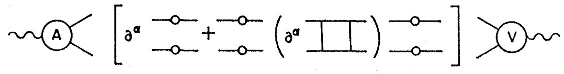

FIG. 5. Alternative expression for the quantity of Fig. 2.

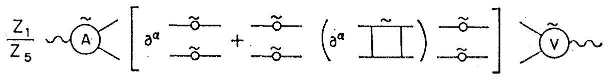

FIG. 6. Renormalized version of Fig. 5.

sion for the quantity depicted in Fig. 2. This expression is summarized by Fig. 5. This may now be expressed in renormalized quantities which we indicate by a tilde. Assuming $\Gamma_5{}^\mu$ to be multiplicatively renormalizable by $Z_5{}^{-1}$, as well as G by Z_1 and Γ^μ by $Z_2=Z_1$, we have the renormalized version of Fig. 5. This is given in Fig. 6. When Z_1/Z_5 is finite the above involves only finite quantities.

In lowest-order perturbation theory the quantity in parentheses in Fig. 6 vanishes, and only the derivative of the propagator is left. We have not evaluated this formula.

PHYSICAL REVIEW VOLUME 182, NUMBER 5 25 JUNE 1969

Kinematic Structure of Vertex Functions*

T. L. TRUEMAN
Brookhaven National Laboratory, Upton, New York 11973
(Received 20 February 1969)

An invariant-amplitude expansion is obtained for vertex functions of fields belonging to arbitrary representations of the Lorentz group between states of arbitrary spins. The method is based on analysis of the singularities of the Lorentz-group parameters defining the vertex function. The restrictions due to parity and subsidiary conditions are also given.

I. INTRODUCTION

THE purpose of this paper is to give an expansion for vertex functions in terms of functions which are completely free from kinematic singularities and constraints. Following common practice, we will refer to these functions as invariant amplitudes, although the distinguishing feature is the absence of kinematic singularities and constraints. The method we employ is the same as that used in an earlier paper on kinematic constraints and singularities of scattering amplitudes.[1] It is based on an analysis of the singularities of the Lorentz-group parameters defining the amplitude as a function of the scalar variables. Since the vertex depends on only one variable, t, this method allows a complete removal of all singularities and constraints simultaneously.

The vertex functions we will study are

$$F_{\lambda_1\lambda_2}{}^{JM}(p_1,p_2)=\langle p_2,s_2,\lambda_2|\phi_{JM}{}^{j_0\sigma}|p_1,s_1,\lambda_1\rangle\,. \quad (1.1)$$

There are two legs on the mass shell, with arbitrary spins and masses, and one leg off the mass shell. The latter is taken to belong to the $j_0\sigma$ representation of the Lorentz group.[2–4] (For the finite, nonunitary representations $\phi_{JM}{}^{[a,b]}$, $j_0=a-b$ and $\sigma=a+b+1$.)

There are many reasons for studying the structure of these functions for such general cases. To cite a few: (a) One can use them to construct one-particle exchange or pole terms which correspond to very-high-spin particles and which satisfy the conditions required by Lorentz invariance. (b) Many results which are true for arbitrary σ and physical J may presumably be extended to complex values of J and so be applied to factorized Regge residues; thus, one could study the behavior of the Regge residues in a very direct way. (c) The results may be a useful step in obtaining the kinematic structure of more complex amplitudes. (d) Further understanding of the significance of singularities and constraints, such as those which result from subsidiary conditions, may be obtained. A number of authors have studied this problem using a variety of methods and have obtained rather general results.[2,5–7] The method employed here is different from all of the preceding ones and the results are obtained in a substantially different and, we believe, more useful form.

In Sec. II, we review the multipole expansion and the difficulties with it. We then obtain an expansion for $F_{\lambda_1\lambda_2}{}^{JM}(p_{1R},p_2)$, where $p_{1R}=(m_1,\mathbf{0})$, in terms of invariant amplitudes. In Sec. III, we discuss the restrictions due to parity conservation. One of the important features of our expansion is that the form of these restrictions is very simple. In Sec. IV, we carry out the transformation to the center-of-mass amplitudes

* Work performed under the auspices of U. S. Atomic Energy Commission.

[1] T. L. Trueman, Phys. Rev. **173**, 1684 (1968).
[2] H. Joos, Fortschr. Physik **10**, 65 (1962).
[3] J. Strathdee, J. F. Boyce, R. Delbourgo, and A. Salam, Trieste Report, 1967 (unpublished).
[4] A. Sciarrino and M. Toller, J. Math. Phys. **8**, 1252 (1967).
[5] M. Scadron, Phys. Rev. **165**, 1640 (1968).
[6] M. Bander, Phys. Rev. **173**, 1568 (1968).
[7] M. S. Marinov, Ann. Phys. (N.Y.) **49**, 357 (1968).

$F_{\lambda_1\lambda_2}{}^{JM}(p_{1c},-p_{2c})$, $\mathbf{p}_{1c}+\mathbf{p}_{2c}=0$, and discuss their $t=0$ behavior. In Sec. V, we discuss the restrictions due to subsidiary conditions such as current conservation. The Appendix gives the relation of our results to those obtained by other methods for simple examples.

II. EXPANSION OF VERTEX FUNCTION

The object that we are studying in this section is

$$F_{\lambda_1\lambda_2}{}^{JM}(p_{1R},p_2)=\langle s_2,\lambda_2|e^{iK_z\zeta}\phi_{JM}{}^{j0\sigma}|s_1,\lambda_1\rangle\,, \quad (2.1)$$

where the states $|s_i,\lambda_i\rangle$ denote states at rest with spin projection λ_i in the z direction. K_z denotes the generator of pure Lorentz transformations in the z direction, and ζ is the parameter of the Lorentz transformation. Its relation to $t=(p_1-p_2)^2$ is given by

$$\cosh\tfrac{1}{2}\zeta=\left(\frac{(m_1+m_2)^2-t}{4m_1m_2}\right)^{1/2},$$
$$\sinh\tfrac{1}{2}\zeta=\left(\frac{(m_1-m_2)^2-t}{4m_1m_2}\right)^{1/2}. \quad (2.2)$$

By direct count, if there are no constraints on the $\phi_{JM}{}^{j0\sigma}$, the number of independent amplitudes is found to be

$$(2J+1)(2s+1)\,,\quad \text{for}\quad J\leq|s_1-s_2|$$
$$(2J+1)(2s+1)-(J-|s_1-s_2|)(J+1-|s_1-s_2|)\,,\quad \text{for}\quad |s_1-s_2|\leq J\leq s_1+s_2$$
$$(2s_1+1)(2s_2+1)\,,\quad \text{for}\quad s_1+s_2\leq J$$
$$s=\min(s_1,s_2)\,.$$

If there are constraints, such as parity conservation, there will be fewer independent amplitudes.

The usual way of parametrizing vertex functions is by multipole amplitudes. This was done for scalar and vector fields by Durand, DeCelles, and Marr[8] and generalized to arbitrary fields by de Rafael.[9] The difficulty with this is already apparent for the case of the scalar field. The procedure is to expand $e^{iK_z\zeta}$ in powers of $iK_z\zeta$. The Wigner-Eckart theorem is used to reduce the λ dependence of the matrix elements and obtain an expansion of the form

$$F_{\lambda_1\lambda_2}{}^{00}(p_{1R},p_2)=(-1)^{\lambda_1}\sum_l\begin{pmatrix}s_2 & l & s_1\\ \lambda_2 & 0 & -\lambda_1\end{pmatrix}M_l(t)\,.$$

This expression is very suitable near $\zeta=0$ or $t=(m_1-m_2)^2$. In fact, the amplitudes $\mathfrak{M}_l(t)$ defined by

$$\mathfrak{M}_l(t)=M_l(t)/(\sinh\tfrac{1}{2}\zeta)^l$$

are free from kinematic singularities and constraints at $t=(m_1-m_2)^2$. On the other hand, at $t=(m_1+m_2)^2$ they may have branch points and are in general constrained. Instead of expanding in powers of $iK_z\zeta$, one can just as well expand in powers of $iK_z(\zeta-i\pi)$. One can easily derive a generalization of the Wigner-Eckart theorem to matrix elements of the form $\langle s_2,\lambda_2|e^{\pi K_z}K_z{}^l|s_1,\lambda_1\rangle$. The result is an expansion that is useful near $t=(m_1+m_2)^2$ but not near $t=(m_1-m_2)^2$:

$$F_{\lambda_1\lambda_2}{}^{00}(p_{1R},p_2)=\sum_l\begin{pmatrix}s_2 & l & s_1\\ \lambda_2 & 0 & -\lambda_1\end{pmatrix}(\cosh\tfrac{1}{2}\zeta)^l\mathfrak{M}_l(t)\,,$$

with $\mathfrak{M}_l(t)$ regular and unconstrained at $t=(m_1+m_2)^2$. Thus at $t=(m_1+m_2)^2$ all of the $M_l(t)$ are expressible in terms of one amplitude $\mathfrak{M}_0(t)$. This phenomenon is well known in the case of the electromagnetic form factors $G_E(t)$ and $G_M(t)$ which are equal at $t=4m^2$. The problem then is to find functions analogous to the electromagnetic form factors $F_1(t)$ and $F_2(t)$, which are independent for all t. The earlier papers do just this.[2,5–7] This involves going to a spinor basis for the on-mass-shell particles as well as expanding those amplitudes. One still has the computational problem of going back to the physical amplitudes. (Although this step may be circumvented by trace techniques, one must make up for that by using projection operators when the complete density matrix is desired.) In any case, compact formulas for the general case have not yet been obtained by these methods.

The procedure we will use is quite different. First, notice that, according to Eq. (2.2), only the points $t=(m_1\pm m_2)^2$ are interesting. (In particular, the point $t=0$ does not become interesting until we go to the center-of-mass amplitudes.) We have already seen that the multipole expansions about these two points are quite different. Experience[1,10] suggests that we look at expansions about these two points for slightly different amplitudes: those quantized normal to the direction of motion. Thus, consider

$$\langle s_2,\lambda_2|e^{iK_x\zeta}\phi_{JM}{}^{j0\sigma}|s_1,\lambda_1\rangle =d_{\lambda_2\lambda_2'}{}^{s_2}(\tfrac{1}{2}\pi)d_{MM'}{}^{J}(\tfrac{1}{2}\pi)d_{\lambda_1\lambda_1'}{}^{s_1}(\tfrac{1}{2}\pi)\times F_{\lambda_1'\lambda_2'}{}^{JM'}(p_{1R},p_2)\,, \quad (2.3)$$

and let us combine s_1 with J to get s_3:

$$\langle s_2,\lambda_2|e^{iK_x\zeta}|s_3,\lambda_3\rangle\equiv\sum_{\lambda_1,M}(-1)^{\lambda_3}\begin{pmatrix}s_1 & J & s_3\\ \lambda_1 & M & -\lambda_3\end{pmatrix}\times\langle s_2\lambda_2|e^{iK_x\zeta}\phi_{JM}{}^{j0\sigma}|s_1,\lambda_1\rangle\,. \quad (2.4)$$

Now if we expand the exponential about $t=(m_1\pm m_2)^2$, we conclude in the usual manner that

$$\langle s_2,\lambda_2|e^{iK_x\zeta}|s_3,\lambda_3\rangle=(\cosh\tfrac{1}{2}\zeta)^m(\sinh\tfrac{1}{2}\zeta)^nA_{\lambda_2\lambda_3}{}^{Js_3}(t)\,,$$

[8] L. Durand, III, P. C. DeCelles, and R. B. Marr, Phys. Rev. 126, 1882 (1962).

[9] E. de Rafael, Ann. Inst. Henri Poincaré 5, 83 (1966).

[10] G. Cohen-Tannoudji, A. Morel, and H. Navelet, Ann. Phys. (N.Y.) 46, 239 (1968).

where

$$m=\max(|s_2-s_3|,\ |\lambda_2+\lambda_3|)$$

and

$$n=\max(|s_2-s_3|,\ |\lambda_2-\lambda_3|).$$

$A_{\lambda_2\lambda_3}{}^{Js_3}(t)$ is kinematically regular at $t=(m_1\pm m_2)^2$. Of course, they are not independent there. In fact they are not independent anywhere. It is clear, however, that A's with different values of m and n are not kinematically related since they are matrix elements of different powers of K_x when $t=(m_1\pm m_2)^2$. This is a useful guide in selecting a subset of the A's in terms of which all the others can be expressed for any value of t. When $s_3\leq s_2$, choose the $2s_3+1$ amplitudes

$$\langle s_2,\ s_2|e^{iK_x\zeta}|s_3,\ s_3-l\rangle \equiv(\cosh\tfrac{1}{2}\zeta)^{s_2+s_3-l}(\sinh\tfrac{1}{2}\zeta)^{s_2-s_3+l}A_l{}^{Js_3}(t),\quad 0\leq l\leq 2s_3 \tag{2.5a}$$

and for $s_3\geq s_2$ choose the $2s_2+1$ amplitudes

$$\langle s_2,\ s_2-l|e^{iK_x\zeta}|s_3,\ s_3\rangle \equiv(\cosh\tfrac{1}{2}\zeta)^{s_2+s_3-l}(\sinh\tfrac{1}{2}\zeta)^{s_3-s_2+l}A_l{}^{Js_3}(t),\quad 0\leq l\leq 2s_2. \tag{2.5b}$$

Rotational invariance alone allows all of the $A_{\lambda_2\lambda_3}{}^{Js_3}(t)$ to be expressed in terms of these $A_l{}^{Js_3}(t)$ for all t.

The desired expansion formula can be obtained from Eqs. (5) by inverting Eqs. (3) and (4). First, for those terms with $s_3\leq s_2$, look at

$$\langle s_2,s_2|e^{iK_x\zeta}|s_3,\ s_3-l\rangle =\sum_{M',\lambda_1',\lambda_2',\lambda_3'}(-1)^{\lambda_3'}\begin{pmatrix}s_1 & J & s_3\\ \lambda_1' & M' & -\lambda_3'\end{pmatrix}d_{s_2,\lambda_2'}{}^{s_2}(\tfrac{1}{2}\pi)\times d_{s_3-l,\lambda_3'}{}^{s_3}(\tfrac{1}{2}\pi)F_{\lambda_1'\lambda_2'}{}^{JM'}(p_{1R},p_2). \tag{2.6}$$

Multiply both sides by $d_{s_3-l,\lambda_2}{}^{s_3}(\frac{1}{2}\pi)$ and sum on all values of l. This projects out $\lambda_3'=\lambda_2=\lambda_2'$ and we may then divide out

$$d_{s_2,\lambda_2}{}^{s_2}(\tfrac{1}{2}\pi)=\frac{(-1)^{s_2-\lambda_2}}{2^{s_2}}\left(\frac{(2s_2)!}{(s_2+\lambda_2)!(s_2-\lambda_2)!}\right)^{1/2}$$

to obtain

$$\sum_l\langle s_2,s_2|e^{iK_x\zeta}|s_3,\ s_3-l\rangle d_{s_3-l,\lambda_2}{}^{s_3}(\tfrac{1}{2}\pi)\left(\frac{(s_2+\lambda_2)!(s_2-\lambda_2)!}{(2s_2)!}\right)^{1/2} =\sum_{M,\lambda_2}\begin{pmatrix}s_1 & J & s_3\\ \lambda & M & -\lambda_2\end{pmatrix}(-1)^{\lambda_2}F_{\lambda_1\lambda_2}{}^{JM}(p_{1R},p_2).$$

There is an obvious corresponding formula for $s_3\geq s_2$; the important thing is that the right-hand side is the same as it is here. Thus if we multiply by the $3j$ symbol times $2s_3+1$ and sum on s_3, we obtain the desired expansion:

$$F_{\lambda_1\lambda_2}{}^{JM}(p_{1R},p_2) =\sum_{s_3,l}d_{L-l,\lambda_2}{}^{L}(\tfrac{1}{2}\pi)[(L'+\lambda_2)!(L'-\lambda_2)!]^{1/2}\times\begin{pmatrix}s_1 & J & s_3\\ \lambda_1 & M & -\lambda_2\end{pmatrix}(\cosh\tfrac{1}{2}\zeta)^{s_2+s_3-l}\times(\sinh\tfrac{1}{2}\zeta)^{|s_2-s_3|+l}A_l{}^{Js_3}(t), \tag{2.7}$$

where $L=\min(s_2,s_3)$ and $L'=\max(s_2,s_3)$, and we have absorbed a few harmless factors into the definition of $A_l{}^{Js_3}(t)$.

Thus, in this special frame, we have obtained an expansion of the vertex function for arbitrary spins and general fields in terms of amplitudes free from kinematic singularities and constraints. The Lorentz-group properties of $\phi_{JM}{}^{j_0\sigma}$ played no role in this analysis. These come in when we want the vertex function in some other frame. The vertex function in any frame is simply expressed in terms of this vertex function through two Wigner rotations matrices for s_1 and s_2 and the $j_0\sigma$ representation of the Lorentz group corresponding to the transformation from this special frame to the desired frame. There is no point in writing that expression here. It is very simple; all the complications come in the evaluation of the representation functions. We will do this in the important—and simple—transformation to the center-of-mass system in Sec. IV.

In the Appendix, we show how these A's are related to the invariant amplitudes when tensor wave functions or Bargmann-Wigner wave functions are used for the case of scalar or pseudoscalar currents. The result is surprisingly simple.

III. RESTRICTIONS DUE TO PARITY CONSERVATION

It will be most convenient to use the reflection operator in the xy plane denoted by Y:

$$Y=e^{-i\pi J_y}P. \tag{3.1}$$

It has the property that[11,2]

$$Y|s_i,\lambda_i\rangle=\eta_i(-1)^{s_i-\lambda_i}|s_i,\ -\lambda_i\rangle, \tag{3.2}$$

$$Y\phi_{JM}{}^{j_0\sigma}Y^{-1}=\eta(-1)^{M-j_0}\phi_{J-M}{}^{-j_0\sigma}, \tag{3.3}$$

where η_i denotes the intrinsic parity of particle i and η denotes the "intrinsic parity" of the field $\phi_{JM}{}^{j_0\sigma}$. Only in the case $j_0=0$ is this η meaningful. When $j_0\neq 0$, it can always be set equal to 1 by redefining the fields. Evidently only when $j_0=0$ do we get a restriction on the vertex functions; namely,

$$F_{\lambda_1\lambda_2}{}^{JM}(p_{1R},p_2)=\eta\eta_1\eta_2(-1)^{s_1-s_2}F_{-\lambda_1,-\lambda_2}{}^{J,-M}(p_{1R},p_2)\quad\text{if}\quad j_0=0. \tag{3.4}$$

[11] M. Jacob and G. C. Wick, Ann. Phys. (N.Y.) 7, 404 (1959).

From Eq. (2.6) we get a very simple condition on the invariant amplitudes:

$$A_l^{Js_3}(t)=0 \quad \text{unless} \quad (-1)^l=\eta\eta_1\eta_2(-1)^{s_2-s_3-J}. \quad (3.5)$$

One of the main virtues of this expansion is the simplicity of the parity condition.

In many cases, when $j_0\neq 0$, it is useful to consider at the same time the matrix elements of $\phi_{JM}{}^{j_0\sigma}$ and $\phi_{JM}{}^{-j_0\sigma}$. This is familiar in the case of the Dirac field which is the direct sum of $\phi^{1/2,3/2}$ and $\phi^{-1/2,3/2}$. Parity then relates the invariant amplitudes in the expansion of the corresponding two vertex functions:

$$A_l^{Js_3(j_0\sigma)}(t)=(-1)^l\eta\eta_1\eta_2(-1)^{s_2-s_3-J}A_l^{Js_3(-j_0\sigma)}(t). \quad (3.6)$$

This is the only restriction due to parity unless there are additional constraints relating $\phi^{j_0\sigma}$ to $\phi^{-j_0\sigma}$. See Sec. V.

IV. CENTER-OF-MASS AMPLITUDES AND $t=0$ BEHAVIOR

Let us now continue the amplitudes $F_{\lambda_1\lambda_2}{}^{JM}(p_{1R},p_2)$ from negative t to $t>(m_1+m_2)^2$. For definiteness, we choose the path in t such that $\sinh\zeta$ is negative at the end of the continuation. Let ω be the parameter of the Lorentz transformation from the rest frame of p_1 to the center-of-mass frame, with $\mathbf{p}_1$ in the $+z$ direction. The parameter ω is related to t by

$$\begin{aligned}\cosh\omega&=(t+m_1^2-m_2^2)/2m_1t^{1/2},\\ \sinh\omega&=T/2m_1t^{1/2}=-(m_2/t^{1/2})\sinh\zeta,\\ T&=+\{[t-(m_1+m_2)^2][t-(m_1-m_2)^2]\}^{1/2}.\end{aligned} \quad (4.1)$$

Then denote $F_{\lambda_1\lambda_2}{}^{JM}(p_1,p_2)$, so continued and in the center-of-mass frame, by $F_{\lambda_1\lambda_2}{}^{JM}(t)$. Its relation to the continuation of $F_{\lambda_1\lambda_2}{}^{JM}(p_{1R},p_2)$ is

$$\begin{aligned}F_{\lambda_1\lambda_2}{}^{JM}(t)&=\langle s_2,\lambda_2|e^{iK_z\zeta}e^{iK_z\omega}\phi_{JM}{}^{j_0\sigma}e^{-iK_z\omega}|s_1,\lambda_1\rangle\\ &=\sum_{J'} d_{J'MJ}{}^{j_0\sigma}(-\omega)F_{\lambda_1\lambda_2}{}^{J'M}(p_{1R},p_2). \quad (4.2)\end{aligned}$$

$d_{J'MJ}{}^{j_0\sigma}(\omega)$ is the $j_0\sigma$ representation of a pure Lorentz transformation in the z direction. For the finite-dimensional representation $[a,b]$ it has the simple form

$$d_{J'MJ}{}^{[a,b]}(-\omega)=\sum_{m_a m_b}\begin{pmatrix} a & b & J\\ m_a & m_b & -M\end{pmatrix}\begin{pmatrix} a & b & J'\\ m_a & m_b & -M\end{pmatrix}\times[(2J+1)(2J'+1)]^{1/2}e^{-(m_b-m_a)\omega}. \quad (4.3)$$

The insertion of Eq. (2.7) into (4.2) yields an invariant-amplitude expansion for $F_{\lambda_1\lambda_2}{}^{JM}(t)$. Note that the kinematic-singularity structure is somewhat different for $F_{\lambda_1\lambda_2}{}^{JM}(t)$ because $d_{J'MJ}{}^{j_0\sigma}(-\omega)$ has singularities at $t=(m_1\pm m_2)^2$. In addition, it has singularities at $t=0$. Thus, the expected singularities of the vertex at $t=0$ result entirely from the center-of-mass transformation and depend on the field representation and not on the external particles. We are assuming here that $m_1\neq m_2$. Equal masses will be mentioned later.

The important property of $d_{J'MJ}{}^{j_0\sigma}(-\omega)$ near $t=0$ is that it goes to infinity at a rate depending only on j_0, σ, and M but not on J and J'.[4] Furthermore, the dependence on J and J' factors into a function depending on J and a function depending on J' ($\mathrm{Re}\sigma\neq 0$). Thus, the behavior of all $F_{\lambda_1\lambda_2}{}^{JM}(t)$ near $t=0$ is independent of J. The similarity of this behavior to that of Regge residues in a daughter sequence is clear.[12] For illustrative purposes, consider a finite-dimensional representation $[a,b]$. The precise behavior near $t=0$ depends on how $t=0$ is approached and on the ratio m_1/m_2. Suppose we choose the path such that

$$e^{-(m_b-m_a)\omega}\sim(t^{1/2})^{m_a-m_b}. \quad (4.4)$$

Then the singularity of $F_{\lambda_1\lambda_2}{}^{JM}(t)$ is determined by the largest value of m_b-m_a compatible with $m_b+m_a=M$. This value is given by

$$\begin{aligned}\max(m_b-m_2)&=(a+b)-|M-(b-a)|\\ &=\sigma-1-|M+j_0|. \quad (4.5)\end{aligned}$$

The maximum value is reached when $m_b=b$ for $b-a\leq M$ and when $m_a=-a$ for $b-a\geq M$. The behavior of $F_{\lambda_1\lambda_2}{}^{JM}(t)$ is then given by

$$F_{\lambda_1\lambda_2}{}^{JM}(t)\approx(t^{1/2})^{|M+j_0|-\sigma+1}\begin{pmatrix} a & b & J\\ m_a & m_b & -M\end{pmatrix}(2J+1)^{1/2}\times\sum_{J'}\begin{pmatrix} a & b & J'\\ m_a & m_b & -M\end{pmatrix}(2J'+1)^{1/2}\times F_{\lambda_1\lambda_2}{}^{J'M}(p_{1R},p_2)|_{t=0}, \quad (4.6)$$

with m_a and m_b taking the appropriate values. The fact that the most singular amplitudes occur for $M=-j_0$ is also very similar to the behavior of Regge residues.[13]

Let us look briefly at the case $m_1=m_2=m$. As $t\to 0$, $\omega\to i\frac{1}{2}\pi$ while $\zeta\to 0$. Then

$$F_{\lambda_1\lambda_2}{}^{JM}(t)\to\sum_{J'} d_{J'MJ}{}^{j_0\sigma}(-i\tfrac{1}{2}\pi)(-1)^{s-\lambda_2}\times\begin{pmatrix} s & J' & s\\ \lambda_1 & M & -\lambda_2\end{pmatrix}A_0{}^{J's}(0), \quad (4.7)$$

where we have also assumed $s_1=s_2=s$. Note that for imaginary arguments, the finite-dimensional representations become the corresponding $O(4)$ representations of real argument. Thus the form (4.7), when used to

[12] D. Z. Freedman and J.-M. Wang, Phys. Rev. **153**, 1596 (1967).

[13] R. F. Sawyer, Phys. Rev. Letters **18**, 1212 (1967); **19**, 209 (1967); G. Cosenja, A. Sciarrino, and M. Toller, Phys. Letters **27B**, 398 (1968); Nuovo Cimento **57**, 253 (1968); M. Le Bellac, *ibid.* **55A**, 318 (1968); A. Capella, A. P. Contogouris, and J. Tran Thanh Van, Phys. Rev. **175**, 1892 (1968); P. DiVechia, F. Drago, and M. L. Paciello, Nuovo Cimento **56**, 1185 (1968).

construct a scattering amplitude, will lead to the same structure as derived by Freedman and Wang.[14]

V. SUBSIDIARY CONDITIONS

One frequently deals with matrix elements of operators which are subject to subsidiary conditions. The most familiar case is that of a conserved vector operator. A natural generalization of this condition to general finite representations $[a,b]$ is that, in the center-of-mass system, the matrix elements of $\phi_{JM}{}^{[a,b]}$ are nonzero only for $J=a+b$:

$$F_{\lambda_1\lambda_2}{}^{JM}(t)=0 \quad \text{for} \quad J\neq a+b=\sigma-1\,. \tag{5.1}$$

This leads to a condition on the invariant amplitudes by way of

$$\sum_{J'} d_{J'MJ}{}^{[a,b]}(-\omega)F_{\lambda_1\lambda_2}{}^{J'M}(p_{1R},p_2)=0 \qquad \text{for} \quad J\neq a+b=\sigma-1\,. \tag{5.2}$$

Notice that, because of the way the $t=0$ behavior factors in Eq. (4.6), such a subsidiary condition will in general require a less singular behavior of the vertices as $t\to 0$. This is familiar in the case of photon coupling to unequal-mass particles. It is also analogous to the behavior of Regge residues for unequal masses in the absence of daughters.[13] The precise behavior required by Eq. (4.6) is

$$F_{\lambda_1\lambda_2}{}^{a+b,M}(t)\sim (t^{1/2})^{\sigma-1-|M-j_0|}\,. \tag{5.3}$$

Probably the simplest way to obtain the desired expansion, which clearly now has a somewhat different structure, is to introduce a new field ϕ' which does not satisfy subsidiary conditions. The field ϕ can be expressed in terms of ϕ' by means of projection operators which do satisfy the subsidiary condition:

$$\phi_{JM}{}^{(p)}=\mathcal{P}_{JM,J'M'}{}^{(p)}(p^2)^{\sigma-1-j_0}\phi_{J'M'}{}'(p)\,. \tag{5.4}$$

[The familiar case, $a=b=\frac{1}{2}$, has $\mathcal{P}_{\mu\nu}(p)=g_{\mu\nu}-p_\mu p_\nu/p^2$.] One can apply the preceding arguments to obtain an expansion for the matrix element of ϕ'. Then Eq. (5.4) can be used to obtain the desired expansion for ϕ. Obviously, in the center-of-mass system,

$$\mathcal{P}_{JM,J'M'}(\mathbf{p}=0)=\delta_{J,a+b}\delta_{J',a+b}\delta_{MM'}\,.$$

The factor $(p^2)^{\sigma-1-j_0}$ is there in order to have the behavior of F consistent with Eq. (5.3) as applied to ϕ and Eq. (4.6) as applied to ϕ'. (Alternatively, the projection operators are singular at $t=0$ when $\mathbf{p}\neq 0$ and the factor is there to avoid building a $t=0$ singularity into ϕ.) In any other reference frame, the operator may be obtained by Lorentz transformation. Thus, for example,

$$F_{\lambda_1\lambda_2}{}^{JM}(p_{1R},p_2)=t^{\sigma-1-j_0}d_{\sigma-1,M,J}{}^{\sigma j_0}(\omega) \times\Big[\sum_{J'} d_{J'M,\sigma-1}{}^{\sigma j_0}(-\omega)F_{\lambda_1\lambda_2}{}'^{J'M}(p_{1R},p_2)\Big], \tag{5.5}$$

where $F(p_{1R},p_2)$ is the original matrix element of ϕ defined in Eq. (2.1), while $F'(p_{1R},p_2)$ is the analogous matrix element of ϕ'. and should have a expansion of the form of Eq. (2.7) with regular and unconstrained A's.

A condition that is in someways similar to Eq. (5.1) is the requirement that a pole of $F_{\lambda_1\lambda_2}{}^{JM}(t)$ at some particular value m^2 of t be present in only one angular momentum state. This is obviously less stringent than Eq. (5.1) since it requires that the field acts as if it has spin J only on the mass shell. In fact, such conditions can always be satisfied without modifying the kinematic structure, provided $m^2\neq 0$. For example, suppose $s_1=s_2=0$. Then, with simplified notation,

$$F^J(t)=d_{J'0J}{}^{j_0\sigma}(-\omega)(\sinh\zeta)^{J'}A_{J'}(t)\,.$$

Evidently, if we set

$$A_J(t)=\sum_K \frac{d_{K0J'}{}^{j_0\sigma}(\omega_K)}{t-m_K{}^2}\left(\frac{1}{\sinh\zeta_K}\right)^{J'},$$

with ω_K and ζ_K the functions ω and ζ with $m_K{}^2$ substituted for t, clearly $F^J(t)$ will have a pole at $t=m_J{}^2$ but not at $t=m_{J'}{}^2$, $J'\neq J$. This form fails if any $m_J{}^2=0$, since then the corresponding residue functions $d_{J0J'}{}^{j_0\sigma}(\omega_J)$ become singular. This obviously results from the $O(3,1)$ symmetry which forces any pole at $t=0$ to be present in all J states belonging to the representation (j_0,σ).[12]

The final condition that we wish to discuss is the generalization of the Dirac equation. This is the requirement that for some value of t, $F_{\lambda_1\lambda_2}{}^{JM}(t)$ has a pole corresponding to a particle of definite parity. That is, one of

$$(t-m_J{}^2)[F_{\lambda_1\lambda_2}{}^{JM}(t)\pm F_{-\lambda_1-\lambda_2}{}^{J,-M}(t)]$$

must vanish at $t=m_J{}^2$. For $j_0=0$ representations this is automatic. When $j_0\neq 0$, all that parity requires is a condition like Eq. (3.6). Thus, we must require, in addition, that

$$F_{\lambda_1\lambda_2}{}^{JM(j_0\sigma)}(t)=\pm F_{\lambda_1\lambda_2}{}^{JM(-j_0,\sigma)} +\text{terms regular at } t=m_J{}^2.$$

This is a condition at one value of t and so, just as in the last case, will not modify the kinematic structure, provided $m_J{}^2\neq 0$. On the other hand, we see immediately from Eq. (4.6) that it cannot hold for $m_J{}^2=0$ without changing the $t=0$ structure when $j_0\neq 0$. Presumably, one could also require definite parity off the mass shell in a way very similar to the way in which definite J is required off the mass shell. This would modify the

[14] D. Z. Freedman and J.-M. Wang, Phys. Rev. **160**, 1560 (1967).

kinematical structure, but since this does not seem to be a very useful condition, we have not carried it through.

ACKNOWLEDGMENTS

Most of this work was done while the author was visiting at SLAC and he would like to thank Professor S. D. Drell and the rest of the Theory Group for their kind hospitality.

APPENDIX

Here we briefly show how our expansion is related to two other possible expansions in the simple case of scalar or pseudoscalar currents between two identical particles of spin s and mass m. As a first example, suppose we represent the particles by Bargmann-Wigner wave functions $U_{\alpha_1\alpha_2\cdots\alpha_{2s}}{}^{(\lambda)}(p)$, $\alpha_i=1, 2, 3, 4$. An invariant-amplitude expansion can be written very simply as

$$\mathcal{A}_0(t)+\mathcal{A}_2(t)\gamma_5{}^{(1)}\gamma_5{}^{(2)}+\mathcal{A}_4(t)\gamma_5{}^{(1)}\gamma_5{}^{(2)}\gamma_5{}^{(3)}\gamma_5{}^{(4)}+\cdots \quad \text{(scalar)}$$

or

$$\mathcal{A}_1(t)\gamma_5{}^{(1)}+\mathcal{A}_3(t)\gamma_5{}^{(1)}\gamma_5{}^{(2)}\gamma_5{}^{(3)}+\cdots \text{(pseudoscalar)}.$$

$\gamma_\mu{}^{(i)}$ denotes the usual 4×4 γ matrix operating on the indices α_i and the identity matrix operating on the indices α_j, $j\neq 1$. There is no need to symmetrize the γ matrices over all the indices since the Bargmann-Wigner wave functions are already symmetric in those indices. Now calculate the matrix elements $\langle s,s|e^{iK_x\zeta}\phi|s, s-l\rangle$. The wave function for spin projection $s-l$ consists of $s-l$ Dirac spinors with spin up and l Dirac spinors with spin down. The form of these submatrix elements is

$$\frac{1}{(2m)^{1/2}}\chi_+{}^\dagger((E+m)^{1/2},\ -\sigma_x(E-m)^{1/2})\begin{pmatrix}1\\0\end{pmatrix}\chi_\lambda =\delta_{\lambda,+}\left(\frac{E+m}{2m}\right)^{1/2},$$

$$\frac{1}{(2m)^{1/2}}\chi_+{}^\dagger((E+m)^{1/2},\ -\sigma_x(E-m)^{1/2})\gamma_5\begin{pmatrix}1\\0\end{pmatrix}\chi_\lambda =-\left(\frac{E-m}{2m}\right)^{1/2}\delta_{\lambda,-}.$$

Thus we have

$$\langle s,s|e^{iK_x\zeta}\phi|s, s-l\rangle =(-1)^l\left[\left(\frac{E+m}{2m}\right)^{1/2}\right]^{s-l}\left[\left(\frac{E-m}{2m}\right)^{1/2}\right]^l\mathcal{A}_l(t)\binom{s}{l}^{-1/2}$$
$$=(-1)^l(\cosh\tfrac{1}{2}\zeta)^{s-l}(\sinh\tfrac{1}{2}\zeta)^l\mathcal{A}_l(t)\binom{s}{l}^{-1/2}.$$

So the invariant amplitudes in this case are the same as those used in Eq. (4.7), up to constant factors.

Another basis for expansion has been given by Scadron,[5] who uses the tensor wave functions for bosons and Rarita-Schwinger wave functions for fermions. There is a direct relation between this expansion and the expansion just given. (See, for example, Marinov.[7]) Let us just do bosons. Then

$$U_{\alpha_1\alpha_2\cdots\alpha_{2s}}(p)=[C^{-1}\gamma^{\mu_1}(1/2m)(p'+m)]_{\alpha_1\alpha_2}\cdots [C^{-1}\gamma^{\mu_s}(1/2m)(p'+m)]_{\alpha_{2s-1},\alpha_{2s}}\phi_{\mu_1\cdots\mu_s},$$

where $\phi_{\mu_1\cdots\mu_s}$ is symmetric in the four-vector indices $\mu_1\cdots\mu_s$. To transform from one basis to the other, all we need do is evaluate

$$\mathrm{Tr}\{[C^{-1}\gamma^{\mu_1}(1/2m)(p'+m)]\gamma_0[C^{-1}\gamma^{\nu_1}(1/2m)(p+m)]^\dagger\gamma_0\} =-(1/m^2)[g_{\mu_1\nu_1}(m^2+p'\cdot p)-p_{\mu_1}p_{\nu_1}']$$

and

$$\mathrm{Tr}\{[C^{-1}\gamma^{\mu_1}(1/2m)(p'+m)]\gamma_0\gamma_5 \times[C^{-1}\gamma^{\nu_1}(1/2m)(p+m)]^\dagger\gamma_0\gamma_5\} =-(1/m^2)[g_{\mu_1\nu_1}(m^2-p'\cdot p)+p_{\mu_1}p_{\nu_1}'].$$

Thus, in the tensor basis the coefficient of the invariant amplitude $\mathcal{A}_l$ is

$$[(-1)^s/m^{2s}][g_{\mu_1\nu_1}(m^2-p'\cdot p)+p_{\mu_1}p_{\nu_1}'] \times[g_{\mu_2\nu_2}(m^2-p'\cdot p)+p_{\mu_2}p_{\nu_2}']\cdots \times[g_{\mu_{l+1}\nu_{l+1}}(m^2+p'\cdot p)-p_{\mu_{l+1}}p_{\nu_{l+1}}']\cdots \times[g_{\mu_s\nu_s}(m^2+p'\cdot p)-p_{\mu_s}p_{\nu_s}'].$$

These are to be evaluated as

$$\phi_{\mu_1\cdots\mu_s}{}^\dagger(p')M_{\mu_1\cdots\mu_s,\nu_1\cdots\nu_s}\phi_{\nu_1\cdots\nu_s}(p).$$

Evidently, the relation between Scadron's invariant amplitudes, the coefficients of terms like $g_{\mu_1\nu_1}g_{\mu_2\nu_2}\cdots g_{\mu_s\nu_s}$, $g_{\mu_1\nu_1}p_{\mu_2}p_{\nu_2}'\cdots g_{\mu_s\nu_s}$, etc., and ours is not so simple as in the Bargmann-Wigner basis, but it is straightforward and it is clear that the relation is nonsingular.