Anomalies of the Axial-Vector Current*

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The anomalies of the neutral axial-vector current, of its divergence, and of its commutators in the presence of electromagnetic interactions are exhibited. It is shown that it is impossible in perturbation theory to maintain minimality of electromagnetic interactions, gauge invariance, and partial conservation of axial-vector current (PCAC). The departures from gauge invariance or PCAC which are necessitated by minimality are established. The relevance of the present considerations to $\pi^0 \to 2\gamma$ decay is discussed.

I. INTRODUCTION

It has been noticed recently[1] that in the σ model the effective coupling constant for $\pi^0 \to 2\gamma$ does not vanish for zero pion mass. However, partial conservation of axial-vector current (PCAC) and gauge invariance, which appear to hold in the σ model, predict the vanishing of this quantity.[2] The reason for this contradiction has been traced to the fact that the Feynman diagram for $j_5{}^\mu \to 2\gamma$ ($j_5{}^\mu$ is the axial-vector current) is ambiguous whenever the above process goes through the triangle graph of Fig. 1, as is the case in the σ model. It is found that this diagram cannot be defined in a fashion which preserves both PCAC and gauge invariance.[3] Therefore, the PCAC argument[1,2] cannot be directly utilized to yield the $\pi^0 \to 2\gamma$ amplitude.

Adler[4] has also examined the triangle graph of Fig. 1 and has pointed out that, if gauge invariance is imposed to give a unique value for the graph, then the departure from PCAC of this diagram can be represented by

$$i\partial_\mu j_5{}^\mu = j_5 + (e^2/16\pi^2) F^{\mu\nu}\tilde{F}_{\mu\nu}. \tag{1.1}$$

Here j_5 is the naive value for the divergence, calculated from the straightforward application of the equations of motion, $F^{\mu\nu}$ is the electromagnetic field tensor, and $\tilde{F}^{\mu\nu}$ is its dual:

$$\tilde{F}^{\mu\nu} = \varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}. \tag{1.2}$$

The purpose of this paper is to comment further on the ambiguities associated with the singular nature of the axial-vector current, when it is constructed from fermion fields. We give in Sec. II an operator proof that when $j_5{}^\mu$ is defined in a gauge-invariant fashion, its divergence will not in general be given by j_5. An additional term will contribute whenever the matrix elements of $j_5{}^\mu F_{\mu\nu}$ are superficially linearly (or more) divergent, as is the case for the triangle graph. In these instances this additional term has matrix elements which suggest the form (1.1). It is further shown that a gauge-noninvariant definition of $j_5{}^\mu$ can be given such that the divergence coincides with the "naive" value j_5. In either case, the current has finite matrix elements; viz., the superficial divergence disappears. Furthermore, the $\pi^0 \to 2\gamma$ decay amplitude is unambiguously determined, regardless of the definition of the current, and the invariant coupling does not vanish since either PCAC or gauge invariance (or both) is lost.[5] The question of whether (1.1) is valid to all orders in the strong interactions is discussed, and different anomalies are demonstrated for more general interactions.

Historically, the first derivation of Eq. (1.1) for *external* electromagnetic fields was given by Schwinger,[6] who recognized that the equivalence between pseudovector and pseudoscalar π-N coupling could be exhibited in perturbation theory only when additional terms are present in $\partial_\mu j_5{}^\mu$. His technique employed gauge-invariant differentiation and gave (1.1). It is now rec-

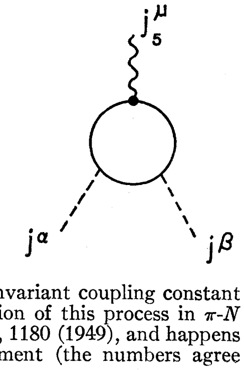

FIG. 1. The process $j_5{}^\mu \to 2\gamma$ proceeding through a fermion loop.

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[1] J. S. Bell and R. Jackiw, Nuovo Cimento **60**, 47 (1969).

[2] D. G. Sutherland, Nucl. Phys. **B2**, 433 (1967).

[3] A regularization procedure can be given which preserves PCAC and gauge invariance (see Ref. 1). However, this procedure probably leads to an unrenormalizable theory. We speculate that the scheme outlined in Ref. 1 is equivalent to introducing into the σ model a nonminimal interaction proportional to $\phi F^{\mu\nu}\tilde{F}_{\mu\nu}$ which, in lowest order, at least, cancels the finite effective-coupling constant.

[4] S. L. Adler, Phys. Rev. **177**, 2426 (1969).

[5] The value in the σ model for the invariant coupling constant of $\pi^0 \to 2\gamma$ coincides with the calculation of this process in π-N theory by J. Steinberger, Phys. Rev. **76**, 1180 (1949), and happens to be in good agreement with experiment (the numbers agree within 20%).

[6] J. Schwinger, Phys. Rev. **82**, 664 (1951).

ognized that his considerations are relevant to PCAC arguments.[7]

In Sec. III we show that the same anomalous behavior of the triangle graph which is responsible for the breakdown of PCAC or of gauge invariance implies that *interaction-picture* current commutators are anomalous. The divergence of the current in the interaction picture is, however, *not* anomalous. [Equation (1.1) holds in the Heisenberg picture.] Pion decay is recalculated in the interaction picture, and the nonvanishing of the coupling is seen to follow from the *anomalous* commutators and contact terms. The argument demonstrates that Schwinger terms and seagull terms *do not* cancel and that the Schwinger term *can* contribute to low-energy theorems, contrary to popular belief. It will be also seen that whereas the Heisenberg-picture argument requires point splitting in defining j_5^μ, the interaction-picture derivation does not make use of this technique, but adds seagull terms to T products.

Section IV is directed to general remarks concerning the implications of the above for physical theory.

II. AXIAL-VECTOR CURRENT AND ITS DIVERGENCE IN THE HEISENBERG PICTURE

To establish the anomalous Heisenberg-picture divergence equation, we define

$$j_5^\mu(x,\epsilon,a)=\bar\psi(x+\tfrac12\epsilon)\gamma^5\gamma^\mu\psi(x-\tfrac12\epsilon)\times\exp\left(iea\int_{x-\epsilon/2}^{x+\epsilon/2}A_\alpha(y)dy^\alpha\right),\quad(2.1)$$

where A_α is the electromagnetic potential. When $a=1$, the above is gauge-invariant. The local current is obtained by choosing ϵ to be small, averaging over the directions of ϵ, and letting $\epsilon^2\equiv\epsilon_\mu\epsilon^\mu\to0$:

$$j_5^\mu(x,a)\equiv\lim_{\epsilon^2\to0}\bar{j}_5^\mu(x,\epsilon,a).\quad(2.2)$$

Similarly, the divergence is given by

$$\partial_\mu j_5^\mu(x,a)\equiv\lim_{\epsilon^2\to0}\partial_\mu\bar{j}_5^\mu(x,\epsilon,a).\quad(2.3)$$

The equation of motion for ψ is

$$i\gamma\cdot\partial\psi=-eA\psi+\mathcal{O}\psi,\quad(2.4a)$$

where $\mathcal{O}$ is some operator containing the mass terms, as well as other interactions. The anomaly one seeks is independent of masses. The effects of the other interactions will be examined at the end of this section. Thus we set $\mathcal{O}$ to zero for the present, and we deal with massless fermions which have only electromagnetic interactions ($j_5=0$).

$$i\gamma\cdot\partial\psi=-eA\psi.\quad(2.4b)$$

By virtue of (2.4b), it follows that the divergence of (2.1) is

$$i\partial_\mu j_5^\mu(x,\epsilon,a)=ej_5^\mu(x,\epsilon,a)\times\left[A_\mu(x+\tfrac12\epsilon)-A_\mu(x-\tfrac12\epsilon)-a\partial_\mu\int_{x-\epsilon/2}^{x+\epsilon/2}A_\nu(y)dy^\nu\right]$$
$$=ej_5^\mu(x,\epsilon,a)\epsilon^\alpha[\partial_\alpha A_\mu(x)-a\partial_\mu A_\alpha(x)+O(\epsilon)].\quad(2.5)$$

The naive result, $\partial_\mu j_5^\mu=0$, follows if ϵ is set equal to zero in (2.5). This is legitimate when $j_5^\mu(x,\epsilon,a)$ is well behaved as $\epsilon\to0$. However, if a matrix element of $j_5^\mu(x,\epsilon,a)[\partial_\alpha A_\mu(x)-a\partial_\mu A_\alpha(x)]$ is linearly divergent, then a nonvanishing result may remain as $\epsilon^2\to0$. Since the dimension of j_5^μ is (length)$^{-3}$, one may expect a cubic divergence. However, the pseudovector character of j_5^μ reduces the divergence by two powers and leaves a possible linear divergence. We now show that this is indeed the case for selected matrix elements.

We consider first the case that A_μ is an *external* electromagnetic field, and determine the singular portion of the exact vacuum-expectation value of j_5^μ. We have

$$\langle\Omega|i\partial_\mu j_5^\mu(x,\epsilon,a)|\Omega\rangle=e\langle\Omega|j_5^\mu(x,\epsilon,a)|\Omega\rangle\epsilon^\alpha\times[\partial_\alpha A_\mu(x)-a\partial_\mu A_\alpha(x)+O(\epsilon)]$$
$$=-e\,\mathrm{Tr}\gamma^5\gamma^\mu G(x-\tfrac12\epsilon,\,x+\tfrac12\epsilon)\epsilon^\alpha\times[\partial_\alpha A_\mu(x)-a\partial_\mu A_\alpha(x)+O(\epsilon)],\quad(2.6a)$$

$$G(x-\tfrac12\epsilon,\,x+\tfrac12\epsilon)=\langle\Omega|T\psi(x-\tfrac12\epsilon)\bar\psi(x+\tfrac12\epsilon)|\Omega\rangle,\quad\epsilon_0>0.\quad(2.6b)$$

The portion of G which is singular in ϵ may be calculated by perturbation theory. It may be verified that (in the absence of other interactions)

$$G(x-\tfrac12\epsilon,\,x+\tfrac12\epsilon)=S(-\epsilon)+ie\int d^4y\,S(x-y-\tfrac12\epsilon)\gamma^\alpha S(y-x-\tfrac12\epsilon)A_\alpha(y)$$
$$-e^2\int d^4yd^4z\,S(x-y-\tfrac12\epsilon)\gamma^\alpha S(y-z)\gamma^\beta\times S(z-x-\tfrac12\epsilon)A_\alpha(y)A_\beta(z)+O(\ln\epsilon),\quad(2.7a)$$

$$S(x)\equiv i\int\frac{d^4p}{(2\pi)^4}\frac{e^{-ip\cdot x}}{\not p+i\eta}.\quad(2.7b)$$

[7] After completing this investigation, we received unpublished reports from C. R. Hagen, Phys. Rev. 177, 2622 (1969) and K. G. Wilson (to be published); this refers to non-Lagrangian models for current algebra in which some of the topics discussed here are treated.

The momentum representation of (2.7a) is

$$G(x-\tfrac{1}{2}\epsilon,\ x+\tfrac{1}{2}\epsilon)=S(-\epsilon)+ie\int\frac{d^4pd^4q}{(2\pi)^8}$$
$$\times e^{i\epsilon\cdot p}e^{-ix\cdot q}S(p+\tfrac{1}{2}q)\gamma^\alpha S(p-\tfrac{1}{2}q)A_\alpha(q)$$
$$-e^2\int\frac{d^4pd^4q_1d^4q_2}{(2\pi)^{12}}e^{i\epsilon\cdot p}e^{-ix\cdot(q_1+q_2)}\{S[p+\tfrac{1}{2}(q_1+q_2)]\gamma^\alpha$$
$$\times S[p-\tfrac{1}{4}(q_1-q_2)]\gamma^\beta S[p-\tfrac{1}{4}(q_1+q_2)]\}$$
$$\times A_\alpha(q_1)A_\beta(q_2)+O(\ln\epsilon).\quad(2.8)$$

For reasons of C invariance, only the contribution to G linear in A is of interest. With ϵ finite we may compute $\epsilon^\alpha G$ by an integration by parts, so that we find, with the suitable trace taken,

$$-\mathrm{Tr}[\gamma^5\gamma^\mu G(x-\tfrac{1}{2}\epsilon,\ x+\tfrac{1}{2}\epsilon)\epsilon^\alpha]_{\epsilon\to0}$$
$$=\tfrac{1}{2}ie\partial_\nu A_\lambda(x)\int\frac{d^4p}{(2\pi)^4}e^{i\epsilon\cdot p}\frac{\partial}{\partial p^\alpha}$$
$$\times\left\{\mathrm{Tr}\left[\gamma^5\gamma^\mu\frac{\partial S(p)}{\partial p^\nu}\gamma^\lambda S(p)-S(p)\gamma^\lambda\frac{\partial S(p)}{\partial p^\nu}\right]\right\}$$
$$+O(\epsilon\ln\epsilon).\quad(2.9)$$

Now when $\epsilon\to0$, the resulting p integral, which is purely a surface integral, is finite, if we integrate over angles of p first (or equivalently, if we average over ϵ before taking the limit $\epsilon\to0$). We then obtain, by explicitly evaluating (2.9) and substituting into (2.6a),

$$\langle\Omega|i\partial_\mu j_5^\mu(x,a)|\Omega\rangle=(e^2/16\pi^2)\tilde{F}^{\mu\alpha}[\partial_\alpha A_\mu(x)-a\partial_\mu A_\alpha(x)]$$
$$=(e^2/16\pi^2)\times\tfrac{1}{2}(1+a)\tilde{F}^{\mu\alpha}F_{\mu\alpha}.\quad(2.10)$$

When the gauge-invariant definition is chosen for j_5^μ, viz., $a=1$, we obtain

$$\langle\Omega|i\partial_\mu j_5^\mu(x,1)|\Omega\rangle=(e^2/16\pi^2)\tilde{F}^{\mu\nu}F_{\mu\nu},\quad(2.11)$$

which coincides with (1.1). This was first derived by Schwinger,[6] using a method different from ours. However, gauge-noninvariant definitions of j_5^μ will give other results. In particular, the choice $a=-1$ gives

$$\langle\Omega|i\partial_\mu j_5^\mu(x,\ -1)|\Omega\rangle=0.\quad(2.12)$$

Thus the naive equation may be regained at the expense of gauge invariance. We show below that pion decay in the σ model is not affected by the choice of a. It should be emphasized that the present results are exact, and valid to all orders in the external electromagnetic field.

Next we consider the situation when the electromagnetic field is a dynamical operator. We calculate the two-photon–vacuum matrix element of $i\partial_\mu j_5^\mu$:

$$T^{\mu\nu}(p,q,\epsilon,a)=\int d^4yd^4z\ e^{ip\cdot y}e^{iq\cdot z}p^2q^2$$
$$\times\langle\Omega|TA^\mu(y)A^\nu(z)i\partial_\mu j_5^\mu(x,a,\epsilon)|\Omega\rangle$$
$$=e\epsilon^\alpha\int d^4yd^4ze^{ip\cdot y}e^{iq\cdot z}p^2q^2\langle\Omega|TA^\mu(y)A^\nu(z)j_5^\omega(x,\epsilon,a)$$
$$\times[\partial_\alpha A_\omega(x)-a\partial_\omega A_\alpha(x)]|\Omega\rangle+O(\epsilon^2).\quad(2.13)$$

The vacuum-expectation value is calculated to lowest-order perturbation theory. We find that

$$T^{\mu\nu}(p,q,\epsilon,a)=-e^2\ \mathrm{Tr}\gamma^5\gamma^\omega\epsilon^\alpha e^{ix\cdot(p+q)}\int\frac{d^4r}{(2\pi)^4}$$
$$\times e^{i\epsilon\cdot r}[S(r-\tfrac{1}{2}q)\gamma^\beta S(r+\tfrac{1}{2}q)(g_\omega{}^\mu g_\beta{}^\nu p_\alpha-ag_\alpha{}^\mu g_\beta{}^\nu p_\omega)$$
$$+S(r-\tfrac{1}{2}p)\gamma^\beta S(r+\tfrac{1}{2}p)(g_\beta{}^\mu g_\omega{}^\nu q_\alpha-ag_\beta{}^\mu g_\alpha{}^\nu q_\omega)]$$
$$+O(e^3)+O(\epsilon\ln\epsilon).\quad(2.14)$$

Evaluating the trace and keeping only the most singular part leaves

$$T^{\mu\nu}(p,q,\epsilon,a)=4e^2\varepsilon^{\omega\beta\theta\phi}\epsilon^\alpha$$
$$\times[q_\phi p_\alpha g_\omega{}^\mu g_\beta{}^\nu-aq_\phi p_\omega g_\alpha{}^\mu g_\beta{}^\nu+p_\phi q_\alpha g_\beta{}^\mu g_\omega{}^\nu-ap_\phi q_\omega g_\beta{}^\mu g_\alpha{}^\nu]$$
$$\times e^{ix\cdot(p+q)}\int\frac{d^4r}{(2\pi)^4}\frac{e^{i\epsilon\cdot r}}{r^4}r_\theta+O(\epsilon\ln\epsilon)+O(e^3).\quad(2.15\mathrm{a})$$

The integral is evaluated as before, by partial integration. Averaging over ϵ leaves, in the limit $\epsilon^2\to0$,

$$T^{\mu\nu}(p,q,a)=-(e^2e^{ix\cdot(p+q)}/4\pi^2)$$
$$\times(1+a)\varepsilon^{\mu\nu\alpha\beta}p_\alpha q_\beta+O(e^3).\quad(2.15\mathrm{b})$$

This is equivalent to setting directly in (2.13)

$$i\partial_\mu j_5^\mu(x,a)=(e^2/16\pi^2)\tfrac{1}{2}(1+a)\tilde{F}^{\mu\nu}F_{\mu\nu}+O(e^3).\quad(2.16\mathrm{a})$$

Thus we have regained the analog of (1.1) for this matrix element. It is clear that this lowest-order calculation is equivalent to the external-field description. Note that (2.16a) may be rewritten in terms of renormalized quantities

$$i\partial_\mu j_5^\mu(x,a)=(e^2{}_R/16\pi^2)\tilde{F}_R{}^{\mu\nu}F_{R\mu\nu},$$
$$e_R{}^2=z_3e^2,\quad(2.16\mathrm{b})$$
$$F^{\mu\nu}=z_3{}^{1/2}F_R{}^{\mu\nu}.$$

We now show that the matrix elements of the axial-vector current and not only its divergence are finite. We consider only the case of external electromagnetic field, and calculate

$$\langle\Omega|j_5(x,\epsilon,a)|\Omega\rangle=-\mathrm{Tr}\gamma^5\gamma^\mu G(x-\tfrac{1}{2}\epsilon,\ x+\tfrac{1}{2}\epsilon)$$
$$\times[1+iea\epsilon_\alpha A^\alpha(x)+O(\epsilon^2)].\quad(2.17)$$

Again, the singular terms are the ones of interest and for this purpose we may use the Born expansion (2.7a).

Since we are computing the axial-vector vertex, only the terms proportional to $A_\mu A_\nu$ are of interest because of C invariance. We obtain, by using the technique of integration by parts as above for the terms of the form ϵ (singular parts),

$$\langle\Omega|j_5{}^\mu(x,\epsilon,a)|\Omega\rangle = -\frac{e^2}{2}\int\frac{d^4q_1}{(2\pi)^4}\frac{d^4q_2}{(2\pi)^4}e^{-i(q_1+q_2)\cdot x}A_\alpha(q_1)A_\beta(q_2)\times\int\frac{d^4p}{(2\pi)^4}e^{ip\cdot\epsilon}\bigg\{\mathrm{Tr}\bigg[\gamma^5\gamma^\mu S\Big(p+\frac{q_1+q_2}{2}\Big)\gamma^\alpha\times S\Big(p-\frac{q_1-q_2}{2}\Big)\gamma^\beta S\Big(p-\frac{q_1+q_2}{2}\Big)\bigg] + a\frac{\partial}{\partial p^\alpha}\mathrm{Tr}[\gamma^5\gamma_\mu S(p-\tfrac{1}{2}q_2)\gamma^\beta S(p+\tfrac{1}{2}q_2)] + \begin{pmatrix} q_1\leftrightarrow q_2\\ \alpha\leftrightarrow\beta\end{pmatrix}\bigg\}. \quad (2.18a)$$

We first notice that when $a=1$ (gauge-invariant definition of the axial-vector current) that *without averaging on* ϵ, the limit $\epsilon\to 0$ is well defined. This is because, when $a=1$, the integrand in (2.18a), as $p\to\infty$, is of order $1/p^5$ and therefore convergent. When $a\neq 1$, the integrand in (2.18a) is of order $1/p^3+1/p^4$ the contribution of $1/p^3$ vanishes on symmetric integration, and the $1/p^4$ contribution is finite and unambiguous if we symmetrically integrate over p (or average over ϵ). This is the only *a priori* reason for taking the gauge-invariant definition of the axial-vector current, in that it is finite and unambiguous without the necessity of any special treatment of the parameter ϵ as $\epsilon\to 0$. However, when $a\neq 1$, if we symmetrically integrate over p, the $1/p^3$ and $1/p^4$ contributions vanish, and again the integral is finite. Finally, we may reproduce the results of our calculation of the divergence of the axial-vector current using this form. We find that when we let $a\to -1+(a+1)$ that the term proportional to $(a+1)$ may be calculated explicitly. The answer is (in the limit $\epsilon\to 0$)

$$\langle\Omega|j_5{}^\mu(x,a)|\Omega\rangle = -(ie^2/16\pi^2)(1+a)\tilde{F}^{\mu\beta}(x)A_\beta(x) + \langle\Omega|\tilde{j}_5{}^\mu(x)|\Omega\rangle. \quad (2.18b)$$

The additional term is independent of a, and its form is

$$\langle\Omega|\tilde{j}_5{}^\mu(x)|\Omega\rangle = -\frac{e^2}{2}\int\frac{d^4q_1d^4q_2}{(2\pi)^8}e^{-i(q_1+q_2)\cdot x}A_\alpha(q_1)A_\beta(q_2)\times\int\frac{d^4p}{(2\pi)^4}\bigg\{\mathrm{Tr}\bigg[S\Big(p-\frac{q_1+q_2}{2}\Big)\gamma^5\gamma^\mu S\Big(p+\frac{q_1+q_2}{2}\Big)\gamma^\alpha\times S\Big(p-\frac{q_1-q_2}{2}\Big)\gamma^\beta\bigg] - \frac{\partial}{\partial p^\alpha}\mathrm{Tr}[\gamma^5\gamma_\mu S(p-\tfrac{1}{2}q_2)\gamma^\beta\times S(p+\tfrac{1}{2}q_2)] + \begin{pmatrix}\alpha\leftrightarrow\beta\\ q_1\leftrightarrow q_2\end{pmatrix}\bigg\}. \quad (2.18c)$$

In this form we can show, *without* the use of a translation of the variable p, that the traditional form of the divergence of the axial-vector current can be used. That is, $(m=0)$ $\partial_\mu\langle\Omega|\tilde{j}_5{}^\mu|\Omega\rangle=0$. Thus we reproduce our previous result, namely,

$$i\partial_\mu\langle\Omega|j_5{}^\mu(x,a)|\Omega\rangle = (e^2/16\pi^2)\tfrac{1}{2}(1+a)\tilde{F}^{\mu\beta}(x)F_{\mu\beta}(x). \quad (2.19)$$

Evidently $\tilde{j}_5{}^\mu$ satisfies the naive divergence equation, as can be seen by comparing with (2.11).

It is seen that the current $j_5{}^\mu(x,a)$ turns out to be finite, regardless of whether gauge invariance $(a=1)$ or PCAC $(a=-1)$ is imposed. This was already discovered by examination of the triangle graph.[1] Note that even when gauge invariance is imposed,

$$\langle\Omega|j_5{}^\mu(x,1)|\Omega\rangle = (-ie^2/8\pi^2)\tilde{F}^{\mu\beta}(x)A_\beta(x) + \langle\Omega|\tilde{j}_\beta{}^\mu|\Omega\rangle. \quad (2.20)$$

$\tilde{j}_\beta{}^\mu$ is not separately gauge-invariant, since $\tilde{F}^{\mu\beta}(x)A_\beta(x)$ varies with gauge transformations. Since $\tilde{j}_\beta{}^\mu$ does not depend on a, it is gauge-dependent regardless of choice of a.

We have discovered the startling result that although the theory possess two formal symmetries, gauge invariance and PCAC, the solution given by perturbation theory cannot maintain conservation of both of the currents, but only of one.[8] From the formal point of view there appears to be no way to select which conservation law should be preserved. Consulting nature does not help since it is not known whether the *neutral* hadronic axial-vector current is observable. However, gauge invariance does possess the formal advantage discussed above, viz., the axial-vector current is finite independent of ϵ averaging.

It is next shown that the $\pi^0\to 2\gamma$ decay amplitude does not depend on the choice of a. Let us generalize the previous considerations to the case $j_5\neq 0$. Evidently we have

$$j_5{}^\mu(x,a) = (-ie^2/16\pi^2)(1+a)\times\tilde{F}^{\mu\beta}(x)A_\beta(x) + \tilde{j}_5{}^\mu(x), \quad (2.21)$$

$$i\partial_\mu j_5{}^\mu(x,a) = (e^2/16\pi^2)\tfrac{1}{2}(1+a)\times\tilde{F}^{\mu\beta}(x)F_{\alpha\beta}(x) + j_5(x), \quad (2.22a)$$

$$i\partial_\mu\tilde{j}_5{}^\mu(x) = j_5(x). \quad (2.22b)$$

It is, of course, arbitrary which pseudoscalar operator is used as an interpolating field for the pion. We may use $i\partial_\mu j_5{}^\mu$ or $i\partial_\mu\tilde{j}_5{}^\mu = j_5$. However, in order to obtain useful information about physical pion amplitudes, from expressions evaluated at zero pion mass, it is necessary that the operator lead to expressions which extrapolate smoothly from $k^2=0$ to $k^2=m_\pi{}^2$. We shall now argue that $i\partial_\mu\tilde{j}_5{}^\mu = j_5$ is the proper operator for this purpose. First, in the σ model, j_5 is proportional to the canonical

[8] Similar incompatibility of conservation of *two* currents has been encountered in two-dimensional model field theories; see K. Johnson, Phys. Letters **5**, 253 (1963).

pion field, which has smooth matrix elements. Second, $i\partial_\mu j_5^\mu$ leads to a vanishing invariant decay constant. In those theories which lead to a nonvanishing effective coupling constant (such as the σ model), $i\partial_\mu j_5^\mu$ is manifestly not smooth. (Only these theories are of physical interest.) Finally, as we shall now see, the choice j_5 gives an answer independent of the ambiguities of j_5^μ. Accepting j_5 as the proper interpolating field, we see that according to (2.22a) the pion field is proportional to the divergence of $j_5^\mu(x,a)$ minus $(e^2/16\pi^2)[(1+a)/2] \times \tilde{F}^{\alpha\beta}(x)F_{\alpha\beta}(x)$. The current may be written as a gauge-invariant part, independent of a, plus a gauge-dependent part:

$$j_5^\mu(x,a)=j_5^\mu(x,1)+(ie^2/16\pi^2)(1-a)\tilde{F}^{\mu\beta}(x)A_\beta(x).$$

At zero pion mass, the gauge-invariant portion $\partial_\mu j_5^\mu(x,1)$ does not contribute, according to the standard PCAC argument. Therefore the only remaining terms are

$$i\partial_\mu[ie^2/16\pi^2(1-a)\tilde{F}^{\mu\beta}(x)A_\beta(x)] -(e^2/16\pi^2)\tfrac{1}{2}(1+a)\tilde{F}^{\alpha\beta}(x)F_{\alpha\beta}(x) =(-e^2/16\pi^2)\tilde{F}^{\alpha\beta}(x)F_{\alpha\beta}(x). \quad (2.23)$$

It is seen that this is independent of a, and leads to a nonvanishing decay constant; it may be verified[1] that it agrees with the direct calculation in the σ model, where use is made of the canonical pion field. We stress that the above calculation is performed in the Heisenberg picture, and makes use of the anomalous divergence. The interaction-picture calculation will shed different light on the problem.

We conclude this section by a discussion of the modifications of the above results by other interactions. These modifications may arise in two ways. In the first place the basic divergence equation [Eq. (2.5)] may have additional terms. Also, higher-order corrections can change the value of the matrix elements of $i\partial_\mu j_5^\mu$. To study the effect of the former, let us consider the more general equation of motion (2.4a):

$$i\gamma\cdot\partial\psi=-eA\psi+\mathcal{O}\psi. \quad (2.24)$$

$\mathcal{O}$ is a Dirac matrix, which we assume has no vector component, the latter being already exhibited in A^μ. Hermiticity requires that $\bar{\mathcal{O}}=\mathcal{O}$. (We ignore internal symmetries.) The generalization of (2.5) is (we restrict ourselves now to gauge-invariant definition of the axial-vector current, and suppress $a=1$)

$$i\partial_\mu j_5^\mu(x,\epsilon)=j_5(x,\epsilon)+e\epsilon^\alpha j_5^\mu(x,\epsilon)F_{\alpha\mu}(x) +\tfrac{1}{2}\epsilon^\alpha\bar{\psi}(x+\tfrac{1}{2}\epsilon)[\partial_\alpha\mathcal{O}(x),\gamma^5]\psi(x-\tfrac{1}{2}\epsilon) +O(\bar{\psi}(x+\tfrac{1}{2}\epsilon)\psi(x-\tfrac{1}{2}\epsilon)\epsilon^2), \quad (2.25)$$

$$j_5(x,\epsilon)\equiv\bar{\psi}(x+\tfrac{1}{2}\epsilon)[\mathcal{O}(x),\gamma^5]_+\psi(x-\tfrac{1}{2}\epsilon) \times\exp\left(ie\int_{x-\epsilon/2}^{x+\epsilon/2}A_\alpha(y)dy^\alpha\right).$$

We shall ignore the $O(\bar{\psi}(x+\frac{1}{2}\epsilon)\psi(x-\frac{1}{2}\epsilon)\epsilon^2)$ term; that is, we shall assume that there are no problems in defining the product $\mathcal{O}(x)\psi(x)$.[9] Decomposing $\mathcal{O}(x)$ as

$$\mathcal{O}(x)=S(x)+\sigma^{\alpha\beta}T_{\alpha\beta}(x)+\gamma^5\gamma^\mu B_\mu(x)+\gamma^5A(x) \quad (2.26)$$

yields

$$i\partial_\mu j_5^\mu(x,\epsilon)=j_5(x,\epsilon)+e\epsilon^\alpha j_5^\mu(x,\epsilon)F_{\alpha\mu}(x) +\epsilon^\alpha j^\mu(x,\epsilon)\partial_\alpha B_\mu(x). \quad (2.27)$$

Hence additional anomalous terms are present only when there exists pseudovector coupling. Since the axial-vector current is not conserved, such coupling would lead to a unrenormalizable theory. To avoid this we assume that $B=0$. For the same reason we take $T_{\alpha\beta}=0$. Therefore, for renormalizable theories without internal symmetries, the anomalous term in the divergence equation retains the form (2.5).

Next we discuss higher-order corrections to the matrix elements of $i\partial_\mu j_5^\mu(x,\epsilon)=j_5+e\epsilon^\alpha j_5^\mu(x,\epsilon)F_{\alpha\mu}(x)$. When the electromagnetic field is a dynamical variable, we are unable to make a statement valid to all orders of electrodynamics. However, for practical applications, it is sufficient to consider only the second order, which, as was seen above, is equivalent to treating the electromagnetic field as an external variable. We shall now limit the discussion to the external-field problem, and work to all orders in the other interactions.

In the presence of other interactions, (2.5) and (2.6) are still valid. However, (2.7) must be replaced (to order e, which is sufficient since we seek $\partial_\mu j_5^\mu$ to order e^2) by

$$G(x-\tfrac{1}{2}\epsilon,\, x+\tfrac{1}{2}\epsilon)=G(-\epsilon)+ie \times16\int d^4y\frac{d^4p d^4q}{(2\pi)^8}e^{i\epsilon\cdot p}e^{-2ix\cdot q}e^{2iy\cdot q}G(p+q) \times\Gamma^\alpha(p+q,\, p-q)G(p-q)A_\alpha(y)+O(e^2). \quad (2.28)$$

Here $G(p)$ is the complete, unrenormalized momentum-space propagator, and $\Gamma^\alpha(p+q,\, p-q)$ is the complete, unrenormalized electromagnetic vertex function, both in the absence of electrodynamics. The vertex is normalized by the Ward identity

$$\Gamma^\alpha(p,p)=-i\partial^\alpha G^{-1}(p)=iG^{-1}(p)\partial^\alpha G(p)G^{-1}(p). \quad (2.29)$$

Since we seek the singular part of the integral as $\epsilon\to0$, only the large p portion of the integrand is needed. We shall assume that the complete vertex does not introduce

[9] Ignoring the difficulties of operator products $\mathcal{O}(x)\psi(x)$ is, of course, not satisfactory since these too will be singular. To analyze these products, one needs to commit oneself to specific interactions, a program which would be beyond the scope of the present investigation. We speculate, however, that these singularities, which are, of course, those that are handled by conventional renormalization theory, will not affect our results as long as the theory and the axial-vector current are renormalizable. Furthermore, in the Appendix we present an alternative approach which can take into account these singularities in some cases.

any new ultraviolet divergences; that is, for large p,

$$\Gamma^{\alpha}(p,p)\to \text{const},$$
$$\Gamma^{\alpha\beta}(p)\equiv\frac{\partial}{\partial q_\beta}\Gamma^{\alpha}(p+q,\,p-q)\bigg|_{q=0}\sim\frac{1}{p},\ \text{etc.} \tag{2.30}$$

The integrand may be expanded in powers of q as before:

$$G(x-\tfrac{1}{2}\epsilon,\,x+\tfrac{1}{2}\epsilon)=G(-\epsilon)+ie\int\frac{d^4p d^4q}{(2\pi)^8}e^{i\epsilon\cdot p}e^{-ix\cdot q}A_\alpha(q)$$
$$\times\{i\partial^\alpha G(p)+\tfrac{1}{2}iq_\beta[\partial^\beta G(p)G^{-1}(p)\partial^\alpha G(p)$$
$$-\partial^\alpha G(p)G^{-1}(p)\partial^\beta G(p)]+\tfrac{1}{2}q_\beta G(p)\Gamma^{\alpha\beta}(p)G(p)\}$$
$$+O(e^2)+O(\ln\epsilon). \tag{2.31a}$$

We have made use of (2.29) to simplify the above. Note that the Ward-Takahashi identity implies that the symmetric part of $\Gamma^{\alpha\beta}$ is zero. The first two terms in the curly brackets reproduce the corresponding terms of (2.7) when use is made of the asymptotic formula

$$G(p)\to 1/\not{p}, \tag{2.31b}$$

which is accurate for our purposes of extracting the singular ϵ behavior. Therefore, instead of (2.8), we now have

$$-\mathrm{Tr}\gamma^5\gamma^\mu G(x-\tfrac{1}{2}\epsilon,\,x+\tfrac{1}{2}\epsilon)=\frac{-e}{4\pi^2}\frac{\epsilon_\beta}{\epsilon^2}\tilde{F}^{\mu\beta}(x)$$
$$-\tfrac{1}{4}eF_{\alpha\beta}(x)\int\frac{d^4p}{(2\pi)^4}e^{i\epsilon\cdot p}\,\mathrm{Tr}\gamma^5\gamma^\mu\not{p}^{-1}\Gamma^{\alpha\beta}(p)\not{p}^{-1}. \tag{2.31c}$$

The vertex is decomposed into its matrix components

$$\Gamma^\alpha=G_1{}^\alpha+\gamma_\mu G_2{}^{\mu\alpha}+\sigma_{\mu\nu}G_3{}^{\mu\nu\alpha}+\gamma^5\gamma_\mu G_4{}^{\mu\alpha},$$
$$\Gamma^{\alpha\beta}=G_1{}^{\alpha\beta}+\gamma_\mu G_2{}^{\mu\alpha\beta}+\sigma_{\mu\nu}G_3{}^{\mu\nu\alpha\beta}+\gamma^5\gamma_\mu G_4{}^{\mu\alpha\beta}, \tag{2.32a}$$
$$G_1{}^{\alpha\beta}(p)\equiv(\partial/\partial q^{\alpha\beta})G_1{}^\alpha(p+q,\,p-q)\,|_{q=0},$$

etc. Only the induced pseudovector survives the trace:

$$\mathrm{Tr}\gamma^5\gamma^\mu\not{p}^{-1}\Gamma^{\alpha\beta}(p)\not{p}^{-1}=4G_4{}^{\mu\alpha\beta}(p)/p^2. \tag{2.32b}$$

$G_4{}^{\mu\alpha}(p+q,\,p-q)$ is given by $\varepsilon^{\mu\alpha\beta\gamma}q_\beta p_\gamma G_4(p^2,q^2,p\cdot q)$. Therefore,

$$G_4{}^{\mu\alpha\beta}(p)=\varepsilon^{\mu\alpha\beta\gamma}p_\gamma G_4(p^2,0,0)\equiv\varepsilon^{\mu\alpha\beta\gamma}p_\gamma F(p^2). \tag{2.32c}$$

Inserting these expressions into (2.31c) leaves

$$-\mathrm{Tr}\gamma^5\gamma^\mu G(x-\tfrac{1}{2}\epsilon,\,x+\tfrac{1}{2}\epsilon)$$
$$=(-e/4\pi^2)(\epsilon_\beta/\epsilon^2)\tilde{F}^{\mu\beta}[1+f(\epsilon^2)], \tag{2.33a}$$
$$\langle\Omega|i\partial_\mu j_5{}^\mu(x)|\Omega\rangle$$
$$=(e^2/16\pi^2)[1+f(0)]\tilde{F}^{\mu\nu}(x)F_{\mu\nu}(x), \tag{2.33b}$$
$$f(\epsilon^2)=-\frac{1}{(2\pi)^2}\int\frac{d^4p}{p^2}e^{i\epsilon\cdot p}F(p^2)p\cdot\epsilon. \tag{2.33c}$$

Whether or not $f(0)$ vanishes is now seen to depend on the asymptotic behavior of $F(p^2)$. Presumably $F(p^2)$ satisfies an unsubtracted dispersion relation, since there is no direct pseudovector coupling. However, this would indicate that $F(p^2)\propto 1/p^2$ for large p, and $f(0)$ is not zero unless the proportionality constant vanishes.

Unfortunately, perturbation theory cannot be used in connection with the above formula, because of divergences. Indeed the above is ambiguous and one can get any one of three answers, depending upon how one evaluates the expressions (2.28) or (2.33c). If one introduces a cutoff and then lets the cutoff go to infinity in (2.28), one arrives at an infinite result, since the unrenormalized quantities occurring in (2.28) are infinite. If one introduces a cutoff, and maintains it at a finite value, letting p^2 go to infinity first, then (2.30) and (2.31) are true and $F(p^2)$ is given (for scalar and pseudoscalar interactions) by

$$F(p^2)\to c(\mu^2)/p^2-c(\Lambda^2)/p^2. \tag{2.34a}$$

Here μ^2 is the mass of the scalar and pseudoscalar particles and Λ^2 is the cutoff. However, $c(\mu^2)$ turns out to be a constant independent of the masses, and

$$F(p^2)\to p^{-2}\times 0. \tag{2.34b}$$

Finally, the third answer can be obtained by introducing the cutoff only in the calculation of $F(p^2)$ and letting Λ^2 go to infinity before p^2. Then

$$F(p^2)\to c/p^2 \tag{2.35}$$

and $f(0)\neq 0$.

Adler[4] has given an argument to the end that there exist no higher-order effects. He introduced a cutoff, calculated the divergence, and then let the cutoff go to infinity. This is seen in the present context to be equivalent to the second method above. However, we believe that this method may not be reliable because of the dependence on the order of limits. In the Appendix, we present a formula of perturbation theory which is free of divergences, when the axial-vector vertex is multiplicatively renormalized by a renormalization constant whose divergent part coincides with renormalization constant of the vector vertex. This desirable state of affairs occurs when the naive divergence of axial-vector current j_5 is multiplicatively renormalizable, as in the neutral-vector-meson-exchange model.

III. AXIAL-VECTOR CURRENT, ITS DIVERGENCE, AND COMMUTATORS IN THE INTERACTION PICTURE

Section II demonstrated how the anomalous behavior of the axial-vector current modifies the derivation of $\pi^0\to 2\gamma$ in the Heisenberg picture. However, the study of electromagnetic effects is conventionally carried out in the electromagnetic-interaction picture, so that the operators are governed by equations of motion which do not involve electrodynamics. We now carry out the analysis of the axial-vector current, its divergence, and its commutators in the interaction picture. The electromagnetic field will always be considered external.

We consider the vacuum-expectation value of the axial-vector current. To order e^2 this is given by

$$\langle j_5{}^\mu(x)\rangle_H = -\tfrac{1}{2}e^2 \int d^4y d^4z \langle\Omega| T^* j_5{}^\mu(x) j^\sigma(y) j^\rho(z) |\Omega\rangle \times A_\sigma(y) A_\rho(z) . \quad (3.1)$$

The left-hand side involves the Heisenberg-picture vacuum and current, indicated by the subscript H, while the right-hand side involves interaction-picture operators. Using intuition based on simpler cases in electrodynamics, one usually ignores the seagull terms which are defined to be the difference between the T product and the T^* product in calculating divergences. That is, one ignores Schwinger terms in the commutators of the currents, trusting that they will cancel against the divergence of the seagull terms. In the present instance, we find that this cancellation does not occur. Thus we must find the explicit form of these terms. We have

$$\langle j_5{}^\mu(x)\rangle_H = -\tfrac{1}{2}e^2 \int d^4y d^4z \langle\Omega| T j_5{}^\mu(x) j^\sigma(y) j^\rho(z) |\Omega\rangle \times A_\sigma(y) A_\rho(z) + S^\mu(x) , \quad (3.2)$$

where S^μ is the seagull term. The T product is neither gauge- nor Lorentz-invariant. Let us first require that the axial-vector current be gauge-invariant: invariant under the interaction picture transformation $A_\sigma(y) \to A_\sigma(y) + \partial_\sigma \Lambda(y)$. The seagull term is uniquely determined by requiring that summing it with the T product yields a gauge-invariant T^* product.

To exhibit the effect of these transformations, we need to know the anomalous commutators of all the currents. These may be determined by the method of Johnson and Low.[10] We define

$$M^{\mu\sigma\rho}(p,q) = -i \int e^{ip\cdot(y-x)} e^{iq\cdot(z-x)} \times \langle\Omega| T j_5{}^\mu(x) j^\sigma(y) j^\rho(z) |\Omega\rangle d^4y d^4z . \quad (3.3)$$

[10] K. Johnson and F. E. Low, Progr. Theoret. Phys. (Kyoto) Suppl. Nos. **37** & **88**, 74 (1966); we warn against a misunderstanding which might arise from the form that these equations take. When one computes the divergence of (3.2), the time differentation is applied to the *result* of doing the double time integrations of the T product. The form of the answer in terms of commutators is correct only if the equal-time commutators are defined as in (3.4a) and (3.4b), rather than from canonical commutation relations. The canonical rules would lead to vanishing commutators. Even when care is taken to yield the Schwinger term, no operator Schwinger term can be obtained since the currents are free-field currents. The anomalous commutators arise from an additional contribution which comes when the other time integral gets "pinched" between the two currents which form the "equal"-time commutator written in (3.4a). This contribution is included in the momentum-space calculation which is performed below, since when q is kept fixed the full effect of the other time integration is retained. More discussion of this point is given in the above-mentioned paper by Johnson and Low.

TABLE I. Elements of the current commutator tensor.

			$N_1{}^{\mu\sigma\rho}(\mathbf{p},q)$
$\mu=0$	$\sigma=0$		$(ie^2/2\pi^2) p_i q_j \varepsilon^{ij\rho}$
	$\sigma=k$	$\rho=0$	$(-ie^2/4\pi^2) p_i q_j \varepsilon^{ijk}$
		$\rho=l$	$(-ie^2/4\pi^2) q_0 p_i \varepsilon^{ikl} + (-ie^2/4\pi^2) q_0 q_i \varepsilon^{ikl}$
$\mu=i$	$\sigma=0$	$\rho=0$	$(-ie^2/4\pi^2) p_j q_k \varepsilon^{jki}$
		$\rho=l$	$(ie^2/4\pi^2) q_0 p_j \varepsilon^{jli}$
			$N_2{}^{\mu\sigma\rho}(\mathbf{p},q)$
$\sigma=0$	$\rho=0$		$(ie^2/2\pi^2) p_j q_\kappa \varepsilon^{jk\mu}$
	$\rho=k$	$\mu=0$	$(-ie^2/4\pi^2) p_i q_j \varepsilon^{ijk}$
		$\mu=l$	$(-ie^2/4\pi^2) q_0 p_j \varepsilon^{jkl}$

The quantity

$$N_1{}^{\mu\sigma\rho}(\mathbf{p},q) \equiv \int e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} e^{iq\cdot(z-x)} \times \langle\Omega| T[j_5{}^\mu(x), j^\sigma(y)]|_{x_0=y_0} j^\rho(z) |\Omega\rangle d^3y d^4z \quad (3.4a)$$

is given by the coefficient of the $1/p^0$ term in the asymptotic form of $M^{\mu\sigma\rho}(p,q)$ as $p_0 \to \infty$. Similarly, the quantity

$$N_2{}^{\mu\sigma\rho}(\mathbf{p},q) \equiv \int e^{-i\mathbf{p}\cdot(\mathbf{y}-\mathbf{z})} e^{iq\cdot(x-z)} \times \langle\Omega| T[j^\rho(z), j^\sigma(y)]_{z_0=y_0} j_5{}^\mu(x) |\Omega\rangle d^3y d^4z \quad (3.4b)$$

is given by the coefficient of $1/p^0$ in the asymptotic form of $M^{\mu\sigma\rho}(p, -p-q)$ as $p_0 \to \infty$. $M^{\mu\sigma\rho}$ has been calculated by Rosenberg.[11] From his explicit form we may determine $N_1{}^{\mu\sigma\rho}$ and $N_2{}^{\mu\sigma\rho}$. Some components of these objects, relevant for our calculation, are summarized in Table I.

The unique seagull term which makes the T^* product gauge- and Lorentz-invariant is

$$S^0(x) = -(e^2/8\pi^2) A_i(x) F_{jk}(x) \varepsilon^{ijk} ,$$
$$S^m(x) = (e^2/8\pi^2) A_j(x) F_{k0}(x) \varepsilon^{mjk} . \quad (3.5)$$

We are now in a position to calculate the divergence of $\langle j_5{}^\mu(x)\rangle_H$. The contributions to this are of three kinds. First, there is the divergence of $\partial_\mu j_5{}^\mu$ which for simplicity we take to be zero in the interaction picture. Next, there is the contribution of the commutators $C(x)$ arising from taking the derivative through the T product. Finally, there is the divergence of the seagull term. Thus

$$\langle \partial_\mu j_5{}^\mu \rangle_H = C(x) + \partial_\mu S^\mu(x) . \quad (3.6a)$$

$C(x)$ is evaluated from Table I and $\partial_\mu S^\mu(x)$ from (3.5). We find, as expected, that

$$i\langle \partial_\mu j_5{}^\mu \rangle_H = (e^2/16\pi^2) F^{\mu\nu} \tilde{F}_{\mu\nu} . \quad (3.6b)$$

It is seen that the Schwinger terms arising from the commutator do not cancel the seagull term. See the Note added in proof for details.

[11] L. Rosenberg, Phys. Rev. **129**, 2786 (1963).

We may now also understand the gauge-noninvariant definitions of $j_5{}^\mu$ which preserve PCAC. In the present context, they manifest themselves as additional seagull terms which are Lorentz-invariant but not gauge-invariant. For example, if we add to S^μ the term $-(e^2/8\pi^2)\tilde{F}^{\mu\nu}A_\nu$, then PCAC is regained.

In conclusion, we demonstrate how pion decay is calculated in the interaction picture. We seek the interaction-picture matrix element

$$\langle\Omega|T^*\phi(x)j^\alpha(y)j^\beta(z)|\Omega\rangle = \langle\Omega|T\phi(x)j^\alpha(y)j^\beta(z)|\Omega\rangle. \quad (3.7a)$$

There are no seagull terms since the T product is already gauge- and Lorentz-invariant. Use is now made of the interaction-picture equation

$$\partial_\mu j_5{}^\mu \propto \phi \quad (3.7b)$$

to write the above as

$$\langle\Omega|T\phi(x)j^\alpha(y)j^\beta(z)|\Omega\rangle \propto \langle\Omega|T\partial_\mu j_5{}^\mu(x)j^\alpha(y)j^\beta(x)|\Omega\rangle = \partial_\mu\langle\Omega|Tj_5{}^\mu(x)j^\alpha(y)j^\beta(z)|\Omega\rangle - C^{\alpha\beta}(x,y,z). \quad (3.7c)$$

Here $C^{\alpha\beta}(x,y,z)$ is the contribution of the commutators, and is related in an obvious fashion to the $C(x)$ function introduced above. We now rewrite the above again:

$$\langle\Omega|T\phi(x)j^\alpha(y)j^\beta(z)|\Omega\rangle \propto \partial_\mu[\langle\Omega|Tj_5{}^\mu(x)j^\alpha(y)j^\beta(z)|\Omega\rangle + S^{\mu\alpha\beta}(x,y,z)] - \partial_\mu S^{\mu\alpha\beta}(x,y,z) - C^{\alpha\beta}(x,y,z). \quad (3.7d)$$

Here $S^{\mu\alpha\beta}$ is the seagull term, related to the S^μ introduced previously. The term in square brackets is gauge- and Lorentz-invariant, and hence it does not contribute to the invariant $\pi^0\to 2\gamma$ coupling constant at zero pion mass. The only surviving terms are the $\partial S+C$, which, as we have seen, do not sum to zero. Thus the low-energy theorem for $\pi^0\to 2\gamma$ crucially depends on Schwinger terms and seagull terms.

IV. SUMMARY AND CONCLUSION

We summarize our results. It has been demonstrated that the Heisenberg equation of motion for the fermion axial-vector current contains terms which are not given by straightforward application of canonical reasoning. These terms depend on the nature of the interactions of the fermions, and on the method chosen to define the singular product of two fermion fields. In a definite model the nature of the modification can be determined, but in general only to lowest order in interactions. The decay of the pion into two photons, which can be calculated directly and unambiguously in the σ model, is found to be independent of the choice of definition of the current.

In the electromagnetic-interaction picture, it is found that the time-ordered product of an axial-vector current with two-vector currents is singular, and that the removal of this singularity which is effected by the addition of seagull terms modifies the divergence equation for the T^* product.

The present investigation shows that vector-current-conservation symmetry, chiral symmetry, and minimality of vector interactions are mutually incompatible, at least in perturbation theory. In a previous investigation of the problem, minimality was abandoned and both symmetries were preserved.[1] In the present instance, we maintain minimality, and abandon one of the two symmetries. We note that analogous results will hold when other renormalizable vector couplings of the fermions are present, but the explicit form of the additional terms in the divergence of the axial-vector current will depend upon these couplings. In conclusion, we also mention that the present considerations are not modified if the field-current identity is assumed. The point is that the divergence of the field-current operator is formally proportional to the divergence of the current constructed by Noether's theorem. Hence if the divergence of the latter is anomalous, so is the divergence of the former.

Note added in proof. We wish to expand upon the remark in Ref. 10. The definition of the T product of operators requires special care when the operators do not commute at equal times. However, when the equal-time operator products "exist" (have a finite, well-defined Fourier transform), one may most reasonably define the T product of $A(\mathbf{x},x^0)$ and $B(\mathbf{0},0)$ to be

$$T[A(\mathbf{x},x^0)B(0,0)] \equiv \theta(x^0)A(\mathbf{x},x^0)B(\mathbf{0},0) + \theta(-x^0)B(\mathbf{0},0)A(\mathbf{x},x^0), \quad (\text{N1})$$

where $\theta(0)=\frac{1}{2}$, so that $\theta(x^0)+\theta(-x^0)=1$. By hypothesis,

$$A(\mathbf{x},0)B(0,0) \equiv \lim_{x^0\to+0}[A(\mathbf{x},x^0)B(\mathbf{0},0)]$$

and

$$B(\mathbf{0},0)A(\mathbf{x},0) \equiv \lim_{x^0\to-0} B(\mathbf{0},0)A(\mathbf{x},x^0).$$

The operator products needed here exist in this sense. Accordingly, the function

$$\langle 0|T\{j_5{}^\mu(x)j^\sigma(y)j^\rho(z)\}|0\rangle$$

exists and is unambiguous, since quantities such as

$$\langle 0|j_5{}^\mu(x)j^\sigma(y)j^\delta(z)|0\rangle$$

for $x^0>y^0>z^0$, may be ***unambiguously*** calculated from the appropriate Feynman diagram because the ambiguous terms of the diagram are local in time and vanish when $x^0\neq y^0\neq z^0$. With this definition for products of operators it may be seen that quantities of the form

$$\langle 0|T\{[j(x),j^0(y)]_{x^0=y^0}j(z)\}|0\rangle \quad (\text{N2})$$

vanish identically. [This is not in contradiction with (3.4) since the limit $p^0\to\infty$ is taken for fixed q, thus retaining the full effect of the z^0 integration, in particular when z^0 gets "pinched" between x^0 and y^0. See below.] For, if with x^0, $y^0 > z^0$ we calculate

$$\langle 0|[j(x),j^0(y)]_{x^0=y^0}j(z^0)|0\rangle \equiv \lim_{x^0\to y^0+0}\langle 0|j(x)j^0(y)j(z)|0\rangle - \lim_{x^0\to y^0-0}\langle 0|j(y)j^0(x)j(z)|0\rangle\,,$$

we get zero. To show this we use the momentum-space technique introduced in the text and allow the momentum associated with the relative coordinate x^0-y^0 to become infinite. We thereby obtain a polynomial in the momentum associated with the relative coordinate x^0-z^0 (see Table I) and therefore a function in coordinate space which vanishes when $x^0>z^0$. In the same way we also get zero when $z^0>x^0$. So according to the definition of the T product we get the vanishing of (N2). We may now define

$$T^\mu(x)=-\tfrac{1}{2}e^2\int d^4y d^4z \times M^{\mu\sigma\rho}(y,z)A_\sigma(y+x)A_\rho(z+x)\,, \quad \text{(N3)}$$

where $M^{\mu\sigma\rho}$ is as in (3.3), and where the T product is defined by the rules given above. To find the change produced in (N3), under a change of gauge, we let $\delta A_\sigma=\partial_\sigma\Lambda$ and obtain

$$\delta T^\mu(x)=-\tfrac{1}{2}e^2\int d^4y\int d^4z \times F^{\mu\sigma\rho}(y,z)\left[A_\rho(z+x)\frac{\partial}{\partial y^\sigma}\Lambda(y+x) +A_\sigma(y+x)\frac{\partial}{\partial z^\rho}\Lambda(z+x)\right]. \quad \text{(N4)}$$

In (N4) we may integrate the y and z integrals in either order. However, in any rearrangement of (N4), once we have chosen an order we must use the same order in all terms of the rearrangement. We integrate by parts in (N4) with the rule that we do the z integration first, then y. Thus,

$$\delta T^\mu(x)=\tfrac{1}{2}e^2\int d^4y \times\left[\frac{\partial}{\partial y^\sigma}\int d^4z\, F^{\mu\sigma\rho}(y,z)A_\rho(z+x)\right]\Lambda(y+x) +\tfrac{1}{2}e^2\int d^4y\left[\int d^4z\left(\frac{\partial}{\partial z^\rho}F^{\mu\sigma\rho}(y,z)\right)\Lambda(z+x)\right] \times A^\sigma(y+x)\,,$$

or, if we use the fact that F is symmetrical on $y\leftrightarrow z$, $\sigma\leftrightarrow\rho$,

$$\delta T^\mu(x)=\tfrac{1}{2}e^2\int d^4y \times\left[\frac{\partial}{\partial y^\sigma}\int d^4z\, F^{\mu\sigma\rho}(y,z)A^\rho(z+x)\right]\Lambda(y+x) +\tfrac{1}{2}e^2\int d^4z\left[\int d^4y\left(\frac{\partial}{\partial y^\sigma}F^{\mu\sigma\rho}(y,z)\right)\Lambda(y+x)\right] \times A^\sigma(z+x)\,. \quad \text{(N5)}$$

In the second term of (N5), the y integration is done for fixed z, and it will be immediately seen that this term vanishes by writing it out using the definitions of T products and the fact that quantities of the form of (N2) vanish. If, however, we investigate the first term of (N5), we find that all contributions vanish with the exception of

$$\delta T^\mu(x)=\tfrac{1}{2}e^2\int d^4y\,\Lambda(y+x)\delta(y^0)\int d^3z\,A_\rho(z+x) \times\lim_{\epsilon\to 0}\left[\int_{-\epsilon}^{+\epsilon}dz^0\{\langle 0|T\{j^0(y,\epsilon)j^\rho(z,z^0)j_5{}^\mu(0,0)\}|0\rangle -\langle 0|T\{j_5{}^\mu(0,0)j^\rho(z,z^0)j^0(y,-\epsilon)\}|0\rangle\}\right]. \quad \text{(N6)}$$

Notice that in this term, the effect of the perturbation, namely, $j\cdot A$ gets "pinched" between the currents which would normally produce a commutator. This is the basis of the remark in Ref. 10. It is the fact that this term produces a finite contribution which is responsible for the lack of gauge-Lorentz covariance of the T product and produces the anomalies in the formal manipulation of commutators. To evaluate this pinched contribution most easily, it is convenient to extend the z^0 integration over the entire range, $-\infty$ to $+\infty$; the added terms are zero in virtue of (N2). We then obtain

$$\delta T^\mu(x)=\tfrac{1}{2}e^2\int d^4y\,\Lambda(y+x)\delta(y^0)\int d^4z\,A_\rho(z+x) \lim_{\epsilon\to 0}[\langle 0|T\{j^0(y,\epsilon)j^\rho(z)j_5{}^\mu(0,0)\}|0\rangle -\langle 0|T\{j_5{}^\mu(0,0)j^\rho(z)j^0(y,-\epsilon)\}|0\rangle]. \quad \text{(N7a)}$$

We next move to momentum space to cast (N7a) in the form

$$\delta T^\mu(x)=\tfrac{1}{2}e^2\int\frac{d^3p d^4q}{(2\pi)^7}e^{-i(p+q)\cdot x}\Lambda(p)A_\rho(q) \times\lim_{\epsilon\to+0}\int\frac{dp^0}{2\pi}(e^{ip^0\epsilon}-e^{-ip^0\epsilon})M^{\mu 0\rho}(-p,-q) \quad \text{(N7b)}$$

and by using the standard method to evaluate the limit $\epsilon \to 0$ and the entries in Table I, we obtain finally

$$\begin{aligned} \delta T^0(x) &= (e^2/8\pi^2)\partial_i\Lambda(x)F_{jk}(x)\varepsilon^{ijk}, \\ \delta T^m(x) &= -(e^2/8\pi^2)\partial_j\Lambda(x)F_{k0}(x)\varepsilon^{mjk}. \end{aligned} \tag{N8}$$

If we compute with (3.5) we find that $\delta(T^\mu+S^\mu)=0$. Next we compute the divergence of $T^\mu(x)$. The calculation proceeds exactly as above, yielding only an anomalous contribution when the perturbation becomes pinched between the axial current and vector current in one term. This contribution again may be evaluated with the aid of Table I. We find

$$\partial_\mu T^\mu(x) = (e^2/16\pi^2)\varepsilon^{ijk}(3F_{i0}F_{jk}+2\dot{A}_iF_{jk}+A_i\dot{F}_{jk}), \tag{N9}$$

and when we thereby evaluate $\partial_\mu(T^\mu+S^\mu)$ we obtain (3.6b).

We are grateful to Dr. S. L. Adler for finding a sign error in an earlier version of this calculation.

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APPENDIX

We present here an alternative formula for the vacuum-expectation value of the divergence of the axial-vector current in the presence external electromagnetic interactions which may be useful in perturbation theory. It does not have the compact form of (2.33c); however, t may be free of ambiguities. The relevant integral (2.28) can be written by integration by parts in the form

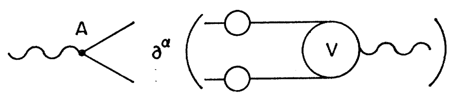

FIG. 2. Diagrammatic representation of Eq. (A1).

$$\begin{aligned} &\lim_{\epsilon\to 0}\int d^4p\ \epsilon^\alpha e^{i\epsilon\cdot p}\ \mathrm{Tr}\gamma^5\gamma^\mu G(p+q)\Gamma^\nu(p+q,\ p-q)G(p-q) \\ &=\lim_{\epsilon\to 0} -i\int d^4p\ \frac{\partial}{\partial p_\alpha}e^{i\epsilon\cdot p}\ \mathrm{Tr}\gamma^5\gamma^\mu G(p+q) \\ &\qquad\times\Gamma^\nu(p+q,\ p-q)G(p-q) \\ &=\lim_{\epsilon\to 0} i\int d^4p\ e^{i\epsilon\cdot p}\frac{\partial}{\partial p_\alpha}[\mathrm{Tr}\gamma^5\gamma^\mu G(p+q) \\ &\qquad\times\Gamma^\nu(p+q,\ p-q)G(p-q)] \\ &=i\int d^4p\ \frac{\partial}{\partial p_\alpha}[\mathrm{Tr}\gamma^5\gamma^\mu G(p+q) \\ &\qquad\times\Gamma^\nu(p+q,\ p-q)G(p-q)]. \end{aligned} \tag{A1}$$

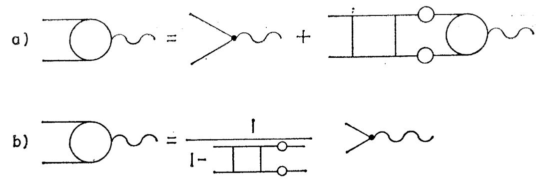

FIG. 3. Bethe-Salpeter equation for axial-vector and vector vertices.

Diagrammatically this has the representation shown in Fig. 2. We now assume that both the axial-vector and vector vertices satisfy a Bethe-Salpeter equation diagrammed in Fig. 3. From Fig. 3(a), we have the result shown in Fig. 4. The equation of Fig. 3(a) has been used in passing from the second equality to the third equality. The momentum p with respect to which we are differentiating must be routed so that it follows the charge. By use of the equation in Fig. 3(b), we obtain an expres-

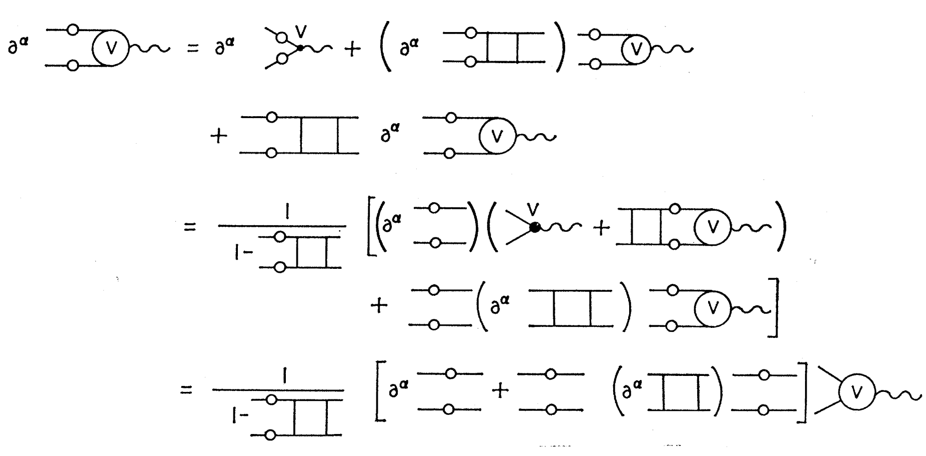

FIG. 4. Manipulation of Fig. 3(b).

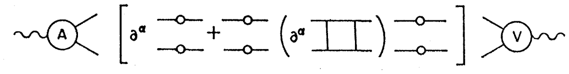

FIG. 5. Alternative expression for the quantity of Fig. 2.

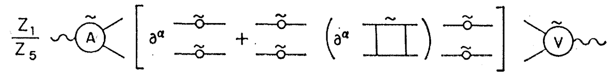

FIG. 6. Renormalized version of Fig. 5.

sion for the quantity depicted in Fig. 2. This expression is summarized by Fig. 5. This may now be expressed in renormalized quantities which we indicate by a tilde. Assuming $\Gamma_5{}^\mu$ to be multiplicatively renormalizable by $Z_5{}^{-1}$, as well as G by Z_1 and Γ^μ by $Z_2=Z_1$, we have the renormalized version of Fig. 5. This is given in Fig. 6. When Z_1/Z_5 is finite the above involves only finite quantities.

In lowest-order perturbation theory the quantity in parentheses in Fig. 6 vanishes, and only the derivative of the propagator is left. We have not evaluated this formula.

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Kinematic Structure of Vertex Functions*

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An invariant-amplitude expansion is obtained for vertex functions of fields belonging to arbitrary representations of the Lorentz group between states of arbitrary spins. The method is based on analysis of the singularities of the Lorentz-group parameters defining the vertex function. The restrictions due to parity and subsidiary conditions are also given.

I. INTRODUCTION

THE purpose of this paper is to give an expansion for vertex functions in terms of functions which are completely free from kinematic singularities and constraints. Following common practice, we will refer to these functions as invariant amplitudes, although the distinguishing feature is the absence of kinematic singularities and constraints. The method we employ is the same as that used in an earlier paper on kinematic constraints and singularities of scattering amplitudes.[1] It is based on an analysis of the singularities of the Lorentz-group parameters defining the amplitude as a function of the scalar variables. Since the vertex depends on only one variable, t, this method allows a complete removal of all singularities and constraints simultaneously.

The vertex functions we will study are

$$F_{\lambda_1\lambda_2}{}^{JM}(p_1,p_2)=\langle p_2,s_2,\lambda_2|\phi_{JM}{}^{j_0\sigma}|p_1,s_1,\lambda_1\rangle\,. \quad (1.1)$$

There are two legs on the mass shell, with arbitrary spins and masses, and one leg off the mass shell. The latter is taken to belong to the $j_0\sigma$ representation of the Lorentz group.[2–4] (For the finite, nonunitary representations $\phi_{JM}{}^{[a,b]}$, $j_0=a-b$ and $\sigma=a+b+1$.) There are many reasons for studying the structure of these functions for such general cases. To cite a few: (a) One can use them to construct one-particle exchange or pole terms which correspond to very-high-spin particles and which satisfy the conditions required by Lorentz invariance. (b) Many results which are true for arbitrary σ and physical J may presumably be extended to complex values of J and so be applied to factorized Regge residues; thus, one could study the behavior of the Regge residues in a very direct way. (c) The results may be a useful step in obtaining the kinematic structure of more complex amplitudes. (d) Further understanding of the significance of singularities and constraints, such as those which result from subsidiary conditions, may be obtained. A number of authors have studied this problem using a variety of methods and have obtained rather general results.[2,5–7] The method employed here is different from all of the preceding ones and the results are obtained in a substantially different and, we believe, more useful form.

In Sec. II, we review the multipole expansion and the difficulties with it. We then obtain an expansion for $F_{\lambda_1\lambda_2}{}^{JM}(p_{1R},p_2)$, where $p_{1R}=(m_1,\mathbf{0})$, in terms of invariant amplitudes. In Sec. III, we discuss the restrictions due to parity conservation. One of the important features of our expansion is that the form of these restrictions is very simple. In Sec. IV, we carry out the transformation to the center-of-mass amplitudes

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