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Sum Rules, the $SU(2)\otimes SU(2)$ Charge Algebra, and Scattering Lengths for $\pi+\pi \to \pi+\pi$ *

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We discuss the Veneziano model for $\pi\pi$ scattering in connection with the hypothesis of partially conserved axial-vector current, the $SU(2)\otimes SU(2)$ charge algebra, the scattering-length ratio a_0/a_2, and the finite-energy sum rules of Dolen, Horn, and Schmid. Following Dashen and Weinstein, and abstracting from the model, it is proposed that $SU(2)\otimes SU(2)$ is a symmetry of the system, and that the strength of the symmetry-breaking interaction is proportional to the deviation of the intercept of the ρ trajectory from $\frac{1}{2}$.

I. INTRODUCTION

RECENTLY, a model for two-body scattering processes has been proposed by Veneziano.[1] The model has various defects, all of which we believe are due to violations of unitarity which accompany the narrow-resonance approximation, and all of which we ignore here.[2]

In Sec. II, we define the model, as applied to $\pi+\pi\to\pi+\pi$. In Sec. III, we begin with some general remarks about $SU(2)\otimes SU(2)$ symmetry and its breaking. We discuss the Adler consistency condition, the value of the derivative of the $I=1$ amplitude at threshold, the scattering length ratio a_0/a_2, the Adler $\pi\pi$ sum rule, and the superconvergent $I=2$ sum rule, evaluated at $t=0$. In Sec. IV, we briefly check the superconvergence of the $I=2$ sum rules for $t<0$. In Sec. V we discuss finite-energy sum rules (FESR) of the Dolen-Horn-Schmid type,[3] for $I=1$ and $I=2$. In Sec. VI, we summarize our conclusions.

II. THE MODEL DEFINED

We work in the t channel, taking for the isospin amplitudes[4]

$$X^t=\begin{pmatrix}A_0{}^t\\A_1{}^t\\A_2{}^t\end{pmatrix}=g\begin{pmatrix}-\frac{1}{2}F_0(\alpha(s),\alpha(u))+\frac{3}{2}F_0(\alpha(t),\alpha(s))+\frac{3}{2}F_0(\alpha(t),\alpha(u))\\F_0(\alpha(t),\alpha(u))-F_0(\alpha(t),\alpha(s))\\F_0(\alpha(s),\alpha(u))\end{pmatrix},\qquad(2.1)$$

where $\alpha(x)=a+bx$, and where

$$F_0(x,y)=\Gamma(1-x)\Gamma(1-y)/\Gamma(1-x-y).\qquad(2.2)$$

The choice (2.1) insures that Bose statistics, isospin conservation, and crossing symmetry are properly incorporated, while (2.2) implies average Regge asymptotic behavior, in the sense suggested by Veneziano.[1,5] The form of (2.2) leads to the violation of unitarity mentioned above. We do not attempt to improve on the

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[1] G. Veneziano, Nuovo Cimento **57**, 190 (1968).

[2] This is a rather strong statement in view of the variety of pathologies involved. Those defects known to us are (a) nonuniqueness of the choice of amplitude; (b) additive fixed poles in the angular momentum plane, in the $I=0$ and 2 amplitudes, at negative wrong-signature integers; (c) violation of factorization by the satellite states lying below the leading trajectory [further, there is the phenomenological problem that some of these states, e.g., the ρ' (1^-) degenerate with f^0, do not show themselves in the data, and that a $\pi^+\pi^-$ state at $\approx$1050 MeV may exist that is not predicted by the model]; (d) the neglect of the Pomeranchon.

With respect to (a), an infinite set of amplitudes of the form

$$\Gamma(M-x)\Gamma(N-y)/\Gamma(M+N-K-x-y)+(M\leftrightarrow N),\quad K\geqslant 1,\ M\geqslant K,\ N\geqslant K$$

can be used for $\pi^+\pi^-$ scattering. [See, e.g., Ref. 1; J. Mandula, Caltech Report No. CALT-68-178 (unpublished); and S. Mandelstam, Phys. Rev. Letters **21**, 1724 (1968).] Except for the term with $M=N=K=1$, which we use in the text, each *individual* term generates negative-resonance widths, and has an angular behavior of its pole residues which does not match the average Regge behavior $\alpha(s)^{\alpha(t)}$. We expect the nonuniqueness will be removed if all internal states are considered as external states and consistency is achieved.

The fixed poles of (b) are discussed by D. Sivers and J. Yellin, Ann. Phys. (N. Y.) (to be published), and have been independently discovered by M. A. Virasoro (private communication to S. Mandelstam) and G. Veneziano (private communication). See also S. Mandelstam and L.-L. Wang, Phys. Rev. **160**, 1490 (1967). There is phenomenological evidence that an additive fixed pole exists in the $B^{(-)}$ amplitude of πN charge-exchange scattering. See R. Dolen *et al.*, *ibid.* **166**, 1768 (1968); R. Roskies, *ibid.* **175**, 1933 (1968).

With respect to (c), we conjecture that factorization cannot be implemented even with a (convergent) infinite sum, unless one is willing to introduce new leading trajectories. For example, several workers have observed that, starting with the $M=N=K=1$ term only, for $\pi\pi\to\pi\pi$, consistency requires the existence of an abnormal C, isoscalar trajectory, degenerate with, and containing the same spin-parity content as the π trajectory.

As for (d), we have found it impossible to introduce the Pomeranchon as an ordinary Regge trajectory without accepting in addition (possibly nonleading) $I=2$ Regge trajectories.

[3] R. Dolen, D. Horn, and C. Schmid, Phys. Rev. **166**, 1768 (1968).

[4] J. Shapiro and J. Yellin, University of California Lawrence Radiation Laboratory Report No. UCRL-18500, 1968 (unpublished); J. A. Shapiro, Phys. Rev. **179**, 1345 (1969).

[5] The asymptotic behavior and other simple properties of the model are considered in great detail by J. Yellin, University of California Lawrence Radiation Laboratory Report No. UCRL-18637 (unpublished).

narrow-resonance approximation here.[6,7] However, we avoid the use of the asymptotic averaging procedure, in order to illustrate the problems involved in a strict application of (2.1) and (2.2).

For convenience we define the variables w, η, τ, z, z_u, and z_s $[x=\alpha(s),\ y=\alpha(u)]$:

$$\tau=1-x-y\,, \tag{2.3a}$$

$$\eta=\tfrac{1}{2}(x-y)\,, \tag{2.3b}$$

$$w=\tfrac{1}{2}+\tau=\tfrac{3}{2}-x-y\,, \tag{2.3c}$$

$$z=2\eta/\tau\,, \tag{2.3d}$$

$$z_u=1+2\tau/(y-\tfrac{1}{2})\,, \tag{2.3e}$$

$$z_s=1+2\tau/(x-\tfrac{1}{2})\,. \tag{2.3f}$$

We will constantly refer to the important special case $\mu\equiv m_\pi=0$, $b=1$ GeV^{-2}, $a=\frac{1}{2}$, below, as case P.

For case P, $\tau=t$, $\eta=\nu=\frac{1}{2}(s-u)$, $w=\alpha(\tau)$, $z=\cos\theta_t$, $z_s=\cos\theta_s$, and $z_u=\cos\theta_u$.

We will also need the constants

$$D=x+y+w=(s+t+u)b+3a=4\mu^2b+3a\,, \tag{2.3g}$$

$$\lambda=4\mu^2b\,, \tag{2.3h}$$

$$\delta=a-\tfrac{1}{2}\,. \tag{2.3i}$$

To formulate sum rules, we will use the expansions[8]

$$F_0(x,y)=\sum_{K=1}^{\infty}\frac{(-1)^K\Gamma(K+\tau)}{\Gamma(K)\Gamma(\tau)}\times\left(\frac{1}{\eta+\frac{1}{2}(1-\tau)-K}+\frac{1}{-\eta+\frac{1}{2}(1-\tau)-K}\right) \tag{2.4}$$

and

$$F_0(w,y)\pm F_0(w,x)=\sum_{K=1}^{\infty}\frac{\Gamma(K+\tau+\frac{1}{2})}{\Gamma(K)\Gamma(\tau+\frac{1}{2})}\left(\frac{1}{y-K}\pm\frac{1}{x-K}\right)=\sum_{K=1}^{\infty}\frac{\Gamma(K+\tau+\frac{1}{2})}{\Gamma(K)\Gamma(\tau+\frac{1}{2})}\times\left(\frac{1}{-\eta+\frac{1}{2}(1-\tau)-K}\pm\frac{1}{\eta+\frac{1}{2}(1-\tau)-K}\right), \tag{2.5}$$

which follow from the properties of the hypergeometric series ${}_2F_1$.

III. PCAC, $SU(2)\otimes SU(2)$ CHARGE ALGEBRA, AND $\pi\pi$ SCATTERING LENGTHS

We assume that the amplitude (2.1) is, up to the narrow-resonance approximation, a representation of reality.[2] Now let us try and make our model consistent with a theory in which broken $SU(2)\otimes SU(2)$ symmetry is relevant for $\pi\pi$ interactions.[9]

According to Dashen and Weinstein (DW),[10] such a theory make sense in the symmetric limit only if the pion then becomes a Goldstone boson, while the pion decay constant f_π, the nucleon mass, and g_A, the $q^2=0$ limit of axial-vector form factor, remain nonzero. In this picture[10] the $\pi\pi$ scattering amplitude $T(p_1,p_2,p_3,p_4)$ can be written, suppressing isospin indices,

$$T(p_i)\cong\epsilon A_1+B_0\xi^2\,, \tag{3.1}$$

where ϵ measures the strength of the $SU(2)\otimes SU(2)$ symmetry breaking, and ξ is a scaling factor such that for fixed P_i, $p_i=\xi P_i$. The constant B_0 appears on the left-hand side of the Adler $\pi\pi$ sum rule, and is the derivative $(d/d\nu)A_1{}^t$ $[\nu=\frac{1}{2}(s-u)]$ evaluated at $s=u=t=0$.

$$B_0=1/8\pi f_\pi{}^2\,. \tag{3.2}$$

If we assume how the symmetry-breaking interaction transforms under $SU(2)\otimes SU(2)$, we can compute ϵA_1. If it transforms as $(\frac{1}{2},\frac{1}{2})$, we get the Weinberg[11] result

$$\epsilon A_1=\mu^2/8\pi f_\pi{}^2=\mu^2B_0\,, \tag{3.3}$$

which leads to the scattering-length ratio $a_0/a_2=-\frac{7}{2}$.

If we choose $\delta=a-\frac{1}{2}=0$ in (2.1) our $\pi\pi$ amplitude vanishes quadratically as $p_i\rightarrow 0$. We therefore conjecture that the strength of the symmetry-breaking interaction is proportional to δ. Case P is then $SU(2)\otimes SU(2)$-symmetric. At $p_i=s=t=u=0$, (2.1)

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[6] What would happen to the correct $\pi\pi$ scattering amplitude if one made the narrow-resonance approximation on it is by no means clear. Three possibilities are the following. (i) Factorization may or may not be destroyed but one gets the degenerate towers of resonances of the present model; (ii) large mass shifts occur, in which, for example, the π and ρ become degenerate; $SU(6)_W$ symmetry is appropriate [S. Mandelstam (private communication)]; (iii) the satellite states arise from continuum in the actual physical amplitude and should therefore be ignored. K. Bardakci (private communication).

[7] Some problems involved in an attempt to go seriously beyond the narrow-resonance approximation are discussed by R. Roskies, Phys. Rev. Letters **21**, 1851 (1968).

[8] These expansions are discussed at some length by D. Sivers and J. Yellin, Ann. Phys. (N. Y.) (to be published). The series (2.4) is interesting because both sets of poles are simultaneously exhibited and each set individually lacks the duality property. The expansion (2.4) converges absolutely for $\tau<0$, while (2.5) converges for $\tau<-\frac{1}{2}$. We use the discontinuity of these sums here, when we leave the region of convergence.

[9] Details of the calculations in Secs. III–V and a review of the relevant PCAC and charge-algebra results are contained in J. Yellin, University of California Lawrence Radiation Laboratory Report No. UCRL-18664 (unpublished).

[10] R. Dashen and M. Weinstein, Phys. Rev. (to be published). See also Goldstone's original work: J. Goldstone, Nuovo Cimento **19**, 154 (1961); J. Goldstone, A. Salam, and S. Weinberg, Phys. Rev. **127**, 965 (1962). We thank Dr. Dashen for several very informative discussions.

As emphasized by Dashen and Weinstein, though the introduction of $SU(2)\otimes SU(2)$ symmetry does not at present lead to new results, e.g., for $\pi\pi$ scattering, it gives, in contrast to previous formulations, an exact meaning to PCAC, and this opens up the possibility of computing PCAC corrections in the future. If our guess about the connections between the intercepts of Regge trajectories and the symmetry-breaking interaction is correct, this leads to many possibilities in precisely that direction.

[11] S. Weinberg, Phys. Rev. Letters **17**, 616 (1966); N. N. Khuri, Phys. Rev. **153**, 1477 (1966). Weinberg makes the explicit assumption that the symmetry-breaking interaction transforms like $(\frac{1}{2},\frac{1}{2})$ under $SU(2)\otimes SU(2)$.

tells us that

$$X^t = \frac{g\Gamma(\frac{1}{2}-\delta)\Gamma(\frac{1}{2}-\delta)}{\Gamma(-2\delta)}\begin{bmatrix}\frac{5}{2}\\0\\1\end{bmatrix}$$

$$\cong -2\delta g\pi\begin{bmatrix}\frac{5}{2}\\0\\1\end{bmatrix}, \qquad (3.4)$$

to first order in δ.

Furthermore, the derivative relation is

$$(d/d\nu)A_1{}^t|_{\nu=0} = gb\pi. \qquad (3.5)$$

Putting in isospin and comparing (3.2)–(3.5), we have

$$(\epsilon A_1/B_0)_{\rm DW} = \mu^2 = (\epsilon A_1/B_0)_{\rm model} = -\delta/b. \qquad (3.6)$$

We do not belabor this result further here numerically except to express our satisfaction that δ comes out small. We note that (3.6) can be written $\alpha(\mu^2)=\frac{1}{2}$, which is the assumption recently made by Lovelace.[12,13]

The model gives us two more hints that δ is small[4]: (1) the widths of the satellite states become negative as δ becomes less than -0.007 for physical μ; (2) the derivatives at threshold of the s-wave amplitudes become unreasonably large if $\delta > +0.1$.

If our identification of case P with $SU(2)\otimes SU(2)$ symmetry is correct, the case-P mass spectrum should give us a clue to that effect. However, all we know in advance is that the mass spectrum consists of degenerate isospin multiplets, since the symmetry generated by the axial-vector charges is realized by the appearance of massless pions.[10] Though there are probably simple words to describe the spectrum, we have been unable to find them, and leave this as a subject for future investigation.

In the next two subsections we show that the resonance contributions to the Adler $\pi\pi$ sum rule[14] in the model are in qualitative agreement with phenomenological estimates, and that the superconvergent $I=2$ sum rule, evaluated at $t=0$ in the model, yields $\Gamma_{\epsilon\pi\pi}/\Gamma_{\rho\pi\pi} = \frac{9}{2}$. The latter result also follows from the $SU(2)\otimes SU(2)$, $I=1$ and 2 sum rules, if one assumes them to be saturated with $\epsilon(0^+)$ and $\rho(1^-)$.[15]

A. Adler $\pi\pi$ Sum Rule

From (2.1) and (2.5),

$$g^{-1}A_1{}^t(\eta,\tau) = \frac{\Gamma(\frac{1}{2}-\eta+\frac{1}{2}\tau)\Gamma(\frac{1}{2}-\tau)}{\Gamma(-\eta-\frac{1}{2}\tau)} - \frac{\Gamma(\frac{1}{2}+\eta+\frac{1}{2}\tau)\Gamma(\frac{1}{2}-\tau)}{\Gamma(\eta-\frac{1}{2}\tau)}$$

$$= \sum_{K=1}^{\infty} \frac{\Gamma(K+\tau+\frac{1}{2})}{\Gamma(\tau+\frac{1}{2})\Gamma(K)} \times\left(\frac{1}{-\eta+\frac{1}{2}(1-\tau)-K} - \frac{1}{\eta+\frac{1}{2}(1-\tau)-K}\right). \qquad (3.7)$$

Taking $d/d\eta|_{\eta=0}$ and $\tau=0$ in (3.7), we have the Adler sum rule for case P:

$$\pi = \sum_{K=1}^{\infty} \frac{\Gamma(K-\frac{1}{2})}{\Gamma(K)\Gamma(\frac{1}{2})(K-\frac{1}{2})} = 2+\frac{1}{3}+\frac{3}{20}+\frac{5}{56}+\cdots. \qquad (3.8)$$

From (3.8) we see that in this model[16] the sum rule is saturated 64% by (ρ,ϵ), 11% by (f,ρ',ϵ'), 5% by the g family, etc. According to Gilman and Harari,[17] the bump at the f^0 mass contributes less than 10%, and the g a few percent, to the sum rule, so we are in qualitative agreement with experiment.[16,18]

B. $I=2$, $t=0$ Sum Rule

Up to factors of π, the discontinuity in η arising from $A_2{}^t$ can be read off from (2.5) and (2.1):

$$D_2{}^t(\eta,\tau) = \sum_{K=1}^{\infty}(-1)^K B^{-1}(K+\tau)[\delta(\eta+\tfrac{1}{2}(1-\tau)-K) - \delta(-\eta+\tfrac{1}{2}(1-\tau)-K)]$$

$$= \tfrac{1}{4}[P_0(z_s)-P_1(z_s)]\delta(\eta-\tfrac{1}{2}) + \tfrac{3}{8}[P_2(z_s)-P_1(z_s)]\delta(\eta-\tfrac{3}{2})+\cdots + \{z_s\to z_u,\ \eta\to-\eta\}\cdots, \qquad (3.9)$$

where $B(a,b)=\Gamma(a)\Gamma(b)/\Gamma(a+b)$.

[12] C. Lovelace, Phys. Letters **28B**, 265 (1968). With reference to Lovelace's fit of the $\bar{p}n\to 3\pi$ Dalitz plot, we are informed by Dr. E. L. Berger (private communication) that a more detailed comparison with the data using Lovelace's expression yields the result that the zero at $\tau=0$ must be moved to $\tau=-1.8$ GeV^2. This tends to cast grave doubts on the validity of mass extrapolations made in Lovelace's manner.

Our approach is rather orthogonal to Lovelace's in that we believe the use of the narrow-resonance approximation makes these results qualitative only. Because of the zero in the model a_2 at $\delta=0$, we cannot check the consistency of a_0/a_2, in the model, with $-\frac{7}{2}$, without additional information, such as our conjecture about the symmetry breaking.

[13] M. Ademollo, G. Veneziano, and S. Weinberg, Phys. Rev. Letters **22**, 83 (1969), have generalized the argument about the PCAC zero in $\pi\pi\to\pi\pi$ to all hadronic amplitudes and suggest a rule which spaces certain Regge intercepts by half-integers. From our point of view this is a manifestation of the fact that for exact $SU(2)\otimes SU(2)$ symmetry the intercepts are *precisely* integral or half-integral.

More details of the extension are contained in the work of C. Goebel, M. Blackmon, and K. C. Wali (unpublished).

[14] S. L. Adler, Phys. Rev. **137**, 1022 (1965).

[15] F. J. Gilman and H. Harari, Phys. Rev. **165**, 1803 (1968). In their work the assumption of $SU(2)\otimes SU(2)$ symmetry breaking through the $(\frac{1}{2},\frac{1}{2})$ representation plays an essential role.

[16] In connection with the derivative of the $I=1$ amplitude, it is interesting that in the model the quantity $L=\frac{1}{6}(2a_0-5a_2)$ satisfies $\mu L = \frac{1}{2}\pi\lambda g(1+4.7\delta+1.4\lambda+\text{quadratic terms})$ for small δ and λ, so that for $b\cong 1$ GeV^{-2}, $g\cong 1$, and $\delta\cong 0$, $\mu L\cong 0.125$ as compared with the charge-algebra result 0.10.

[17] See Ref. 15, p. 1823, paragraph 10 and footnote 67.

[18] It will be noted that the KSRF relation (Ref. 15, p. 1817 and footnote 53) reads $0.92\cong 1.0$ from experiment (using $\Gamma_{\rho\pi\pi}\cong 112$ MeV, and f_π as given by $\Gamma_{\pi\mu\nu}$) and $0.64\cong 1.0$ from our model. In order to derive the $\frac{9}{2}$ ratio for $\Gamma_{\epsilon\pi\pi}/\Gamma_{\rho\pi\pi}$ from the charge-algebra sum rules one needs the KSRF relation [or the arguments of S. Weinberg, Phys. Rev. **178**, 2604 (1969)] in addition to the $I=1$ and 2 sum rules. We shed no light on this situation here.

The usual superconvergent $I=2$ sum rule reads

$$\int_{\nu_0}^{\infty} \nu d\nu\, D_2{}^t(\nu,0)=0. \qquad (3.10)$$

From (3.9) it can be seen that for case P, at each mass corresponding to η=half-integer there is a degenerate tower of states with spins running from 0 to 2η. For example, at $\eta=\pm\frac{1}{2}$ we have $\rho(1^-)$ and $\epsilon(0^+)$, while at $\eta=\pm\frac{3}{2}$ we have $f(2^+)$, $\rho'(1^-)$, and $\epsilon'(0^+)$, etc.

Along $\tau=0$, the amplitude $A_2{}^t$ and its discontinuity vanish, and the contribution of each tower to the sum rule (3.10) is zero, so that the resonances in every tower cancel each other. The cancellation at $z_s=0$ is explicitly exhibited in (3.9).

Since there are no $I=2$ poles, $D_2{}^t$ crosses into

$$D_2{}^t=\tfrac{1}{3}D_0{}^s-\tfrac{1}{2}D_1{}^s$$
$$=\tfrac{1}{3}\sum_{J=0}^{\infty}(2J+1)b_0(J,\nu)P_J(z_s)\times\tfrac{1}{2}[1+(-1)^J]$$
$$-\tfrac{1}{2}\sum_{J=1}^{\infty}(2J+1)b_1(J,\nu)P_J(z_s)\tfrac{1}{2}[1-(-1)^J], \qquad (3.11)$$

where the u, left-hand discontinuity, has been suppressed. Assuming that $D_2{}^t|_{t=0}=0$, and that we have degenerate towers, as in the model, then at the lowest-mass tower,

$$\tfrac{1}{3}b_0(0,\nu_1)-\tfrac{3}{2}b_1(1,\nu_1)=0, \qquad (3.12)$$

yielding the ratio $\Gamma_{\epsilon\pi\pi}/\Gamma_{\rho\pi\pi}=\frac{9}{2}$. Any model having (a) degenerate towers with the proper spin content, (b) a zero in $A_2{}^t$ along $t=0$, and (c) no $I=2$ poles, will yield the $\frac{9}{2}$ ratio.

As pointed out by Gilman and Harari,[15] the $\frac{9}{2}$ ratio also comes out of the $I=1$ and 2 charge-algebra sum rules, provided one assumes they are saturated by ρ and ϵ only.

IV. $I=2$ SUM RULES FOR $\tau<0$

As τ gets negative, poles in $F_0(x,y)$ begin to move out into the unphysical, double spectral region and the s and u poles cross.[19]

We can check that each sum in (3.9) still separately superconverges. For simplicity, consider $\tau=-N$, $(N=1, 2, \cdots)$ and take any odd moment. Then we should have

$$\int d\eta \sum_{J=1}^{\infty}\eta^{2P+1}(-1)^J B^{-1}(-N,J)\times\delta(\eta+\tfrac{1}{2}-J-\tfrac{1}{2}N)=0. \qquad (4.1)$$

Because we have chosen $\tau=-N$, the sum truncates at N, and changing variables, (4.1) becomes

$$\sum_{Q=-(N-1)/2}^{+(N-1)/2}\frac{Q^{2P+1}\Gamma(N+1)}{\Gamma(\frac{1}{2}(N+1)-Q)\Gamma(\frac{1}{2}(N+1)+Q)}=0, \qquad (4.2)$$

showing the cancellation explicitly.

V. SUM RULES FOR $\tau>0$ (FESR)

A. $I=2$ Sum Rules

We now consider the lowest-moment FESR on the right-hand discontinuity in (3.8),[20]

$$\frac{1}{2}\int_{-U}^{+U}\eta d\eta\sum_{K=1}^{\infty}(-1)^K B^{-1}(K,\tau)\times\delta(\eta+\tfrac{1}{2}-K-\tfrac{1}{2}\tau)=0. \qquad (5.1)$$

Let us check and see in what sense (5.1) holds. In (5.1) we choose U so that the highest mass pole included has $K=N$. $(-\frac{1}{2}+N+\frac{1}{2}\tau\leqslant U\leqslant -\frac{1}{2}+N+1+\frac{1}{2}\tau.)$ The left-hand side then becomes

$$\tfrac{1}{2}\sum_{K=1}^{N}(-1)^K B^{-1}(\tau,K)(2K-1+\tau)$$
$$=(-1)^N B^{-1}(\tau,N)\tfrac{1}{2}(N+\tau)$$
$$=\tfrac{1}{2}(-1)^N T_{N+1}(\tau)/\Gamma(N), \qquad (5.2)$$

where $T_N(x)=\Gamma(x+N)/\Gamma(x)$.

Equation (5.2) can be easily proved by induction. The sum changes sign and grows in absolute value as each new resonance is included, so that there are violent cancellations.[21] If we give a finite width to the resonances, we can always find a point intermediate between any pair of neighboring resonances such that the sum vanishes. This remains true for all the moments.

B. $I=1$ Sum Rules for $\tau>0$ (FESR)

The $I_t=1$ discontinuity, from (2.1) and (2.5), is

$$D_1{}^t=\sum_{J=1}^{\infty}\frac{\Gamma(\frac{1}{2}+\tau+J)}{\Gamma(J)\Gamma(\frac{1}{2}+\tau)}\times[\delta(\eta+\tfrac{1}{2}-\tfrac{1}{2}\tau-J)+\delta(-\eta+\tfrac{1}{2}-\tfrac{1}{2}\tau-J)]. \qquad (5.3)$$

Just as for the $I=2$ case, we take U such that $N+\frac{1}{2}\tau-\frac{1}{2}<U<N+1+\frac{1}{2}\tau-\frac{1}{2}$, and for the zeroth-mo-

[19] This situation is interesting because it is in this amplitude that the fixed poles in J occur. One can check that they are present by examining the Schwarz sum rules [J. Schwarz, Phys. Rev. **159**, 1269 (1967)]. This has been done by Veneziano (private communication). The superconvergence of the $\pi^+\pi^-$ amplitude at $t<0$ has been used by Schmid to construct an amplitude agreeing with (2.1). See C. Schmid, Phys. Letters **28B**, 348 (1969); Nuovo Cimento (to be published). We thank Dr. Schmid for several very helpful private communications.

[20] C. Schmid and J. Yellin, Phys. Rev. (to be published), give a detailed discussion of the FESR in connection with 0^-0^- scattering. Further references are given there. Veneziano (Ref. 1) also discusses FESR for large limits of integration, and in an average sense.

[21] This is to be expected because the $I=1$ and 0 resonance contributions are opposite in sign and we are going out of the physical region where the Legendre series diverges. See Ref. 20 and S. Mandelstam, Phys. Rev. **166**, 1539 (1968), Sec. VI, for opposing views on whether or not one should formulate the FESR at positive t.

Because of the oscillating behavior the $I=2$ FESR were not used for numerical work in Ref. 20. However, in the $I=0$ case the oscillations occur about the Regge term rather than zero, and the resulting relations were used numerically.

ment FESR we have

$$\frac{1}{2}\int_{-U}^{+U} d\eta\, D_1{}^t(\eta,\tau)=\sum_{J=1}^{N}\frac{\Gamma(\frac{1}{2}+\tau+J)}{\Gamma(J)\Gamma(\frac{1}{2}+\tau)}=\frac{T_{N+1}(\tau+\frac{1}{2})}{\Gamma(N)(\tau+\frac{3}{2})}, \quad (5.4)$$

which one easily can prove by induction. If we expand in powers of N, we have

$$\frac{1}{2}\int_{-U}^{+U} d\eta\, D_1{}^t(\eta,\tau)=\frac{N^{3/2+\tau}}{(\tau+\frac{3}{2})\Gamma(\tau+\frac{1}{2})}\left(1+\frac{(\frac{3}{2}+\tau)(\frac{1}{2}+\tau)}{2N}+O(N^{-2})\right), \quad (5.5)$$

or, inserting $\alpha(t)=\tau+\frac{1}{2}$, the right-hand side takes the familiar FESR form[20]

$$\frac{N^{\alpha(t)+1}}{[\alpha(t)+1]\Gamma(\alpha(t))}\left(1+\frac{[\alpha(t)+1]\alpha(t)}{2N}+\cdots\right). \quad (5.6)$$

At $\alpha(t)=1$ (i.e., at $t=m_\rho{}^2$) this becomes

$$\tfrac{1}{2}N^2(1+1/N+\cdots), \quad (5.7)$$

so that we commit a 50% error if we choose to keep the leading trajectory only on the right-hand side of the FESR, and take $N=2$. (This means that we keep the ρ and f families on the left.) Let us see what happens on the left if we keep only ρ and f. Rewriting the first term in (5.3) as Legendre polynomials in z_s, we have

$$\begin{aligned}\sum_{J=1}^{\infty}&\frac{\Gamma(\frac{1}{2}+\tau+J)}{\Gamma(J)\Gamma(\frac{1}{2}+\tau)}\delta(\eta+\tfrac{1}{2}-\tfrac{1}{2}\tau-J)\\&=(\tfrac{1}{2}+\tau)\delta(\eta-\tfrac{1}{2}-\tfrac{1}{2}\tau)+(\tfrac{1}{2}+\tau)(\tfrac{3}{2}+\tau)\delta(\eta-\tfrac{3}{2}-\tfrac{1}{2}\tau)+\cdots\\&=\tfrac{1}{4}[P_0(z_s)+P_1(z_s)]\delta(\eta-\tfrac{1}{2}-\tfrac{1}{2}\tau)\\&\quad+\tfrac{3}{8}[P_2(z_s)+P_1(z_s)]\delta(\eta-\tfrac{3}{2}-\tfrac{1}{2}\tau)+\cdots, \quad (5.8)\end{aligned}$$

so that the resonances cancel in the backward direction, as they should. At $\tau=\frac{1}{2}$, the ρ and f contributions to the left-hand side of the FESR are, from (5.8), using $z_s=1+2\tau/(x-\frac{1}{2})$,

$$\tfrac{1}{4}\times3+\tfrac{3}{8}\times11/3=17/8, \quad (5.9)$$

while the ϵ and ρ' contribute

$$\tfrac{1}{4}\times1+\tfrac{3}{8}\times5/3=\tfrac{7}{8}, \quad (5.10)$$

making a total of 3, which checks with (5.4).

Therefore, while the exact relation reads $3=3$, the FESR[3] at $t=m_\rho{}^2$, with ρ and f^0 on the left, and ρ on the right, reads $17/8\cong2$, since compensating errors have been made.

The $I=0$ sum rule, which is suspect in any case because we have neglected the Pomeranchon,[22] contains the oscillating object already associated with the $I=2$ sum rule. The same calculation as was performed here for the $I=1$ case can be done for $I=0$, and is left as an exercise for the enterprising reader.[23]

VI. SUMMARY AND CONCLUSION

We have shown that the Veneziano model, applied to $\pi+\pi\to\pi+\pi$, is consistent with the $SU(2)\otimes SU(2)$ charge algebra and with PCAC.

We conjecture that the underlying $SU(2)\otimes SU(2)$ symmetry of the $\pi\pi$ system is broken by an interaction which moves the intercept of the ρ trajectory away from $\frac{1}{2}$. We then find that $\delta=a-\frac{1}{2}$ is small if one is to get consistency with the results of Dashen and Weinstein[10] and of Weinberg.[11] Without this conjecture we are unable to check the model's consistency with the scattering-length ratio $a_0/a_2=-\frac{7}{2}$, because of its great sensitivity, in the model, to the precise value of the intercept of the ρ trajectory. However, if a_0/a_2 is to be appreciably different from $+\frac{5}{2}$, the intercept must in any case be near $\frac{1}{2}$ ($a_0/a_2\cong\frac{5}{2}+6\mu^2b/\delta$ for small δ and μ^2). We have also shown that the model has the qualitative behavior suggested by Dolen, Horn, and Schmid[3] with respect to FESR, evaluated at positive t.

We have made no detailed comparison with experiment because we believe the use of the narrow resonance approximation renders this a futile exercise.[24]

In our view, in order to go further than we have done here, one must attack the problem of including additional features of unitarity.

Note added in proof. In making the remarks in Sec. III about $SU(2)\otimes SU(2)$-symmetry breaking, I have implicitly assumed that the difficulties which arise if one perturbs the $SU(3)\otimes SU(3)$-symmetric limit and tries to compute the mass-squared matrix for the pseudoscalar mesons [see, e.g., R. F. Dashen, Institute for Advanced Study Report, Sec. IV B (unpublished)] do not occur here. I am indebted to H. R. Pagels for pointing this out to me. While I do not believe there is any problem with respect to $\pi\pi$ scattering, the question is a subtle one, and the reader is referred to Sec. 4 of Dashen and Weinstein (Ref. 10) for a thorough discussion.

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[22] For a discussion of FESR and the Pomeranchon, see H. Harari, Stanford Linear Accelerator Center Report No. SLAC-PUB-463, 1968 (unpublished).

[23] What is missing here is an exact way of stating the FESR so that the nonleading contributions can be calculated. The reader is challenged to find one.

[24] This futility is evidenced in some detail in Refs. 4 and 5, with respect to the nonexistence of the satellite state ρ' and the large predicted widths for the satellites of the g.