mu\phi_\alpha=\partial^\mu j_{\mu,\alpha}\,. \quad (5.34)$$

Then, $\phi_\alpha(x)$ obeys the Yang-Feldman equation (5.25) and χ_α has the same form as (5.26) with the only difference that $j_{\mu,\alpha}(x)$ in (5.25) and (5.26) is now given by (5.33b) instead of (5.19b). As before (5.27) leads to $\chi_\alpha=\chi_{\alpha,\rm in}$. On the basis of these considerations we must conclude that the theorem derived in Sec. 4 is valid also for the σ model. Just recently, in a study of an extended σ model including vector and axial-vector mesons, one of us found that the amplitude for forward π-N and π-π scattering vanishes if the pion mass is zero.[19] Obviously, these two special examples are realizations of the general condition for pion gauge invariance in the σ model.

It is clear that the same statements we make about the "linear" σ model defined by the Lagrangian (5.29) can be made for the nonlinear σ model in Ref. 17, the Nishijima model as studied by Nambu and Lurié[13] and similar models investigated by Chang and Gürsey.[20] Also the models of Wess and Zumino[21] and Weinberg,[22] where interactions with vector and axial-vector mesons are included, belong to this category. In some of these models[20–22] and also in the models constructed by Schwinger[23] the authors prefer to express the chiral transformation not in terms of isospin matrices and γ_5's, but in terms of isospin matrices and the pion field. In this way the coupling of the pion with the nucleon field appears from the beginning in the form of nonlinear derivative coupling. It is usually agreed that the two methods of constructing "chiral symmetric" Lagrangians are equivalent. Nevertheless, in order to make our statements about the asymptotic form of the gauge transformations more transparent for this class of models, we shall discuss one particular example, the nonlinear model proposed by Schwinger.[24] The Lagrangian is

$$\mathcal{L}=\tfrac{1}{2}(1+f_0^2\boldsymbol{\phi}^2)^{-1}(\partial\boldsymbol{\phi})^2-\bar{\psi}\{-i\gamma_\mu\partial^\mu+m-(1+f_0^2\boldsymbol{\phi}^2)^{-1} \\ \times[if\gamma_5\gamma_\mu\boldsymbol{\tau}\cdot\partial^\mu\boldsymbol{\phi}+f_0^2\gamma_\mu\boldsymbol{\tau}\cdot(\boldsymbol{\phi}\times\partial^\mu\boldsymbol{\phi})]\}\psi. \quad (5.35)$$

This Lagrangian is invariant under the gauge transformation

$$\psi\to[1+if_0^2\boldsymbol{\tau}\cdot(\boldsymbol{\phi}\times\mathbf{c})]\psi\,, \\ \boldsymbol{\phi}\to\boldsymbol{\phi}+\mathbf{c}+f_0^2[2(\boldsymbol{\phi}\cdot\mathbf{c})\boldsymbol{\phi}-\boldsymbol{\phi}^2\mathbf{c}]. \quad (5.36)$$

The axial-vector current associated with this transformation is

$$A_{\mu,\alpha}=\partial_\mu\phi_\alpha-j_{\mu,\alpha}\,, \quad (5.37a)$$

where

$$j_{\mu,\alpha}=-f\bar{\psi}i\gamma_5\gamma_\mu\tau_\alpha\psi-2f_0^2\bar{\psi}\gamma_\mu(\boldsymbol{\tau}\times\boldsymbol{\phi})_\alpha\psi \\ -f_0^2(2\phi_\alpha\boldsymbol{\phi}\cdot\partial_\mu\boldsymbol{\phi}-3\boldsymbol{\phi}^2\partial_\mu\phi_\alpha)\,, \quad (5.37b)$$

[18] Usually this transformation is written with a variation parameter $v_\alpha=(f/m)c_\alpha$ and the current $A_{\mu,\alpha}$ is divided by f/m. The symmetry is then interpreted as a dynamical $SU(2)$ transformation. But in this form no limit to the noninteracting system can be performed.

[19] G. Kramer, Phys. Rev. **177**, 2515 (1969). The vanishing is a result of the cancellation between the nucleon-exchange and the σ-exchange terms.

[20] P. Chang and F. Gürsey, Phys. Rev. **164**, 1752 (1967); **169**, 1397 (1968), and earlier papers by Gürsey quoted therein.

[21] J. Wess and B. Zumino, Phys. Rev. **163**, 1727 (1967). See also S. Gasiorowicz and D. A. Geffen, Argonne National Laboratory Report No. ANL/HEP 6809 (unpublished).

[22] S. Weinberg, Phys. Rev. **166**, 1568 (1969).

[23] J. Schwinger, Phys. Letters **24B**, 473 (1967); Phys. Rev. **167**, 1432 (1968).

[24] J. Schwinger, Phys. Letters **24B**, 473 (1967). See also D. B. Fairlie and K. Yoshida, Ann. Phys. (N.Y.) **46**, 326 (1968).

so that, as before, the equation of motion of the pion field has the form

$$\partial^\mu \partial_\mu \phi_\alpha = \partial^\mu j_{\mu,\alpha} \tag{5.38}$$

and the generator of the chiral transformations (5.36) is expressed by

$$\chi_\alpha = \int d^3x \, (\partial_0 \phi_\alpha - j_{0,\alpha}). \tag{5.39}$$

Under the condition that asymptotic pion fields exist in this model it can be shown in the same way as for the gradient and the σ model that $\chi_\alpha = \chi_{\alpha,\text{in}}$. Therefore, the asymptotic fields for this Lagrangian are transformed as

$$\psi_{\text{in}} \to \psi_{\text{in}}, \quad \phi_{\alpha,\text{in}} \to \phi_{\alpha,\text{in}} + c_\alpha, \tag{5.40}$$

that is, by the pion gauge transformation considered in Sec. 4. Thus, the pion-connected S-matrix elements calculated with the Schwinger Lagrangian obey the gauge conditions of Sec. 4. Examples (forward π-N and π-π scattering) were already considered by Schwinger. His results yield vanishing amplitudes for these processes if $m_\pi = 0$ and coupling constants are appropriately redefined.[23]

These properties of the σ model and the nonlinear pion models might be compared with the situation in quantum electrodynamics with respect to local guage transformations. There the total Lagrangian is invariant under the familar gauge transformation

$$\begin{aligned} \psi(x) &\to e^{ie\cdot\Lambda(x)}\psi(x), \\ A_\mu(x) &\to A_\mu(x) + \partial_\mu \Lambda(x), \end{aligned} \tag{5.41}$$

which reduces for the asymptotic fields to

$$\begin{aligned} \psi_{\text{in}} &\to \psi_{\text{in}}, \\ A_{\mu,\text{in}} &\to A_{\mu,\text{in}} + \partial_\mu \Lambda(x). \end{aligned} \tag{5.42}$$

We used precisely this formulation of the local gauge invariance in terms of the asymptotic fields to derive the electromagnetic gauge condition in Sec. 4. This is well known in the literature[25-27] and is derived in different ways depending on the formalism used. In Refs. 25 and 27 the generator of the transformations (5.41) and (5.42) has the same form as (3.16) with $A_{\mu,\text{in}}(x)$ replaced by $A_\mu(x)$ in the case of (5.41), but the commutation relations for equal times for the asymptotic field and the interacting fields differ. If the commutation relations for the interacting and the asymptotic fields have the same form, the generator for the transformation (5.41) is

$$\chi_\Lambda = -\int d^3x \big[(\partial_0 \Lambda(x) \partial^\mu A_\mu(x) - \Lambda(x) \partial_0 \partial^\mu A_\mu(x)) - \Lambda(x) j_0(x) \big], \tag{5.43}$$

where $j_\mu(x)$ is the current density. But the Yang-Feldman equation which defines $A_{\mu,\text{in}}(x)$ is then[28]

$$A_\mu(x) = (g_{\mu\nu} + M \partial_\mu \partial_\nu) A_{\text{in}}{}^\nu(x) + \int d^4y \, \Delta_R(x-y) j(y), \tag{5.44}$$

with a constant M obtained in the renormalization procedure. By substituting for $\partial^\mu A_\mu(x)$ in (5.43) the expression derived from (5.44), we see that $\chi_\Lambda = \chi_{\Lambda,\text{in}}$ in quite a similar way as we saw that $\chi_\alpha = \chi_{\alpha,\text{in}}$ in the pion models.

Finally, we remark that it would be interesting to investigate models which are $SU(2) \times SU(2)$ invariant but are not pion gauge-invariant. We conjecture that in such a model the low-energy theorem does not obtain.

ACKNOWLEDGMENTS

One of us (G. K.) would like to thank T. H. Fields and K. C. Wali for the kind hospitality extended to him at Argonne National Laboratory.

---

[25] J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Addison-Wesley Publishing Co., Inc., Reading, Mass., 1955).

[26] W. E. Thirring, *Principles of Quantum Electrodynamics* (Academic Press Inc., New York, 1958).

[27] H. Rollnik, Ref. 12. Concerning asymptotic conditions and equal-time commutation relations this paper is based on Ref. 28.

[28] H. Rollnik, B. Stech, and E. Nunnemann, Z. Physik **159**, 482 (1960). See L. E. Evans and T. Fulton, Nucl. Phys. **21**, 492 (1960), for asymptotic conditions in the Coulomb gauge.