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Gauge Conditions in Gravitational Interactions*

BRUCE M. BARKER
Department of Physics, Louisiana State University, Baton Rouge, Louisiana 70803

AND

MANJIT S. BHATIA
Department of Physics, University of Windsor, Windsor, Ontario, Canada

AND

SURAJ N. GUPTA
Department of Physics, Wayne State University, Detroit, Michigan 48202

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The requirement of asymptotic gauge invariance in the scattering of a graviton by a spinless particle and in the annihilation of two spinless particles into two gravitons is analyzed, and it is shown that the necessary gauge conditions are satisfied by the scattering matrix elements for these processes. The scattering and annihilation cross sections for various graviton polarization states are also obtained.

1. INTRODUCTION

THE quantum theory of gravitation in flat space[1] has several features similar to those of quantum electrodynamics; in particular both theories share the property of invariance under gauge transformations. Some consequences of gauge invariance in S-matrix theory have been discussed by Weinberg,[2] and recently the gauge invariance in the scattering of a graviton by a spinless particle has been examined by Jackiw.[3] We shall, however, show that, by imposing only the requirement of asymptotic gauge invariance, we obtain weaker gauge conditions than those used by Jackiw, and we shall verify our gauge conditions by investigating the scattering of a graviton by a spinless particle and the annihilation of two spinless particles into two gravitons. The cross sections for these processes will also be obtained and compared with the earlier results.[3–7]

We shall essentially confine ourselves to the consideration of asymptotic gauge invariance. An examination of the unitary condition for the S matrix in the annihilation of two particles into two gravitons reveals a peculiar difficulty observed by Feynman,[8] but we shall not discuss this aspect of the annihilation process here.

According to the Lorentz-covariant expansion procedure,[9] we can express the gravitational coupling terms for neutral spinless particles as

$$L_{\text{int}}=-\tfrac{1}{2}\kappa:h_{\mu\nu}[(\partial_\mu U_0)(\partial_\nu U_0)-\tfrac{1}{2}\delta_{\mu\nu}(\partial_\rho U_0)(\partial_\rho U_0)-\tfrac{1}{2}\delta_{\mu\nu}\mu^2 U_0U_0]:-\tfrac{1}{16}\kappa^2\mu^2\times:(h_{\mu\mu}h_{\nu\nu}-2h_{\mu\nu}h_{\mu\nu})U_0U_0:+O(\kappa^3), \quad (1)$$

and for charged spinless particles as

$$L_{\text{int}}=-\kappa:h_{\mu\nu}[(\partial_\mu U^*)(\partial_\nu U)-\tfrac{1}{2}\delta_{\mu\nu}(\partial_\rho U^*)(\partial_\rho U)-\tfrac{1}{2}\delta_{\mu\nu}\mu^2U^*U]:-\tfrac{1}{8}\kappa^2\mu^2:(h_{\mu\mu}h_{\nu\nu}-2h_{\mu\nu}h_{\mu\nu})U^*U:+O(\kappa^3), \quad (2)$$

while the graviton-graviton coupling terms are

$$L_{\text{int}}=-\tfrac{1}{2}\kappa:(h_{\mu\nu}-\tfrac{1}{2}\delta_{\mu\nu}h_{\rho\rho})[\tfrac{1}{2}(\partial_\mu h_{\alpha\beta})(\partial_\nu h_{\alpha\beta})-(\partial_\nu h_{\mu\alpha})(\partial_\alpha h_{\lambda\lambda})+(\partial_\beta h_{\mu\alpha})(\partial_\alpha h_{\nu\beta})+\tfrac{1}{2}(\partial_\alpha h_{\lambda\lambda})(\partial_\alpha h_{\mu\nu})-(\partial_\alpha h_{\mu\beta})(\partial_\alpha h_{\nu\beta})]:+O(\kappa^2). \quad (3)$$

The above coupling terms are given as ordered products, in which U_0 and U represent the field operators for neutral and charged spinless particles, and $h_{\mu\nu}$ represents the field operator for gravitons. The coupling constant κ is related to Newton's constant of gravitation G as $\kappa^2=16\pi G/c^4$, while μ is related to the spinless particle mass m as $\mu=mc/\hbar$.

In the initial and final states the contribution of the *observable* gravitons to the scattering operator in the interaction picture is obtained by means of the Fourier decomposition

$$h_{\mu\nu}=V^{-1/2}\sum_{\mathbf{k}}\left(\frac{c\hbar}{2k_0}\right)^{1/2}[a_{\mu\nu}(\mathbf{k})e^{i(\mathbf{k}\cdot\mathbf{r}-k_0x_0)}+a_{\mu\nu}^*(\mathbf{k})e^{-i(\mathbf{k}\cdot\mathbf{r}-k_0x_0)}], \quad (4)$$

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[8] R. P. Feynman, Acta Phys. Polon. **24**, 697 (1963). The unitary condition leads to the necessity of introducing "fictitious quanta." However, the fictitious quanta do not affect the S-matrix elements for observable gravitons as long as we deal only with the tree diagrams, and therefore the cross sections derived in our paper will remain unaffected.

[9] The Lorentz-covariant expansion procedure for the Lagrangian density was originally introduced in Ref. 1, and we have followed the procedure given there. However, we have verified that a slightly different expansion procedure, based on $g_{\mu\nu}=\eta_{\mu\nu}+\kappa h_{\mu\nu}$, leads to the same S-matrix elements for the processes considered in this paper.

with

$$k_0=|\mathbf{k}|\,, \tag{5}$$

$$a_{\mu\nu}(\mathbf{k})=e_{\mu\nu,+}(\mathbf{k})a_+(\mathbf{k})+e_{\mu\nu,-}(\mathbf{k})a_-(\mathbf{k})\,, \tag{6}$$

$$e_{\mu\nu,\pm}(\mathbf{k})=2^{-1/2}\{[e_\mu{}^{(1)}(\mathbf{k})e_\nu{}^{(1)}(\mathbf{k})-e_\mu{}^{(2)}(\mathbf{k})e_\nu{}^{(2)}(\mathbf{k})] \pm i[e_\mu{}^{(1)}(\mathbf{k})e_\nu{}^{(2)}(\mathbf{k})+e_\mu{}^{(2)}(\mathbf{k})e_\nu{}^{(1)}(\mathbf{k})]\}\,, \tag{7}$$

where $a_+(\mathbf{k})$ and $a_-(\mathbf{k})$ are annihilation operators for gravitons with their spin axes parallel and antiparallel to $\mathbf{k}$, while $\mathbf{e}^{(1)}(\mathbf{k})$ and $\mathbf{e}^{(2)}(\mathbf{k})$ are unit vectors such that $\mathbf{k}$, $\mathbf{e}^{(1)}(\mathbf{k})$, and $\mathbf{e}^{(2)}(\mathbf{k})$ are mutually perpendicular, and $e_4{}^{(1)}=e_4{}^{(2)}=0$. In the intermediate states the contribution of the *observable as well as nonobservable* gravitons to the scattering operator is obtained by means of the contraction

$$h^{\cdot}_{\mu\nu}(x)h^{\cdot}_{\lambda\rho}(x')=-ic\hbar(\delta_{\mu\lambda}\delta_{\nu\rho}+\delta_{\mu\rho}\delta_{\nu\lambda}-\delta_{\mu\nu}\delta_{\lambda\rho})\times D_F(x-x')\,, \tag{8}$$

with

$$D_F(x-x')=\lim_{\epsilon\to+0}\frac{1}{(2\pi)^4}\int dk\, e^{ik\cdot(x-x')}\frac{1}{k^2-i\epsilon}. \tag{9}$$

We shall denote the space-time coordinates as $x_\mu=(x_i,ix_0)$, while an asterisk will be used to denote the complex conjugate of a number or the Hermitian conjugate of an operator.

2. GAUGE CONDITIONS FOR SCATTERING MATRIX ELEMENTS

Let the scattering operator for a process involving an arbitrary number of observable gravitons in the initial and final states be given by

$$S=M_{\mu\nu,\lambda\rho,\cdots,\alpha\beta,\gamma\delta,\cdots}a_{\mu\nu}{}^*(\mathbf{q})a_{\lambda\rho}{}^*(\mathbf{q}')\cdots a_{\alpha\beta}(\mathbf{p})a_{\gamma\delta}(\mathbf{p}')\cdots\,, \tag{10}$$

where M is symmetrical with respect to the indices μ and ν as well as other similar pairs of indices, while $a_{\mu\nu}{}^*(\mathbf{q}),\cdots$ and $a_{\alpha\beta}(\mathbf{p}),\cdots$, which can be expressed in the form (6), satisfy the relations

$$q_\nu a_{\mu\nu}{}^*(\mathbf{q})=0\,,\ a_{\mu\mu}{}^*(\mathbf{q})=0\,,\ p_\beta a_{\alpha\beta}(\mathbf{p})=0\,,\ a_{\alpha\alpha}(\mathbf{p})=0\,, \tag{11}$$

and we have suppressed creation and annihilation operators for particles other than gravitons. Asymptotic gauge invariance requires that (10) be invariant, when it is subjected to transformations of the form[10]

$$\begin{aligned} a_{\mu\nu}{}^*(\mathbf{q}) &\to a_{\mu\nu}{}^*(\mathbf{q})-iq_\mu\lambda_\nu{}^*(\mathbf{q})-iq_\nu\lambda_\mu{}^*(\mathbf{q})\,,\\ a_{\alpha\beta}(\mathbf{p}) &\to a_{\alpha\beta}(\mathbf{p})+ip_\alpha\lambda_\beta(\mathbf{p})+ip_\beta\lambda_\alpha(\mathbf{p}).\end{aligned} \tag{12}$$

In order to derive the gauge conditions imposed by the above requirement, we first consider the simple case

$$S=M_{\mu\nu}a_{\mu\nu}{}^*(\mathbf{k})\,, \tag{13}$$

where invariance under (12) evidently leads to the condition

$$k_\nu M_{\mu\nu}=0. \tag{14}$$

When S is of the form $M_{\mu\nu}a_{\mu\nu}(\mathbf{k})$, we again obtain the condition (14).

Let us next consider the case

$$S=M_{\mu\nu,\lambda\rho}a_{\mu\nu}{}^*(\mathbf{k}')a_{\lambda\rho}(\mathbf{k})\,, \tag{15}$$

which transforms under (12) as

$$\begin{aligned} S\to\ & M_{\mu\nu,\lambda\rho}a_{\mu\nu}{}^*(\mathbf{k}')a_{\lambda\rho}(\mathbf{k})\\ &-iM_{\mu\nu,\lambda\rho}[k_\mu'\lambda_\nu{}^*(\mathbf{k}')+k_\nu'\lambda_\mu{}^*(\mathbf{k}')]a_{\lambda\rho}(\mathbf{k})\\ &+iM_{\mu\nu,\lambda\rho}a_{\mu\nu}{}^*(\mathbf{k}')[k_\lambda\lambda_\rho(\mathbf{k})+k_\rho\lambda_\lambda(\mathbf{k})]\\ &+M_{\mu\nu,\lambda\rho}[k_\mu'\lambda_\nu{}^*(\mathbf{k}')+k_\nu'\lambda_\mu{}^*(\mathbf{k}')]\\ &\times[k_\lambda\lambda_\rho(\mathbf{k})+k_\rho\lambda_\lambda(\mathbf{k})].\end{aligned} \tag{16}$$

The additional terms in (16) vanish, if

$$k_\nu'M_{\mu\nu,\lambda\rho}=0\,,\quad k_\rho M_{\mu\nu,\lambda\rho}=0. \tag{17}$$

However, in view of (11), the additional terms also vanish, if $k_\nu'M_{\mu\nu,\lambda\rho}$ and $k_\rho M_{\mu\nu,\lambda\rho}$ can be expressed in terms of some quantities $f_{\mu\lambda}$ and $f_{\mu\lambda}'$ as

$$\begin{aligned} k_\nu'M_{\mu\nu,\lambda\rho}&=k_\lambda f_{\rho\mu}+k_\rho f_{\lambda\mu}-\delta_{\lambda\rho}k_\sigma f_{\sigma\mu}\,,\\ k_\rho M_{\mu\nu,\lambda\rho}&=k_\mu'f_{\nu\lambda}'+k_\nu'f_{\mu\lambda}'-\delta_{\mu\nu}k_\sigma'f_{\sigma\lambda}'.\end{aligned} \tag{18}$$

The conditions (18) are again sufficient, when S is of the form $M_{\mu\nu,\lambda\rho}a_{\mu\nu}{}^*(\mathbf{k}')a_{\lambda\rho}{}^*(\mathbf{k})$ or $M_{\mu\nu,\lambda\rho}a_{\mu\nu}(\mathbf{k}')a_{\lambda\rho}(\mathbf{k})$. Thus the relations (18), which are evidently weaker than (17), represent the gauge conditions.

By generalizing the above argument, it is found that if the initial and final states contain n observable gravitons, so that

$$S=M_{\mu_1\nu_1,\cdots,\mu_n\nu_n}a_{\mu_1\nu_1}{}^*(\mathbf{k}^{(1)})\cdots a_{\mu_n\nu_n}(\mathbf{k}^{(n)})\,, \tag{19}$$

then the gauge conditions are given by

$$k_{\nu_p}{}^{(p)}M_{\mu_1\nu_1,\cdots,\mu_n\nu_n}=\sum_{l\neq p}(k_{\mu_l}{}^{(l)}f_{\nu_l\mu_p\cdots}{}^{(l,p)}+k_{\nu_l}{}^{(l)}f_{\mu_l\mu_p\cdots}{}^{(l,p)}-\delta_{\mu_l\nu_l}k_{\lambda_l}{}^{(l)}f_{\lambda_l\mu_p\cdots}{}^{(l,p)})\,, \tag{20}$$

where the indices of f represented by dots can be obtained from the indices of M by dropping μ_l, ν_l, μ_p, and ν_p.

3. GRAVITON SCATTERING BY SPINLESS PARTICLE

The diagrams for the scattering of a graviton by a spinless particle are shown in Fig. 1, where the propagation four-vectors[11] of the graviton and the particle are denoted by k and p in the initial state and by k' and p' in the final state. The scattering operator for

[10] The asymptotic gauge transformation is given by

$$h_{\mu\nu}\to h_{\mu\nu}+\partial_\mu\Lambda_\nu+\partial_\nu\Lambda_\mu,$$

where $h_{\mu\nu}$ is the free-field operator for any observable graviton in the initial or final state, and Λ_μ is an arbitrary massless free-field operator.

[11] The propagation four-vector p_μ of a free particle of mass m satisfies the relation $p_\mu{}^2=-\mu^2$ with $\mu=mc/\hbar$, and the momentum and energy of the particle are $\hbar\mathbf{p}$ and $c\hbar p_0$, respectively.

these diagrams can be expressed as

$$S=iV^{-2}(2\pi)^4\delta(p+k-p'-k')(p_0p_0'k_0k_0')^{-1/2}(\tfrac{1}{4}c\hbar\kappa^2)\times(A_{\mu\nu,\lambda\rho}+A_{\mu\nu,\lambda\rho}'+B_{\mu\nu,\lambda\rho}+C_{\mu\nu,\lambda\rho})\times a_{\mu\nu}^*(\mathbf{k}')a_{\lambda\rho}(\mathbf{k})a^*(\mathbf{p}')a(\mathbf{p}), \quad (21)$$

where $a(\mathbf{p})$ and $a^*(\mathbf{p}')$ are the annihilation and creation operators for the spinless particle, while $A_{\mu\nu,\lambda\rho}$, $A_{\mu\nu,\lambda\rho}'$, $B_{\mu\nu,\lambda\rho}$, and $C_{\mu\nu,\lambda\rho}$, which represent the contributions of the diagrams (a), (a′), (b), and (c), respectively, are given by

$$A_{\mu\nu,\lambda\rho}=[(k'+p')^2+\mu^2]^{-1}\times[p_\mu'p_\nu'+\tfrac{1}{2}(p_\mu'k_\nu'+k_\mu'p_\nu')-\tfrac{1}{2}\delta_{\mu\nu}(k'\cdot p')]\times[p_\lambda p_\rho+\tfrac{1}{2}(p_\lambda k_\rho+k_\lambda p_\rho)-\tfrac{1}{2}\delta_{\lambda\rho}(k\cdot p)], \quad (22)$$

$$A_{\mu\nu,\lambda\rho}'=[(p'-k)^2+\mu^2]^{-1}\times[p_\mu p_\nu-\tfrac{1}{2}(p_\mu k_\nu'+k_\mu'p_\nu)+\tfrac{1}{2}\delta_{\mu\nu}(k'\cdot p)]\times[p_\lambda'p_\rho'-\tfrac{1}{2}(p_\lambda'k_\rho+k_\lambda p_\rho')+\tfrac{1}{2}\delta_{\lambda\rho}(k\cdot p')], \quad (23)$$

$$B_{\mu\nu,\lambda\rho}=[(k'-k)^2]^{-1}(2p_\kappa'p_\sigma+\delta_{\kappa\sigma}\mu^2)\times[-k_\alpha(k_\beta'-k_\beta)\delta_{\mu\nu,\lambda\rho,\kappa\sigma,\alpha\beta}+k_\alpha'(k_\beta'-k_\beta)\times\delta_{\lambda\rho,\mu\nu,\kappa\sigma,\alpha\beta}+k_\alpha'k_\beta\delta_{\kappa\sigma,\mu\nu,\lambda\rho,\alpha\beta}], \quad (24)$$

$$C_{\mu\nu,\lambda\rho}=\tfrac{1}{4}\mu^2(\delta_{\mu\lambda}\delta_{\nu\rho}+\delta_{\mu\rho}\delta_{\nu\lambda}-\delta_{\mu\nu}\delta_{\lambda\rho}), \quad (25)$$

with[12]

$$\begin{aligned}\delta_{\mu\nu,\lambda\rho,\kappa\sigma,\alpha\beta}=\mathrm{sym}[&\tfrac{1}{2}\delta_{\mu\alpha}\delta_{\nu\beta}\delta_{\lambda\kappa}\delta_{\rho\sigma}-\tfrac{1}{2}\delta_{\mu\lambda}\delta_{\nu\alpha}\delta_{\rho\beta}\delta_{\kappa\sigma}\\&-\tfrac{1}{2}\delta_{\mu\kappa}\delta_{\nu\beta}\delta_{\sigma\alpha}\delta_{\lambda\rho}+\delta_{\mu\lambda}\delta_{\nu\kappa}\delta_{\rho\beta}\delta_{\sigma\alpha}+\tfrac{1}{4}\delta_{\mu\kappa}\delta_{\nu\sigma}\delta_{\lambda\rho}\delta_{\alpha\beta}\\&+\tfrac{1}{4}\delta_{\mu\lambda}\delta_{\nu\rho}\delta_{\kappa\sigma}\delta_{\alpha\beta}-\delta_{\mu\lambda}\delta_{\nu\kappa}\delta_{\rho\sigma}\delta_{\alpha\beta}+\tfrac{1}{4}\delta_{\mu\nu}\delta_{\lambda\kappa}\delta_{\rho\sigma}\delta_{\alpha\beta}\\&+\tfrac{1}{4}\delta_{\mu\nu}\delta_{\lambda\beta}\delta_{\kappa\sigma}\delta_{\rho\alpha}+\tfrac{1}{4}\delta_{\mu\nu}\delta_{\sigma\beta}\delta_{\kappa\alpha}\delta_{\lambda\rho}-\tfrac{1}{2}\delta_{\mu\nu}\delta_{\lambda\kappa}\delta_{\rho\beta}\delta_{\sigma\alpha}\\&-\tfrac{1}{4}\delta_{\mu\nu}\delta_{\lambda\rho}\delta_{\kappa\sigma}\delta_{\alpha\beta}]. \quad (26)\end{aligned}$$

It is understood that the quantity within the square brackets in (26) is to be symmetrized with respect to the pair indices $\mu\nu$, $\lambda\rho$, and $\kappa\sigma$.

After lengthy calculations and considerable simplification, we find that

$$k_\nu'A_{\mu\nu,\lambda\rho}=\tfrac{1}{4}p_\mu'[2p_\lambda p_\rho+p_\lambda k_\rho+k_\lambda p_\rho-\delta_{\lambda\rho}(k\cdot p)],$$

$$k_\nu'A_{\mu\nu,\lambda\rho}'=\tfrac{1}{4}p_\mu[-2p_\lambda'p_\rho'+p_\lambda'k_\rho+k_\lambda p_\rho'-\delta_{\lambda\rho}(k\cdot p')],$$

$$\begin{aligned}k_\nu'B_{\mu\nu,\lambda\rho}=&\tfrac{1}{4}p_\mu'[-2p_\lambda p_\rho-p_\lambda k_\rho-k_\lambda p_\rho+\delta_{\lambda\rho}(k\cdot p)]\\&+\tfrac{1}{4}p_\mu[2p_\lambda'p_\rho'-p_\lambda'k_\rho-k_\lambda p_\rho'+\delta_{\lambda\rho}(k\cdot p')]\\&-\tfrac{1}{4}\mu^2(\delta_{\mu\lambda}k_\rho'+\delta_{\mu\rho}k_\lambda'-\delta_{\lambda\rho}k_\mu')\\&-\tfrac{1}{4}(\delta_{\mu\lambda}k_\rho+\delta_{\mu\rho}k_\lambda-\delta_{\lambda\rho}k_\mu)(k\cdot p)(k'\cdot p)/(k'\cdot k),\end{aligned}$$

$$k_\nu'C_{\mu\nu,\lambda\rho}=-\tfrac{1}{4}\mu^2(-\delta_{\mu\lambda}k_\rho'-\delta_{\mu\rho}k_\lambda'+\delta_{\lambda\rho}k_\mu'),$$

so that

$$k_\nu'(A_{\mu\nu,\lambda\rho}+A_{\mu\nu,\lambda\rho}'+B_{\mu\nu,\lambda\rho}+C_{\mu\nu,\lambda\rho})=k_\lambda f_{\rho\mu}+k_\rho f_{\lambda\mu}-\delta_{\lambda\rho}k_\sigma f_{\sigma\mu}, \quad (27)$$

where

$$f_{\rho\mu}=-\tfrac{1}{4}\delta_{\rho\mu}(k\cdot p)(k'\cdot p)/(k'\cdot k). \quad (28)$$

The relation (27) shows that the scattering operator (21) satisfies the first gauge condition in (18); similarly it can be shown that the second gauge condition in (18) is also satisfied.

[12] The definition (26) enables us to express the graviton-graviton coupling (3) as $-\frac{1}{2}\kappa\delta_{\mu\nu,\lambda\rho,\kappa\sigma,\alpha\beta}:h_{\mu\nu}(\partial_\alpha h_{\lambda\rho})(\partial_\beta h_{\kappa\sigma}):$.

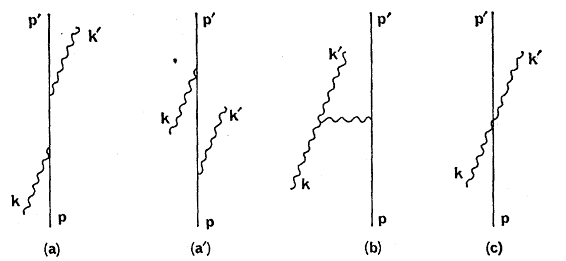

FIG. 1. Graviton scattering by a spinless particle. A wavy line represents a graviton.

In order to derive the scattering cross section, we shall use the laboratory system, so that

$$\mathbf{p}=0, \quad p_0=\mu, \quad \mathbf{k}=\mathbf{k}'+\mathbf{p}', \quad \mu+k_0=k_0'+p_0', \quad (29)$$

and for convenience we shall choose the polarization vectors associated with the gravitons such that

$$\mathbf{e}^{(1)}(\mathbf{k}')=\mathbf{e}^{(1)}(\mathbf{k})=\frac{\mathbf{k}'\times\mathbf{k}}{|\mathbf{k}'\times\mathbf{k}|}, \quad \mathbf{e}^{(2)}(\mathbf{k}')=\frac{\mathbf{k}'\times\mathbf{e}^{(1)}(\mathbf{k}')}{|\mathbf{k}'|}, \quad \mathbf{e}^{(2)}(\mathbf{k})=\frac{\mathbf{k}\times\mathbf{e}^{(1)}(\mathbf{k})}{|\mathbf{k}|}. \quad (30)$$

It is then easy to see that the contributions of the diagrams (a) and (a′) vanish and the contributions of the remaining diagrams take a much simpler form, so that (21) reduces to

$$S=iV^{-2}(2\pi)^4\delta(p+k-p'-k')(\mu p_0'k_0k_0')^{-1/2}(\tfrac{1}{4}c\hbar\kappa^2)\times[\tfrac{1}{2}\mu k_0k_0'/(k_0-k_0')]a_{ij}^*(\mathbf{k}')a_{ij}(\mathbf{k})a^*(\mathbf{p}')a(\mathbf{p})$$

or

$$\begin{aligned}S=iV^{-2}(2\pi)^4\delta(p+k-p'-k')(\tfrac{1}{16}c\hbar\kappa^2)&\times[(\mu k_0k_0'/p_0')^{1/2}/(k_0-k_0')]\{[1+(\mathbf{k}\cdot\mathbf{k}')/k_0k_0']^2\\&\times[a_+^*(\mathbf{k}')a_+(\mathbf{k})+a_-^*(\mathbf{k}')a_-(\mathbf{k})]\\&+[1-(\mathbf{k}\cdot\mathbf{k}')/k_0k_0']^2[a_+^*(\mathbf{k}')a_-(\mathbf{k})\\&+a_-^*(\mathbf{k}')a_+(\mathbf{k})]\}a^*(\mathbf{p}')a(\mathbf{p}). \quad (31)\end{aligned}$$

The resulting scattering cross sections for various polarization states of the gravitons are

$$\frac{d\sigma_{++}}{d\Omega}=\frac{d\sigma_{--}}{d\Omega}=\frac{G^2m^2}{c^4}\frac{\cot^4(\frac{1}{2}\theta)\cos^4(\frac{1}{2}\theta)}{[1+2\epsilon\sin^2(\frac{1}{2}\theta)]^2}, \quad \frac{d\sigma_{+-}}{d\Omega}=\frac{d\sigma_{-+}}{d\Omega}=\frac{G^2m^2}{c^4}\frac{\sin^4(\frac{1}{2}\theta)}{[1+2\epsilon\sin^2(\frac{1}{2}\theta)]^2}, \quad (32)$$

where ϵ is the incident graviton energy in units of mc^2 and θ is the graviton scattering angle. Thus, on averaging over the initial polarization states and summing over the final ones,

$$\frac{d\sigma}{d\Omega}=\frac{G^2m^2}{c^4}\frac{\cot^4(\frac{1}{2}\theta)\cos^4(\frac{1}{2}\theta)+\sin^4(\frac{1}{2}\theta)}{[1+2\epsilon\sin^2(\frac{1}{2}\theta)]^2}, \quad (33)$$

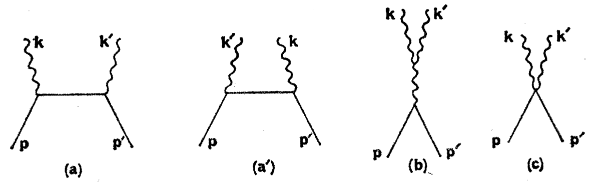

FIG. 2. Annihilation of two neutral or oppositely charged spinless particles into two gravitons.

which reduces in the nonrelativistic approximation to

$$\frac{d\sigma}{d\Omega}=\frac{G^2m^2}{c^4}[\cot^4(\tfrac{1}{2}\theta)\cos^4(\tfrac{1}{2}\theta)+\sin^4(\tfrac{1}{2}\theta)]. \quad (34)$$

The nonrelativistic cross section (34) agrees with all the earlier results,[3–5] but the general form (33) disagrees with the result given by DeWitt.[5] The amplitude for graviton-particle scattering has also been given by Chester[13] without deriving the cross section for this process.

4. ANNIHILATION OF SPINLESS PARTICLES INTO GRAVITONS

The treatment of Sec. 3 can be extended to the annihilation of two neutral or oppositely charged spinless particles with propagation four-vectors p and p' into two gravitons with propagation four-vectors k and k'. The diagrams for this process are shown in Fig. 2, and the scattering operator is given by

$$\begin{aligned}S=iV^{-2}(2\pi)^4\delta(p+p'-k-k')(p_0p_0'k_0k_0')^{-1/2}(\tfrac{1}{4}c\hbar\kappa^2)\\ \times(\bar{A}_{\mu\nu,\lambda\rho}+\bar{A}_{\mu\nu,\lambda\rho}'+\bar{B}_{\mu\nu,\lambda\rho}+\bar{C}_{\mu\nu,\lambda\rho})\\ \times a_{\mu\nu}^*(\mathbf{k}')a_{\lambda\rho}^*(\mathbf{k})a(\mathbf{p}')a(\mathbf{p}), \quad (35)\end{aligned}$$

where, as is well known, $\bar{A}_{\mu\nu,\lambda\rho}$, $\bar{A}_{\mu\nu,\lambda\rho}'$, $\bar{B}_{\mu\nu,\lambda\rho}$, and $\bar{C}_{\mu\nu,\lambda\rho}$ can be obtained by replacing k and p' by $-k$ and $-p'$ in the corresponding quantities of Sec. 3. It is then easy to see that

$$\begin{aligned}k_\nu'(\bar{A}_{\mu\nu,\lambda\rho}+\bar{A}_{\mu\nu,\lambda\rho}'+\bar{B}_{\mu\nu,\lambda\rho}+\bar{C}_{\mu\nu,\lambda\rho})\\ =k_\lambda\bar{f}_{\rho\mu}+k_\rho\bar{f}_{\lambda\mu}-\delta_{\lambda\rho}k_\sigma\bar{f}_{\sigma\mu}, \quad (36)\end{aligned}$$

where

$$\bar{f}_{\rho\mu}=\tfrac{1}{4}\delta_{\rho\mu}(k\cdot p)(k'\cdot p)/(k'\cdot k), \quad (37)$$

so that the gauge conditions (18) are satisfied here in the same manner as in the scattering process.

It is convenient to use the c.m. system for the derivation of the annihilation cross section, so that we have

$$\mathbf{p}'=-\mathbf{p}, \quad \mathbf{k}'=-\mathbf{k}, \quad p_0'=p_0=k_0'=k_0, \quad (38)$$

and we choose the polarization vectors associated with the two gravitons as

$$\begin{aligned}\mathbf{e}^{(1)}(\mathbf{k}')&=\frac{\mathbf{k}'\times\mathbf{p}'}{|\mathbf{k}'\times\mathbf{p}'|}, & \mathbf{e}^{(1)}(\mathbf{k})&=\frac{\mathbf{k}\times\mathbf{p}}{|\mathbf{k}\times\mathbf{p}|}=\mathbf{e}^{(1)}(\mathbf{k}'),\\ \mathbf{e}^{(2)}(\mathbf{k}')&=\frac{\mathbf{k}'\times\mathbf{e}^{(1)}(\mathbf{k}')}{|\mathbf{k}'|}, & \mathbf{e}^{(2)}(\mathbf{k})&=\frac{\mathbf{k}\times\mathbf{e}^{(1)}(\mathbf{k})}{|\mathbf{k}|}=-\mathbf{e}^{(2)}(\mathbf{k}').\end{aligned} \quad (39)$$

[13] A. N. Chester, Phys. Rev. 143, 1275 (1966).

The scattering operator (35) can then be expressed as

$$\begin{aligned}S=-iV^{-2}(2\pi)^4\delta(p+p'-k-k')a(\mathbf{p}')a(\mathbf{p})\\ \times\{[a_+^*(\mathbf{k}')a_+^*(\mathbf{k})+a_-^*(\mathbf{k}')a_-(\mathbf{k})](F_a+F_a'+F_b+F_c)\\ +[a_+^*(\mathbf{k}')a_-^*(\mathbf{k})+a_-^*(\mathbf{k}')a_+(\mathbf{k})](F_a+F_a')\}, \quad (40)\end{aligned}$$

where

$$\begin{aligned}F_a&=-\frac{\kappa^2c\hbar}{128}\frac{[\mu^4-(\mathbf{p}-\mathbf{k})^2(\mathbf{p}+\mathbf{k})^2]^2}{p_0^6[\mu^2+(\mathbf{p}-\mathbf{k})^2]},\\ F_a'&=-\frac{\kappa^2c\hbar}{128}\frac{[\mu^4-(\mathbf{p}+\mathbf{k})^2(\mathbf{p}-\mathbf{k})^2]^2}{p_0^6[\mu^2+(\mathbf{p}+\mathbf{k})^2]}, \quad (41)\\ F_b&=-\tfrac{1}{32}\kappa^2c\hbar[\mu^2(\mu^2-4p_0^2)-(\mathbf{p}-\mathbf{k})^2(\mathbf{p}+\mathbf{k})^2]/p_0^4,\\ F_c&=-\tfrac{1}{4}\kappa^2c\hbar(\mu^2/p_0^2).\end{aligned}$$

The resulting cross sections for various polarization states of the two gravitons are

$$\begin{aligned}d\sigma_{++}/d\Omega=d\sigma_{--}/d\Omega=(G^2E^2/4c^8\beta)[1-\beta^2-\beta^2\sin^2\theta\\ +\beta^4\sin^4\theta/(1-\beta^2\cos^2\theta)]^2, \quad (42)\end{aligned}$$

$$\begin{aligned}d\sigma_{+-}/d\Omega=d\sigma_{-+}/d\Omega=(G^2E^2/4c^8\beta)\\ \times[\beta^4\sin^4\theta/(1-\beta^2\cos^2\theta)]^2, \quad (43)\end{aligned}$$

with

$$E=c\hbar p_0, \quad \beta=|\mathbf{p}|/p_0=v/c, \quad (44)$$

where E is the particle energy, v is its velocity, and θ is the angle between $\mathbf{k}$ and $\mathbf{p}$.

The first and second subscripts of σ in (42) and (43) denote the polarization states of gravitons with the propagation vectors $\mathbf{k}$ and $\mathbf{k}'$, respectively. The total cross section can be obtained by integrating both $d\sigma_{++}$ and $d\sigma_{--}$ over θ from 0 to $\frac{1}{2}\pi$ and integrating either $d\theta_{+-}$ or $d\theta_{-+}$ over θ from 0 to π. Thus,

$$\begin{aligned}\sigma_{++}=\sigma_{--}=(\pi G^2E^2/4c^8\beta^2)\{\beta-3\beta^3+3\beta^5-\beta^7\\ +(\tfrac{1}{2}-2\beta^2+3\beta^4-2\beta^6+\tfrac{1}{2}\beta^8)\ln[(1+\beta)/(1-\beta)]\}, \quad (45)\end{aligned}$$

$$\begin{aligned}\sigma_{+-}=(\pi G^2E^2/2c^8\beta^2)\{7\beta-(53/3)\beta^3+(191/15)\beta^5-\beta^7\\ +(-\tfrac{7}{2}+10\beta^2-9\beta^4+2\beta^6+\tfrac{1}{2}\beta^8)\\ \times\ln[(1+\beta)/(1-\beta)]\}, \quad (46)\end{aligned}$$

and the total cross section is given by

$$\sigma=\sigma_{++}+\sigma_{--}+\sigma_{+-}. \quad (47)$$

In the nonrelativistic approximation the cross section reduces to

$$\sigma=\pi G^2m^2/c^4\beta, \quad (48)$$

while in the extreme relativistic approximation

$$\sigma=8\pi G^2E^2/15c^8. \quad (49)$$

These results are qualitatively in agreement with the earlier estimates of Wheeler[6] and of Ivanenko,[7] but the numerical factors in (48) and (49) disagree with the field-theoretical results given by DeWitt.[5]