Strong-Coupling Solution of the Bronzan-Lee Model*

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An approximate solution of the nonrelativistic Bronzan-Lee model in all sectors is described. By using Tomonaga's intermediate-coupling approximation and then minimizing a strong-coupling approximation to the lowest set of energy eigenvalues, a relatively simple strong-coupling solution is obtained which avoids the restriction to large-source radius inherent in North's strong-coupling treatment. An explicit calculation is carried out for the special case $f=g\gg 1$, in which case one finds an isobar spectrum $E_q \sim (6q^2-1)g^{-2}$, where q denotes the total charge.

In a recent article,[1] North has solved the nonrelativistic Bronzan-Lee model[2] by using the methods of old-fashioned strong-coupling theory.[3] North's solution of the Bronzan-Lee model is obtained by using the same method which he previously used[4] to solve the ordinary Lee model[5] (OLM). North's solution of the OLM is valid provided

$$1 \ll (mg^2)/(mR)^3 \ll (mR)^4,$$

where g is the unrenormalized coupling constant, m is the meson mass, and R^{-1} is the momentum cutoff. In other words, North's solution should be regarded as a *large-source*, strong-coupling approximation.

The present note describes the results of an attempt to remove the restriction to large source radius which was required in Ref. 1. The present calculation is based on Tomonaga's intermediate-coupling approximation[6] (ICA), and in principle the method can be used to obtain numerical results for arbitrary values of the bare coupling constants f and g.[7] However, in order to obtain *explicit* analytical expressions for the ICA results, it is necessary to minimize a strong-coupling approximation to the lowest set of energy eigenvalues (rather than minimize the exact expression, which can only be done numerically on a computer). Thus, explicit results are given below only for the special case $f=g\gg 1$.

In addition to removing the restriction to large R inherent in North's treatment, the present method clarifies the exact nature of the relationship between the spectrum of the Bronzan-Lee model and that of the OLM: The two spectra are essentially identical in the two limiting cases, $f\to 0$ and $(g/f)\to 0$, but for arbitrary values of f and g there does *not* appear to be any simple relationship between the two spectra. The present treatment also indicates that the U^- particle's bare mass has to be set equal to $(fQ)^2\omega^{-1}$ in order to make the physical U^-, physical neutron, and proton mass degenerate. Since this choice is *not* the same as North's, $\epsilon_U=(fgQ^2)^2/(3\omega)$, our results are necessarily different from those given in Ref. 1, in spite of the fact that in the static approximation North's treatment may be regarded as a special case of the intermediate-coupling approximation described below.

The Hamiltonian for the nonrelativistic Bronzan-Lee model may be written in the form[8]

$$H=H_{\text{mes}}+H_{\text{int}}+H_{\text{baryon}}, \tag{1}$$

where

$$H_{\text{mes}}=\sum_k \omega_k a_k^\dagger a_k, \tag{2}$$

$$H_{\text{int}}=\tfrac{1}{2}[g(\lambda_1-i\lambda_2)+f(\lambda_6-i\lambda_7)]\sum_k u_k a_k + \tfrac{1}{2}[g(\lambda_1+i\lambda_2)+f(\lambda_6+i\lambda_7)]\sum_k u_k a_k^\dagger, \tag{3}$$

$$H_{\text{baryon}}=(\tfrac{1}{3}I-\tfrac{1}{2}\lambda_3+\tfrac{1}{6}\sqrt{3}\lambda_8)\epsilon_N+(\tfrac{1}{3}I-\tfrac{1}{3}\sqrt{3}\lambda_8)\epsilon_U. \tag{4}$$

Here $a_k^\dagger$ and a_k are the creation and annihilation operators of the mesons of momentum $\mathbf{k}$, and

$$u_k=(2\pi)^{-3/2}\int u(\mathbf{r})e^{i\mathbf{k}\cdot\mathbf{r}}d^3r, \tag{5}$$

where $u(\mathbf{r})$ is the baryon source function, normalized according to

$$\int u(\mathbf{r})d^3r=1. \tag{6}$$

ω_k denotes the total energy of a nonrelativistic meson with momentum $\mathbf{k}$, $\omega_k=\mathbf{k}^2+\frac{1}{2}$, where energies are expressed in units of $2m$. g is the bare (π^-,P,N) coupling

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[1] G. R. North, Phys. Rev. **168**, 1698 (1968).
[2] J. B. Bronzan, Phys. Rev. **139**, B751 (1965).
[3] See, for example, W. Pauli, *Meson Theory of Nuclear Forces* (Wiley-Interscience, Inc., New York, 1948), 2nd ed.
[4] G. R. North, Phys. Rev. **164**, 2056 (1967).
[5] T. D. Lee, Phys. Rev. **95**, 1329 (1954).
[6] S. Tomonaga, Progr. Theoret. Phys. (Kyoto) **2**, 6 (1947).
[7] The question of whether the ICA represents a valid approximation to the exact solution of the Bronzan-Lee model for arbitrary values of f and g is, of course, a much more difficult question to answer. This problem will not be discussed here, except to say that in the case of exactly soluble models, where a detailed comparison between the ICA and the exact solution is possible, the ICA usually yields fairly good results for arbitrary coupling strengths.

[8] The Hamiltonian written down here is identical to the one given by Eq. (1) of Ref. 1. The traceless, Hermitian 3×3 matrices λ_i ($i=1, 2, \cdots, 8$) are defined in, for example, J. Bernstein, *Elementary Particles and Their Currents* (W. H. Freeman and Co., San Francisco, 1968), p. 213. I denotes the 3×3 identity matrix.

constant, f is the bare (π^-,N,U) coupling constant, and $\epsilon_{N,U}$ may be regarded as the bare N, U masses.

The only processes allowed in the Bronzan-Lee model are

$$P+\pi^- \leftrightarrow N\,, \tag{7a}$$

$$N+\pi^- \leftrightarrow U^-\,, \tag{7b}$$

where North's convention has been adopted [in Bronzan's original article, the baryons are called N, V, U (instead of P, N, U, respectively) and the meson is called θ]. It is clear from the allowed processes (7) that the Bronzan-Lee model conserves the total electric charge of the meson-baryon system. This corresponds to the fact that the operator of total charge,

$$q=\tfrac{1}{2}(\lambda_3+\sqrt{3}\lambda_8)-\sum_k a_k{}^\dagger a_k\,, \tag{8}$$

commutes with the total Hamiltonian (1). Just as in the case of the OLM, the eigenvalues of the total charge operator are restricted to the following integer values: $+1, 0, -1, -2, -3, \cdots, -\infty$.

It can be shown[9] that Tomonaga's[6] intermediate coupling approximation is equivalent to the following substitution of reduced-space operators in the Hamiltonian:

$$a_k \rightarrow f_k a\,, \tag{9a}$$

$$a_k{}^\dagger \rightarrow f_k a^\dagger\,. \tag{9b}$$

The trial function f_k is chosen to minimize the *lowest* set of eigenvalues of the reduced-space Hamiltonian, which turns out to be

$$H_{\rm ICA}=\omega a^\dagger a+\tfrac{1}{2}gQ[(\lambda_1-i\lambda_2)a+\text{H.c.}] +\tfrac{1}{2}fQ[(\lambda_6-i\lambda_7)a+\text{H.c.}]+H_{\rm baryon}\,, \tag{10}$$

where

$$\omega\equiv\sum_k \omega_k f_k{}^2\,, \tag{11}$$

$$Q\equiv\sum_k u_k f_k\,. \tag{12}$$

We also note that the reduced-space operator of total charge is given by

$$q_{\rm ICA}=\tfrac{1}{2}(\lambda_3+\sqrt{3}\lambda_8)-a^\dagger a\,, \tag{13}$$

where the normalization condition,

$$\sum_k f_k{}^2=1\,, \tag{14}$$

has been taken into consideration.

North's strong-coupling treatment of the Bronzan-Lee model leads to a static Hamiltonian of the same mathematical form[10]:

$$H_{\rm North}=wa^\dagger a+\tfrac{1}{2}G[(\lambda_1-i\lambda_2)a+\text{H.c.}] +\tfrac{1}{2}F[(\lambda_6-i\lambda_7)a+\text{H.c.}]+H_{\rm baryon}\,, \tag{15a}$$

where

$$w\equiv N^{-1}\sum_k \omega_k u_k{}^2\,, \tag{15b}$$

$$G\equiv gN^{1/2}\,, \tag{15c}$$

$$F\equiv fN^{1/2}\,, \tag{15d}$$

$$N\equiv\sum_k u_k{}^2\,. \tag{15e}$$

It should be noted that North's strong-coupling approximation may be regarded as a special case (namely, $f_k=N^{-1/2}u_k$) of Tomonaga's intermediate-coupling approximation.[11]

Following the same mathematical procedure as North,[1] we seek simultaneous eigenfunctions of $H_{\rm ICA}$ and $q_{\rm ICA}$ in the form

$$\phi_n=(1+c_n{}^2+d_n{}^2)^{-1/2}\begin{bmatrix}\psi_n\\ c_n\psi_{n-1}\\ d_n\psi_{n-2}\end{bmatrix}\,, \tag{16}$$

where the harmonic oscillator functions ψ_n are defined according to

$$a\psi_n=0\,, \tag{17a}$$

$$\psi_n=(n!)^{-1/2}(a^\dagger)^n\psi_0\,, \tag{17b}$$

$$a^\dagger a\psi_n=n\psi_n\,. \tag{17c}$$

The desired eigenfunctions satisfy

$$H_{\rm ICA}\phi_n=E_n\phi_n\,, \tag{18}$$

$$q_{\rm ICA}\phi_n=-(n-1)\phi_n\,. \tag{19}$$

The matrix equation (18) is equivalent to three simultaneous equations:

$$n\omega+gQc_n\sqrt{n}=E_n\,, \tag{20a}$$

$$gQ\sqrt{n}+c_n[(n-1)\omega+\epsilon_N]+fQd_n(n-1)^{1/2}=c_nE_n\,, \tag{20b}$$

$$fQc_n(n-1)^{1/2}+d_n[(n-2)\omega+\epsilon_U]=d_nE_n\,. \tag{20c}$$

Just as in the case of the ordinary Lee model, ϵ_N is set equal to $g^2Q^2\omega^{-1}$ in order to make $E_1{}^-=0$. $E_1{}^+$ vanishes regardless of the value chosen for the bare mass ϵ_N.

Eliminating c_n and d_n from Eqs. (20), one obtains the following cubic equation for E_n:

$$E_n{}^3+a_2E_n{}^2+a_1E_n+a_0=0\,, \tag{21a}$$

where

$$a_0\equiv -n(n-1)\omega[(n-2)\omega^2+\epsilon_U\omega-f^2Q^2]\,, \tag{21b}$$

$$a_1\equiv \epsilon_U[g^2Q^2\omega^{-1}+(2n-1)\omega]+(3n^2-6n+2)\omega^2 +(n-2)g^2Q^2-(n-1)f^2Q^2\,, \tag{21c}$$

$$a_2\equiv -3(n-1)\omega-g^2Q^2\omega^{-1}-\epsilon_U\,. \tag{21d}$$

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[9] T. D. Lee and D. Pines, Phys. Rev. **92**, 883 (1953).

[10] Equation (15a) is identical to Eq. (2) of Ref. 1.

[11] This statement applies to the static strong-coupling approximation only, i.e., North's treatment is identical to the choice $f_k=N^{-1/2}u_k$ in the ICA if and only if the quasifree fields ϕ' and $\phi'^\dagger$ are formally set equal to zero. However, this is the usual meaning of a strong-coupling approximation since the quasifree fields are of order $(1/g)$ relative to the static field.

In particular, for the sector $n=2$, one finds

$$E_2^3-(3\omega+g^2Q^2\omega^{-1}+\epsilon_U)E_2^2+[\epsilon_U(g^2Q^2\omega^{-1}+3\omega)+2\omega^2-f^2Q^2]E_2+2\omega(f^2Q^2-\epsilon_U\omega)=0. \quad (22)$$

In analogy to the case of the ordinary Lee model (where the value of the bare mass ϵ_N is adjusted so that the physical neutron has the same mass as the proton), one may also adjust the value of the bare mass ϵ_U to make the physical U particle have the same mass as the physical neutron and proton. In other words, the value of ϵ_U is adjusted so that Eq. (22) has one zero root,[12] i.e., we set ϵ_U equal to $f^2Q^2\omega^{-1}$. Then the other two roots of Eq. (22) are easily found to be

$$E_2^{\pm}=\tfrac{1}{2}(3\omega+g^2Q^2\omega^{-1}+f^2Q^2\omega^{-1})\pm\tfrac{1}{2}[(3\omega+g^2Q^2\omega^{-1}-f^2Q^2\omega^{-1})^2+4(f^2Q^2-2\omega^2)]^{1/2}. \quad (23)$$

Note that if the bare U particle is decoupled (i.e., $f\rightarrow 0$), the two roots (23) become identical to the results for the ordinary Lee model. This is consistent with our interpretation of the zero root of Eq. (22) as the mass of the physical U particle.

In connection with the general nature of the relationship between the Bronzan-Lee model and the ordinary Lee model, it is instructive to note that in the ICA the energy eigenvalues are given by

$$\begin{vmatrix} n-y & g'\sqrt{n} & 0 \\ g'\sqrt{n} & n-1+g'^2-y & f'\sqrt{(n-1)} \\ 0 & f'\sqrt{(n-1)} & n-2+f'^2-y \end{vmatrix}=0, \quad (24)$$

where the notation has been simplified somewhat by introducing

$$y\equiv E_n/\omega, \quad (25a)$$

$$g'\equiv g(Q/\omega), \quad (25b)$$

$$f'\equiv f(Q/\omega). \quad (25c)$$

Expanding the determinant, one finds that the energy levels of the Bronzan-Lee model are given by

$$(n-2+f'^2-y)\begin{vmatrix} n-y & g'\sqrt{n} \\ g'\sqrt{n} & n-1+g'^2-y \end{vmatrix}-f'^2(n-1)(n-y)=0, \quad (26)$$

where the two-by-two determinant is the same one which determines the energy spectrum of the ordinary Lee model (OLM), i.e., the energy levels of the OLM are given by

$$\begin{vmatrix} n-y & g'\sqrt{n} \\ g'\sqrt{n} & n-1+g'^2-y \end{vmatrix}=0. \quad (27)$$

It is clear from Eqs. (26) and (27) that the two spectra coincide as the U particle is decoupled (except that the Bronzan-Lee model has an extra set of energy levels given by $y=n-2$). However, for nonvanishing values of the coupling constant f, the presence of the additional term $-f'^2(n-1)(n-y)$ in Eq. (26) implies that, in general, there is no simple relation between the energy spectra of the two models.[13]

The exact roots of the cubic equation (24) may be written in the form[14]

$$y_n^{\mathrm{I}}=(n-1)+\tfrac{1}{3}(f'^2+g'^2)+2(\sqrt{c})\cos\tfrac{1}{3}\phi, \quad (28a)$$

$$y_n^{\mathrm{II}}=(n-1)+\tfrac{1}{3}(f'^2+g'^2)-(\sqrt{c})[\cos\tfrac{1}{3}\phi+\sqrt{3}\sin\tfrac{1}{3}\phi], \quad (28b)$$

$$y_n^{\mathrm{III}}=(n-1)+\tfrac{1}{3}(f'^2+g'^2)-(\sqrt{c})[\cos\tfrac{1}{3}\phi-\sqrt{3}\sin\tfrac{1}{3}\phi], \quad (28c)$$

where

$$c\equiv\tfrac{1}{3}+\tfrac{1}{3}(n-2)f'^2+\tfrac{1}{3}ng'^2+\tfrac{1}{9}(f'^4-f'^2g'^2+g'^4), \quad (28d)$$

$$\phi\equiv\cos^{-1}(-b/2c\sqrt{c}), \quad (28e)$$

$$-\tfrac{1}{2}b=-\tfrac{1}{2}(n-\tfrac{4}{3})f'^2+\tfrac{1}{2}[n-\tfrac{2}{3}]g'^2+\tfrac{1}{6}(n-2)f'^4+\tfrac{1}{6}ng'^4-\tfrac{1}{6}(n-1)(f'g')^2+(1/27)f'^6+(1/27)g'^6-(1/18)(f'^4g'^2+f'^2g'^4). \quad (28f)$$

At this point we introduce the *ansatz*

$$(\alpha\omega_k+\beta)f_k=u_k, \quad (29)$$

where the α and β are to be chosen to minimize the *lowest* set of energy eigenvalues and simultaneously satisfy the normalization condition (14). Equation (29) implies the following relation between ω and Q:

$$\alpha\omega+\beta=Q. \quad (30)$$

The explicit form of the intermediate-coupling solution can be determined in the following manner. It is clear from Eqs. (28a)–(28c) that the lowest set of energy eigenvalues can be expressed as a function of ω and Q. We shall henceforth denote the lowest set of energy

[12] In Ref. 1, North states that ϵ_U is chosen so that one root of the cubic equation for E_2 coincides with the lower-energy eigenvalue (here denoted by E_2^{OLM}) for the $n=2$ sector of the ordinary Lee model. However, the quadratic equation for the energy eigenvalues of the OLM is easily shown to be $E_n^2-[(2n-1)\omega+g^2Q^2\omega^{-1}]E_n+n(n-1)\omega^2=0$. Note that the coefficient of ϵ_U in Eq. (21a) turns out to be $-E_2^2+[g^2Q^2\omega^{-1}+3\omega]E_2-2\omega^2$, which vanishes if $E_2=E_2^{\mathrm{OLM}}$. In other words, one cannot find a *finite* value of ϵ_U such that one of the roots will coincide with the lower (or the upper) root of the OLM. North's choice $\epsilon_U=(fgQ^2)^2/(3\omega^3)$ is very large and represents an approximation to the exact result that ϵ_U must be infinite in order to make one of the roots equal to E_2^{OLM}.

[13] At the end of Ref. 1 North makes the following statement: ". . . we can recover the Lee-model spectrum and renormalized couplings in either of two ways; (1) decoupling the U^- altogether, (2) taking the strong-coupling limit of the U^- bare Yukawa coupling." It is clear from Eqs. (26) and (27) that North's statement about decoupling the U^- altogether is correct. In the limit $(g/f)\rightarrow 0$, the determinant on the left-hand side of Eq. (25) becomes

$$\begin{vmatrix} n-y & 0 & 0 \\ 0 & n-1-y & f'\sqrt{n} \\ 0 & f'\sqrt{n} & n-2+f'^2-y \end{vmatrix},$$

which is obviously equal to $n-y$ times the determinant which characterizes the energy eigenvalues for the OLM (except that n is replaced by $n-1$ and g is replaced by f). Thus, for any finite value of g, one will certainly recover the Lee-model spectrum by going to the limit $f\rightarrow\infty$.

[14] See, for example, *C. R. C. Standard Mathematical Tables*, edited by C. D. Hodgman (Chemical Rubber Publishing Co., Cleveland, Ohio, 1957), 11th ed., pp. 344, 345.

eigenvalues by $E_n(\omega,Q)$. Then, using relation (30) between ω and Q, one can easily determine the total derivative of $E_n(\omega,Q)$ with respect to ω: $(dE_n/d\omega) = (\partial E_n/\partial\omega)+\alpha(\partial E_n/\partial Q)$. The minimization condition $(dE_n/d\omega)=0$, then yields one relation between α and β, and the normalization condition (14) yields a second relation. Hence α and β are uniquely determined for any sector n and for arbitrary values of the coupling constants f and g and of the momentum cutoff R^{-1} (except that the results cannot be extrapolated to the point-source limit, $R=0$).

To see how the calculation goes in practice, let us consider the special case $f=g$ in detail. In this case Eqs. (28d) and (28f) simplify to

$$c=\tfrac{1}{3}+\tfrac{2}{3}(n-1)g'^2+\tfrac{1}{9}g'^4, \tag{31a}$$

$$-\tfrac{1}{2}b=\tfrac{1}{3}g'^2+\tfrac{1}{6}(n-1)g'^4-(1/27)g'^6. \tag{31b}$$

Hence

$$\cos\phi=\frac{-[1-(9/2)(n-1)g'^{-2}-9g'^{-4}]}{[1+6(n-1)g'^{-2}+3g'^{-4}]^{3/2}}. \tag{31c}$$

Expanding the right-hand side of Eq. (31c) in powers of g'^{-2}, one finds

$$\cos\phi=-1+(27/2)(n-1)g'^{-2} -(27/2)[8(n-1)^2-1]g'^{-4}+O(g'^{-6}).$$

In other words, if $(27/2)(n-1)(\omega/gQ)^2$ is very small compared to unity, then the angle ϕ will turn out to be slightly less than π radians, in which case the lowest set of energy levels is given by

$$E_n^{\mathrm{II}}=(n-1)\omega+\tfrac{2}{3}g^2Q^2\omega^{-1} -\tfrac{1}{3}g^2Q^2\omega^{-1}[1+6(n-1)(\omega/gQ)^2 +3(\omega/gQ)^4]^{1/2}(x+\sqrt{3}y), \tag{32a}$$

where

$$x\equiv\cos\tfrac{1}{3}\phi, \tag{32b}$$

$$y\equiv\sin\tfrac{1}{3}\phi. \tag{32c}$$

In the extreme strong-coupling limit, $\frac{1}{3}\phi\to\frac{1}{3}\pi$ radians, and the other two sets of energy levels, E_n^{I} and E_n^{III}, merge together. In this limit the separation between I (or III) and the lower set II is easily found to be

$$E_n^{\mathrm{I}}-E_n^{\mathrm{II}}\cong 3\omega\sqrt{c}\cong g^2Q^2/\omega. \tag{33}$$

We also note that $x\equiv\cos\frac{1}{3}\phi$ is related to $\cos\phi$ by the well-known trigonometric relation

$$4x^3-3x=\cos\phi, \tag{34}$$

where $\cos\phi$ is given by Eq. (31c).

Using Eqs. (31c) and (34), one can easily verify that

$$x=\tfrac{1}{2}+\tfrac{3}{2}(n-1)^{1/2}(\omega/gQ)-\tfrac{3}{4}(n-1)(\omega/gQ)^2 -[(81/16)(n-1)^{3/2}-\tfrac{3}{4}(n-1)^{-1/2}](\omega/gQ)^3+O(g^{-4}). \tag{35}$$

After tedious but straightforward algebra, one finds

$$E_n^{\mathrm{II}}\cong 2[6(n-1)^2-1]\omega^3(gQ)^{-2}. \tag{36}$$

Note that expression (36) is very similar to the strong-coupling result for the lower-energy levels of the ordinary Lee model, namely,[15]

$$E_n^{\mathrm{OLM}}\cong n(n-1)\omega^3(gQ)^{-2}. \tag{37}$$

Minimization of the approximate expression (36) is identical to minimization of the approximate expression (37) for the ordinary Lee model.[16] In the strong-coupling limit considered here and for the special case $f=g$, the only difference between the two models is that the multiplicative factor $n(n-1)$ is replaced by $2[6(n-1)^2-1]$. Otherwise the calculation is identical to the OLM case as discussed in Ref. 16. In particular, one finds

$$E_n^{\mathrm{II}}\cong -3\pi^2[6(n-1)^2-1]R^3g^{-2}, \quad \text{for } R\gg 1$$
$$\cong (27/2)\pi^2[6(n-1)^2-1]R^{-3}g^{-2}, \quad \text{for } R\ll 1.$$

In principle one can minimize the exact expression (32a) for E_n^{II} rather than the strong-coupling approximation (36). As already noted for the case of the ordinary Lee model,[16] α and β then turn out to also depend on the isobar state (i.e., on the value of $n\equiv 1-q$) and on the values of the bare-coupling constants f and g. As one moves away from the strong-coupling regime, one must also ascertain whether or not E_n^{II} remains the *lowest* set of energy levels. Although we have not been able to solve this problem analytically, by using a modern computer one can easily evaluate the ICA results for any desired range of values of the basic parameters n, f, g, and R. In conclusion, we reiterate that the principal advantage of the present strong-coupling version of Tomonaga's ICA is that it eliminates the large-source assumption inherent in North's treatment.

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[15] Equation (37) is identical to Eq. (22) of Ref. 4.
[16] H. H. Nickle, Phys. Rev. 178, 2382 (1969).