"No-Interaction" Theorem in Classical Relativistic Mechanics

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(Received 1 July 1968)

A slight extension is given of the "no-interaction" theorem, as presented by Van Dam and Wigner, for a closed two-body system in classical relativistic mechanics. The proof here is simple and physical.

1. INTRODUCTION

IN this paper we extend slightly the "no-interaction" theorem of relativistic classical mechanics for a closed two-body system as presented by Van Dam and Wigner.[1] The proof given in Sec. 2 is appealing for its simplicity and physical basis. As Currie[2] has shown, the theorem only holds for space of dimensionality greater than 1. In Sec. 3, we present a one-dimensional model with interaction, the Hamiltonian for which, unlike Currie's, reduces to the conventional free form when the interaction vanishes. Section 4 contains a discussion.

2. ZERO INTERACTION

Relativistic Hamiltonian mechanics has been considered by a number of people. Currie's zero-interaction theorem[2] is based on the precepts that physical positions are canonical, that transformations of the inhomogeneous Lorentz group (IHLG) are canonical, and that world lines are manifestly invariant. Hill[3] has attempted to avoid the negative result for a canonical framework by not demanding that positions be canonical. The presentation we give is kinematical, as is also the approach of Van Dam and Wigner,[1] and a canonical framework is not assumed.

A minimal definition of a relativistic closed system is that the total energy E is conserved and the states form a basis for a physical realization of the IHLG. We also assume that E is the time translation generator for the system, so that the space translation generator, which we call the total momentum $\mathbf{P}$, is also conserved. We further make the usual but important assumption for a closed system that $\mathbf{P}$ is just the sum of the individual three-momenta of the particles and that the individual particle four-momenta transform as four-vectors. Finally, the condition is made that the orbits of the particles do not coincide except at a finite number of discrete points and that the motion is continuous; i.e., there are no impulses such as, for example, an elastic collision at a point.

Let the two particles at time t have four-momenta $p_1(t)$, $p_2(t)$ at the space positions $\mathbf{x}_1\equiv(x_1,y_1,z_1)$ and $\mathbf{x}_2\equiv(x_2,y_2,z_2)$, respectively. We now go to a primed frame of reference via a pure Lorentz transformation of arbitrary *infinitesimal* velocity $\mathbf{u}$. There is no loss of generality if we take $\mathbf{u}$ as being in the x direction.

In the primed frame of reference, the total three-momentum is $\mathbf{P}'(t_C')$ and the particle momenta are $p_1'(t_1')$, $p_2'(t_2')$, the times, which are in general all different, referring to time t in the unprimed frame. By hypothesis, we then have in both frames

$$\begin{aligned}\mathbf{P}(t)&=\mathbf{p}_1(t)+\mathbf{p}_2(t)\,,\\ \mathbf{P}'(t')&=\mathbf{p}_1'(t')+\mathbf{p}_2'(t')\,.\end{aligned}\tag{2.1}$$

Putting $c=1$, the momenta in each frame are by hypothesis related by

$$\begin{aligned}P_x'(t_C')&=\gamma[P_x(t)-uE(t)]\,,\\ E'(t_C')&=\gamma[E(t)-uP_x(t)]\,,\\ P_{y,z}'(t_C')&=P_{y,z}(t)\,,\end{aligned}\tag{2.2}$$

where

$$\gamma=(1-u^2)^{-1/2},\quad u=|\mathbf{u}|\,,$$

and similarly for the particles.

If $x_1\neq x_2$, the corresponding events in the primed frame are on a hyperplane with time not equal to a constant. Since u is infinitesimal, the time difference is also infinitesimal. We can then find the force $\mathbf{F}_1$ acting on particle 1:

$$\mathbf{F}_1=\frac{d\mathbf{p}_1'(t_1')}{dt_1'}\approx\frac{\mathbf{p}_1'(t_2')-\mathbf{p}_1'(t_1')}{(t_2'-t_1')}.$$

By three-momentum conservation and Eq. (2.1), we have

$$\mathbf{P}'(t_C')=\mathbf{P}'(t_2')=\mathbf{p}_1'(t_2')+\mathbf{p}_2'(t_2')\,.$$

Eliminating $\mathbf{p}_1'(t_2')$ gives us

$$\mathbf{F}_1=\frac{\mathbf{P}'(t_C')-\mathbf{p}_2'(t_2')-\mathbf{p}_1'(t_1')}{(t_2'-t_1')}.$$

Equations (2.2) then imply

$$F_{1x}=\frac{\gamma\{P_x(t)-p_{1x}(t)-p_{2x}(t)-u[E(t)-E_1(t)-E_2(t)]\}}{t_2'-t_1'},$$

$$F_{1y,z}=\frac{P_{y,z}(t)-p_{1y,z}(t)-p_{2y,z}(t)}{t_2'-t_1'}.$$

Thus, by Eq. (2.1),

$$\mathbf{F}_1=-\gamma\mathbf{u}V/(t_2'-t_1')\,,$$

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[1] H. Van Dam and E. P. Wigner, Phys. Rev. 142, 838 (1966).
[2] D. G. Currie, J. Math. Phys. 4, 1470 (1963).
[3] R. N. Hill, J. Math. Phys. 8, 1756 (1967).

where

$$V=E(t)-E_1(t)-E_2(t), \quad (2.3)$$

$E_{1,2}$ being just the energies of particles 1, 2. Now

$$t_{1,2}'=\gamma(t-\mathbf{u}\cdot\mathbf{x}_{1,2}), \quad (2.4)$$

so that we finally get

$$\mathbf{F}_1=-\mathbf{u}V/\mathbf{u}\cdot(\mathbf{x}_1-\mathbf{x}_2). \quad (2.5)$$

Since **u** is infinitesimal, this is the force in the original frame of reference. It evidently depends on the direction of **u**, which is arbitrary, and this is impossible. We must therefore conclude that there can be no force between the particles. This, of course, does not apply in the one-dimensional case where **u** has essentially only one possible direction.

This proof affords a slight extension of the theorem as presented by Van Dam and Wigner, since we do not need their asymptotic condition and we have used the more general conservation law of total energy.

3. ONE-DIMENSIONAL MODEL EXHIBITING INTERACTION

We consider here only the special case of conservative forces. Then, the force on particle 1 is

$$F_1=-\partial V/\partial x_1=-V/x, \quad (3.1)$$

where

$$x=x_1-x_2. \quad (3.2)$$

Upon integration, we have

$$\log|V|=\log|x|+\text{const}.$$

Thus

$$V=\xi|x|, \quad (3.3)$$

where ξ is a positive or negative constant, its sign corresponding to attraction or repulsion. The force is then

$$F_1=\pm\xi, \quad (3.4)$$

the $\pm$ sign here corresponding to $x\lessgtr 0$.

The orbital equation is found to be

$$x_1(t)=x_1(t_0)+\frac{1}{(\pm)\xi}\times\{[m_1^2+((\pm)\xi(t-t_0)+p_1(t_0))^2]^{1/2}-E_1(t_0)\}, \quad (3.5)$$

where the motion between t and t_0 does not involve the particles crossing. To find the position when such crossing occurs, we just add on the paths between the points of crossing.

Since the force in this model is a conservative one, it is simple to set up the usual Hamiltonian formalism. That is, the Hamiltonian H is just

$$H=(m_1^2+p_1^2)^{1/2}+(m_2^2+p_2^2)^{1/2}+V\equiv H_0+V. \quad (3.6)$$

To complete the canonical formalism, one easily finds the generator of pure Lorentz transformations to be

$$K=x_1E_1+x_2E_2+W\equiv K_0+W, \quad (3.7a)$$

with

$$W=\tfrac{1}{2}\xi|x_1^2-x_2^2|. \quad (3.7b)$$

Note added in proof. It is simple to see that this example also satisfies Currie's world-line conditions,[2] since the Poisson brackets of x_i with V and with W vanish for this case. However, it must be pointed out that the criteria of Sec. 2 for a one-dimensional model are most probably *not* equivalent to those of Currie[2] and of Hill,[3] since it is our expressed purpose to present the problem in a simple noncanonical framework. It might also be noted that the Poisson bracket here of x_1 and x_2 vanishes, and hence this particular example does not satisfy Hill's criterion mentioned in the Introduction. At present, though, we do not know if there is any ingredient in Sec. 2 which would imply that positions are canonical.

4. DISCUSSION

The one-dimensional model is, of course, only of academic interest. An interesting feature of the above model is that the force is not only conservative but constant and either always repulsive or always attractive.

We reiterate the three important conditions underlying the proof of Sec. 2.

(a) Conservation of total instantaneous linear momentum, i.e., the sum of the kinetmatical three-momenta.

(b) The total three-momentum **P** and the total energy E transform as a four-vector.

(c) The kinematical four-momentum of each particle transforms as a four-vector.

We note that, unlike Currie[2] and Hill,[3] we do not use a canonical formulation, nor do we need to consider the total angular momentum explicitly. Condition (b) does imply that the generator of pure Lorentz transformations **K** is such that (**P**,E) transform properly. Currie and Hill do not necessarily have condition (a), but (c) is equivalent to the world-line condition. The approach presented in this paper seems, however, to be closer to a quantum-mechanical one, in that the arguments have been centered more around momentum rather than position. Evidently, for interaction one or more of these conditions must be relaxed.

Recently, Currie,[4] Hill,[3,5] and Van Dam and Wigner[1,6] have constructed formulations for a relativistic dynamics with interaction. They are primarily concerned with the transformation or orbits and, at present, it is not clear how the ultimate and necessary transition

[4] D. G. Currie, Phys. Rev. **142**, 817 (1966).
[5] R. N. Hill, J. Math. Phys. **8**, 201 (1967).
[6] H. Van Dam and E. P. Wigner, Phys. Rev. **138**, B1576 (1965).

to quantum mechanics is to be made. There is already some confusion as to the meaning of "position" in relativistic quantum mechanics.[7] In their formulation, Van Dam and Wigner must introduce an interaction momentum, so that the total three-momentum is no longer the sum of the individual kinematical three-momenta of the particles. This is disturbing in that we would then no longer have the more exacting definition of a closed system as in nonrelativistic theory. Furthermore, it undermines the usual procedure in quantum mechanics of being able to write a Heisenberg bound state vector as simply a linear sum of product state vectors with total momentum equal to that of the bound state. If (a) is relaxed, we should, however, expect it to be satisfied in the nonrelativistic limit.

In relativistic quantum mechanics, it is, so far, condition (c) which is not adhered to.[8,9] This, at least, allows one to preserve the usual concept of a closed system and, in particular, of a bound state. Considering representations, it is possible to understand in a very *heuristic* manner why it might be reasonable to relax (c). Upon transforming a physical state, we expect to describe an irreducible representation of the IHLG. However, if we were to "go into" the system and transform each particle separately without regard to the other particles present, then we might expect to obtain rather a product of representations, which is then reducible. It is true that for a system without bound states the representation based on *all* states is equivalent to the direct product representation. But in that case, since all states are considered, we also have a continuum of masses for the c.m. system. However, this is not true if bound states are present.[9] If condition (c) is relaxed, we are giving up the world-line condition for world lines under interaction.

In quantum mechanics, the particles in the interaction region are off the energy shell or, equivalently, off the mass shell. This is simply the expression of the existence of interaction energy. If we introduce also an interaction momentum, then the question arises of what this can correspond to in quantum mechanics. Would it perhaps mean that particles in the interaction region are in some way "off the momentum shell"? Again, one must take care over the nonrelativistic limit. Another concept which is imcompatible with the introduction of interaction momentum is that, at least hypothetically, of being able to "switch off" the interaction at any time (i.e., on any spacelike hyperplane) without the system's suffering an impulse; that is, the motion remains continuous. A nonrelativistic example is that of a closed system of a charge entering two concentric spheres which are oppositely charged. The interaction may be cut off at any point by giving the inner sphere the appropriate size, and the usual conservation laws arrived at. Of course, this particular example cannot hold relativistically since the spheres need to be rigid.

To sum up: In searching for a relativistic dynamics, we are faced with giving up either condition (a), which would be disturbing to quantum mechanics, or (c), which would be contrary to our feeling for classical relativity. To the authors' knowledge, there does not seem to be any *direct* evidence for the world-line condition holding under interaction. In the example of a charged particle in a uniform field, the radiation at least is neglected.[10] Furthermore, if the path is observed in a cloud chamber, the path taken is really the trail of ions. Besides, one also ignores the effect of creating the ions. Then there is the gravitational force, but this belongs to the general theory of relativity, where the transformation group is indeed quite different. We do believe that the states of a closed system form a basis for a physical realization of the IHLG, since this is just a matter of the transformation of reference frames *in vacuo* and hence, determined by the kinematics of free systems. But it may perhaps not be reasonable, except as an approximation, to investigate the system in more detail just on the basis of special relativity.[11]

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[7] R. Fong and E. G. P. Rowe, Ann. Phys. (N.Y.) **46**, 559 (1968), especially Sec. V. Other references are given in this paper.

[8] L. H. Thomas, Rev. Mod. Phys. **17**, 182 (1945); B. Bakamjian and L. H. Thomas, Phys. Rev. **92**, 1300 (1953); L. L. Foldy, *ibid.* **122**, 275 (1961); F. Coester, Helv. Phys. Acta **38**, 7 (1965).

[9] R. Fong and J. Sucher, J. Math. Phys. **5**, 956 (1964).

[10] See, e.g., P. G. Bergmann, *Introduction to the Theory of Relativity* (Prentice-Hall, Inc., Englewood Cliffs, N. J., 1942), pp. 135–138.

[11] See also D. Bohm, *The Special Theory of Relativity* (W. A. Benjamin, Inc., New York, 1965), p. 109.