Derivation of the Blackbody Radiation Spectrum without Quantum Assumptions

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The Planck radiation law for the blackbody radiation spectrum is derived without the formalism of quantum theory. The hypotheses assume (a) the existence, at the absolute zero of temperature, of classical homogeneous fluctuating electromagnetic radiation with a Lorentz-invariant spectrum; (b) that classical electrodynamics holds for a dipole oscillator; (c) that a free particle in equilibrium with blackbody radiation has the classical mean kinetic energy $\frac{1}{2}kT$ per degree of freedom. The Lorentz invariance of the spectrum of zero-temperature radiation is used to derive the zero-point electromagnetic energy-density spectrum, found to be linear in frequency, $\frac{1}{2}\hbar\omega$ per normal mode. The procedures based on classical theory employed by Einstein and Hopf, which were formerly regarded as giving a rigorous derivation of the Rayleigh-Jeans radiation law, are modified and corrected for electromagnetic zero-point energy to allow a rigorous derivation of the full blackbody spectrum from classical theory without any assumptions of discrete or discontinuous processes.

I. MOTIVATION FOR A NEW DERIVATION OF PLANCK'S LAW

THE theoretical derivation of the blackbody energy spectrum for thermal radiation looms large in the history of physics, not for its own interest, but because of its tremendous impact in demonstrating the failure of classical theory and in suggesting the formulation of a new quantum mechanics. However, it is clear that quantum theory has been developed far beyond the original modifications of classical theory demanded for the derivation of the Planck radiation law. As an interesting illustration that some quantum systems require only a small part of the full quantum theory for their analysis in classical terms, this paper presents a derivation of the blackbody energy spectrum requiring in addition to the usual ideas of classical theory only the assumption of classical, Lorentz-invariant electromagnetic radiation at the absolute zero of temperature. Any notion of discrete or discontinuous processes is unnecessary for the analysis of this particular phenomenon.

II. HISTORICAL REVIEW OF BLACKBODY THEORY

In order to place the new derivation of the blackbody spectrum in its proper theoretical context,[1] we first recall the theoretical analysis of thermal radiation from the point of view of classical theory. During the nineteenth century, attempts to use classical thermodynamics and electrodynamics to analyze thermal radiation achieved major syccesses.[2] Use of Maxwell's electromagnetic radiation pressure in combination with the thermodynamics of the Carnot cycle led Boltzmann to derive the connection between total blackbody energy density U and temperature T:

$$U=\sigma T^4, \tag{1}$$

which had been obtained experimentally by Stefan. Moreover, use of the properties of reflection of electromagnetic radiation from a moving mirror, coupled with thermodynamic arguments, allowed the derivation of Wien's displacement law involving the energy-density spectrum $\rho(\omega,T)$:

$$\rho(\omega,T)=\omega^3F(\omega/T),$$

where

$$U=\int_{\omega=0}^{\infty} d\omega\, \rho(\omega,T), \tag{2}$$

and the function F is still undetermined.

Here, however, classical theory stopped. Any attempt to apply the classical form of the energy equipartition theorem to the energy of mechanical or of electromagnetic vibrations led directly to the Rayleigh-Jeans radiation law

$$\rho(\omega,T)=(\omega^2/\pi^2c^3)kT \tag{3}$$

(k is Boltzmann's constant), with its associated ultraviolet divergence. Attempting to push the contradiction between classical theory and experimental results even one step further, Einstein and Hopf[3] proved that the assumption of energy equipartition for only the kinetic energy of a *free* particle again led rigorously through classical theory to the Rayleigh-Jeans law. The need for some new hypothesis beyond traditional classical theory seemed undeniable.

Planck came upon his radiation law first by making an *ad hoc* modification of the assumed connection between energy and entropy for thermal radiation, and later by assuming that the calculation of the entropy of an oscillator in thermal equilibrium with radiation might be carried out assuming discrete units of energy.

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[1] E. T. Wittaker, in *A History of the Theories of Aether and Electricity, Modern Theories 1900–1926* (Philosophical Library, Inc., New York, 1954), gives an invaluable summary and reference source for the historical arguments related to blackbody radiation.

[2] See the historical survey presented by M. Planck, in *Theory of Heat Radiation* (Dover Publications, Inc., New York, 1959).

[3] A. Einstein and L. Hopf, Ann. Physik **33**, 1105 (1910).

By comparison with experiment or with Wien's displacement law, the discrete energy unit required for agreement was seen to be $\hbar\omega$. The postulate of energy quanta linear in frequency was thus introduced into physics.

In the present paper, we will not assume energy quantization, but rather will add a different postulate to classical theory. We will show that the single untraditional assumption of a Lorentz-invariant spectrum of fluctuating electromagnetic radiation in the universe (electromagnetic zero-point radiation) will allow us to *derive* the zero-point energy spectrum (found to be linear in frequency, $\frac{1}{2}\hbar\omega$ per normal mode), and then, employing precisely the arguments of Einstein and Hopf, which formerly led to the Rayleigh-Jeans law, to arrive at the full Planck blackbody spectrum.

III. ZERO-POINT ENERGY IN QUANTUM THEORY

Before turning to the actual derivations, we wish to make some indications of the connections between the postulate of electromagnetic zero-point energy to be employed here and the usual formulations of quantum theory. Thus in Schrödinger wave mechanics, the presence of ground-state probability distributions having a finite extent in space is usually regarded as an indication of zero-point oscillations of the mechanical systems. The quantization of the simple harmonic oscillator is often written with a ground-state energy $\frac{1}{2}\hbar\omega$, and occasionally this assignment is made to the electromagnetic energy spectrum, although in this later case it is usually dismissed as giving rise to no observable effect. Thus the zero-point energy appears as a result of the formulation of quantum theory, usually in the role of an unnoticed though occasionally annoying hanger-on.

However, Casimir has shown that it is worthwhile to take quantum electromagnetic zero-point energy seriously despite the divergence of zero-point energy density. Accepting the usually ignored assignment of an energy $\frac{1}{2}\hbar\omega$ to each normal mode, Casimir was the first to calculate the quantum electromagnetic attraction of two conducting parallel plates,[4] an effect which has been confirmed by experiment.[5] Moreover, he used the notions of quantum electromagnetic zero-point energy to calculate a number of further effects[6] which had previously been obtained by quantum electrodynamic perturbation theory. The number of such calculations based on Casimir's zero-point energy ideas has been extended,[7] and the results are in agreement with the traditional evaluations (when they exist) from quantum perturbation and dispersion theory.

This paper suggests that electromagnetic zero-point energy should not only be considered seriously, but may even play a fundamental role in physical phenomena. Here we are abstracting *one* aspect of quantum theory, electromagnetic zero-point energy, and will superimpose it on classical theory. Making no initial assumptions about the form of this energy beyond the Lorentz invariance of the spectrum, we will make it the basis for the rigorous derivation of the full blackbody energy spectrum. Since no quantum assumptions are involved, we prefer to speak of *electromagnetic zero-point energy* rather than *quantum electromagnetic zero-point energy*. What is involved is a theory of fluctuations, not of quanta.

We should emphasize that the untraditional fluctuation assumption has connections with alternative formulations of quantum theory which probably go well beyond heuristic ideas. Nelson[8] has shown that the mathematical formalism of nonrelativistic quantum *mechanics* is fully equivalent to classical mechanics on which is superimposed a random walk. There has been some tentative work[9] on the equivalence of quantum electromagnetic theory with classical electromagnetism on which there is superimposed a random walk in the normal coordinates. The use of zero-point energy here is hardly a full fluctuation theory equivalent to some part of quantum electrodynamics, but it forms a nontrivial beginning.

IV. DERIVATION OF THE ELECTROMAGNETIC ZERO-POINT ENERGY SPECTRUM

The one hypothesis which we add to classical theory is the assumption of fluctuating electromagnetic radiation in the universe even at the absolute zero of temperature. In order to maintain the Lorentz invariance of the theory, we require that the spectrum of the radiation shall look the same to all observers moving at constant relative velocity with respect to each other.

The fluctuating electromagnetic radiation can be written in the form of transverse plane waves

$$\mathbf{E}(\mathbf{x},t)=\mathrm{Re}\sum_{\lambda=1}^{2}\int d^3k\,\boldsymbol{\varepsilon}(\mathbf{k},\lambda)h(\omega_{\mathbf{k}})e^{i\omega_{\mathbf{k}}t-i\mathbf{k}\cdot\mathbf{x}-i\theta(\mathbf{k},\lambda)}\,, \qquad (4)$$

$$\mathbf{B}(\mathbf{x},t)=\mathrm{Re}\sum_{\lambda=1}^{2}\int d^3k\frac{\mathbf{k}\times\boldsymbol{\varepsilon}(\mathbf{k},\lambda)}{k}h(\omega_{\mathbf{k}})e^{i\omega_{\mathbf{k}}t-i\mathbf{k}\cdot\mathbf{x}-i\theta(\mathbf{k},\lambda)}\,,$$

$$\boldsymbol{\varepsilon}(\mathbf{k},\lambda)\cdot\mathbf{k}=0,\quad \boldsymbol{\varepsilon}(\mathbf{k},\lambda)\cdot\boldsymbol{\varepsilon}(\mathbf{k},\lambda')=\delta_{\lambda\lambda'}. \qquad (5)$$

Here the random phase $\Theta(\mathbf{k},\lambda)$ is introduced (following Planck[2] and Einstein and Hopf[3]) to indicate the fluctuating character of the radiation. The function $h(\omega_{\mathbf{k}})$ can depend only upon $\omega_{\mathbf{k}}=ck=c|\mathbf{k}|$ because of the assumed isotropy of the radiation.

[4] H. B. G. Casimir, Koninkl. Ned. Akad. Wetenschap. **51**, 793 (1948).

[5] M. J. Sparnaay, Physica **24**, 751 (1958).

[6] H. B. G. Casimir, J. Chimie Phys. **46**, 407 (1949).

[7] T. H. Boyer, Phys. Rev. **174**, 1631 (1968); **174**, 1764 (1968); **180**, 19 (1969).

[8] E. Nelson, Phys. Rev. **150**, 1079 (1966).

[9] T. H. Boyer, Phys. Rev. **174**, 1631 (1968).

In order to connect the function $h(\omega_k)$ with the spectral energy-density function $\rho(\omega_k)$, we compute

$$\frac{1}{8\pi}\langle \mathbf{E}^2+\mathbf{B}^2\rangle=\frac{2}{8\pi}\sum_{\lambda_1=1}^{2}\sum_{\lambda_2=1}^{2}\int d^3k_1\int d^3k_2\boldsymbol{\varepsilon}(\mathbf{k}_1,\lambda_1)\cdot\boldsymbol{\varepsilon}(\mathbf{k}_2,\lambda_2)h(\omega_1)h(\omega_2)\cdot\delta^3(\mathbf{k}_1-\mathbf{k}_2)\delta_{\lambda_1\lambda_2}\times\tfrac{1}{2}, \quad (6)$$

where the initial 2 comes from the equal contributions of **E** and **B**, and the final $\frac{1}{2}$ comes from the correlation between the phases:

$$\langle\cos(\omega_1 t-\mathbf{k}_1\cdot\mathbf{x}-\theta(\mathbf{k}_1,\lambda_1))\cos(\omega_2 t-\mathbf{k}_2\cdot\mathbf{x}-\theta(\mathbf{k}_2,\lambda_2))\rangle=\tfrac{1}{2}\delta^3(\mathbf{k}_1-\mathbf{k}_2)\delta_{\lambda_1\lambda_2}. \quad (7)$$

Thus

$$\frac{1}{8\pi}\langle \mathbf{E}^2+\mathbf{B}^2\rangle=\frac{1}{8\pi}\sum_{\lambda=1}^{2}\int d^3k\, h^2(\omega_k)=\int_{k=0}^{\infty}dk\,k^2h^2(\omega_k)=\int_{\omega=0}^{\infty}d\omega\frac{\omega^2}{c^3}h^2(\omega), \quad (8)$$

and we identify the spectral energy-density function $\rho(\omega)$ as

$$\rho(\omega)=\frac{\omega^2}{c^3}h^2(\omega). \quad (9)$$

Under a Lorentz transformation and along the x axis, transverse plane wave go into transverse plane waves with transformed frequencies and wave numbers. Carrying out the transformation of the fields in (4), we merely rotate the components of **E** and **B** into each other and note the Lorentz-invariant character of the phase of a plane wave,

$$\mathbf{E}'(\mathbf{x}',t')=\mathrm{Re}\sum_{\lambda=1}^{2}\int d^3k\left[\hat{\imath}\epsilon_x+\hat{\jmath}\gamma\left(\epsilon_y-\frac{v}{c}\frac{(\mathbf{k}\times\boldsymbol{\varepsilon})_z}{k}\right)+\hat{k}\gamma\left(\epsilon_z+\frac{v}{c}\frac{(\mathbf{k}\times\boldsymbol{\varepsilon})_y}{k}\right)\right]h(\omega_k)\times\exp[i\omega't'-i\mathbf{k}'\cdot\mathbf{x}'-i\theta(\mathbf{k},\lambda)],$$

$$\mathbf{B}'(\mathbf{x}',t')=\mathrm{Re}\sum_{\lambda=1}^{2}\int d^3k\left[\hat{\imath}\epsilon_x+\hat{\jmath}\gamma\left(\frac{(\mathbf{k}\times\boldsymbol{\varepsilon})_y}{k}+\frac{v}{c}\epsilon_z\right)+\mathbf{k}\gamma\left(\frac{(\mathbf{k}\times\boldsymbol{\varepsilon})_z}{k}-\frac{v}{c}\epsilon_x\right)\right]h(\omega_k)\times\exp[i\omega't'-i\mathbf{k}'\cdot\mathbf{x}'-i\theta(\mathbf{k},\lambda)], \quad (10)$$

with $\boldsymbol{\varepsilon}\equiv\boldsymbol{\varepsilon}(\mathbf{k},\lambda)$. The connection between the primed and unprimed wave vectors is always

$$k_x'=\gamma(k_x-(v/c^2)\omega_k),\quad \omega_{k'}'=\gamma(\omega_k-vk_x),\quad k_y'=k_y,\quad k_z'=k_z, \quad (11)$$

with $\gamma=(1-v^2/c^2)^{-1/2}$. Now we compute the energy density in the primed frame, writing

$$d^3k=d^3k'\gamma(1+vk_{1x}'/\omega_1')$$

in Eq. (10) so as to associate the variable of integration with the frequencies of the plane waves. We have

$$(8\pi)^{-1}\langle \mathbf{E}'^2+\mathbf{B}'^2\rangle=\frac{2}{8\pi}\sum_{\lambda_1=1}^{2}\sum_{\lambda_2=1}^{2}\int d^3k_1'\int d^3k_2'\left[\gamma^2\left(1+\frac{vk_{1x}'}{\omega_1'}\right)^2\right]\times\left[\gamma^2\left(1-\frac{vk_{1x}}{\omega_1}\right)^2\right]h^2(\omega_1)\delta^3(\mathbf{k}_1-\mathbf{k}_2)\delta_{\lambda_1\lambda_2}\times\tfrac{1}{2}$$

$$=\mathrm{const}\int d^3k'h^2(\omega_k)\gamma\left(1-\frac{vk_x}{\omega_k}\right), \quad (12)$$

where the first factor in square brackets arises from transformation of $d^3k_1d^3k_2$, the second factor arises from summing over the polarization vectors, and always the primed and unprimed variables appearing are related by (11).

By invariance of the spectrum of zero-point energy, we mean that if an observer measures the original density of radiation energy with some instrument with a selective frequency filter, then he will find precisely the same energy density when inspecting the trans-transformed distribution with the same filter. Thus the energy density contained in the frequency interval $\omega_k=a$ to $\omega_k=b$ for the untransformed distribution must be identical with the energy density contained in the frequency interval $\omega_{k'}'=a$ to $\omega_{k'}'=b$ for the transformed distribution. The *numbers* a and b can be chosen arbitrarily, but are the *same* for the transformed and untransformed systems. Thus, from Eqs. (8) and (12) we require, for any velocity v and any a and b,

$$\int_{\omega_k=a}^{\omega_k=b}d^3k\,h^2(\omega_k)=\int_{\omega_{k'}'=a}^{\omega_{k'}'=b}d^3k'h^2(\omega_k)\gamma\left(1-\frac{vk_x}{\omega_k}\right), \quad (13)$$

where the primed and unprimed quantities on the right-hand side are related as in Eq. (11). But since the variable of integration is a dummy variable, this is the same as requiring

$$h^2(\omega_{k'}')=h^2(\omega_k)\gamma(1-vk_x/\omega_k). \quad (14)$$

Since from (11) $\omega_{k'}'=\gamma(1-vk_x/\omega_k)\omega_k$, it follows that h^2 must be a *linear* function of ω_k. In other words, by suitably adjusting the normalization constants in the original electromagnetic fields, we must have exactly an electromagnetic zero-point energy $\frac{1}{2}\hbar\omega$ per normal mode if the spectrum is to be Lorentz-invariant. Introducing polar coordinates as in Eq. (8), and then relating the number of normal modes to the variables of integration d^3k, we see that we require

$$\pi^2h^2(\omega)=\tfrac{1}{2}\hbar\omega, \quad (15)$$

and the zero-point spectral density function $\rho(\omega)$ is

$$\rho(\omega)=\hbar\omega^3/2\pi^2c^3. \quad (16)$$

V. CLASSICAL SYSTEM CONSIDERED BY EINSTEIN AND HOPF

A. System Involved

Einstein and Hopf[3] consider a particle of mass m which contains a nonrelativistic electromagnetic dipole oscillator of angular frequency ω_0. For convenience, it is assumed that all particle motion is in the x direction $v=v_x$, and the oscillator dipole points along the z axis. The system is chosen so that the free translational motion enables one to use the ideas of classical statistical mechanics while the oscillator contained in the particle allows for interaction with the electromagnetic radiation. In thermal equilibrium with blackbody radiation at temperature T, the mean-square velocity due to interaction with the radiation must be identical with that determined by classical statistical mechanics for a free particle which, for example, may be thought of as one of the gas atoms of an ideal gas.

B. Forces

During a time interval τ, the particle experiences two forces due to electromagnetic radiation.

(a) The interaction of the oscillator vibrations with the fluctuations of the radiation field leads to a random force and a corresponding fluctuating impulse Δ during a time interval τ.

(b) There is a velocity-dependent average force F_x tending to slow the particle down. Assuming that the velocity of the particle is small compared with the velocity of light c, the force may be written as linear in v, $F_x=-Pv$. Also, if the time interval τ is short so that the change in particle velocity is small, the impulse may be taken as $=-Pv\tau$. We note that the function P will depend only upon the thermal part of the radiation and not on the zero-point radiation. We have postulated that the zero-point spectrum is Lorentz-invariant; thus no uniform motion relative to the zero-point radiation is observable, and no velocity-dependent force depending on the zero-point spectrum is possible.

C. Average Squared Momentum

For any particle having a momentum mv_t at time t, the momentum after a time interval τ is changed by the impulses from forces (a) and (b):

$$mv_{t+\tau}=mv_t+\Delta-Pv_t\tau. \quad (17)$$

In equilibrium, if we average over all particles, the mean-square momentum must be constant in time; thus

$$\langle(mv_t)^2\rangle=\langle(mv_{t+\tau})^2\rangle=\langle(mv_t+\Delta-Pv_t\tau)^2\rangle \quad (18)$$

or

$$0=\langle\Delta^2\rangle+2m\langle v\Delta\rangle-2mP\tau\langle v^2\rangle-2P\tau\langle v\Delta\rangle+P^2\tau^2\langle v^2\rangle, \quad (19)$$

where we have dropped the subscript t on the velocity v_t. Now the velocity v_t is fixed at one instant t while the impulse Δ in the time interval from t to $t+\tau$ may be positive or negative with equal probability. Hence $\langle v\Delta\rangle=0$, and we may drop two of the terms of Eq. (19). Also, the mass m of the particle is at our disposal, so that we may neglect the term $P^2\tau^2\langle v^2\rangle$ compared to $-2mP\tau\langle v^2\rangle$. Thus (19) reduces to the Einstein-Hopf impulse-squared equation

$$\langle\Delta^2\rangle=2mP\tau\langle v^2\rangle. \quad (20)$$

If we assume the classical equipartition theorem $\frac{1}{2}m\langle v^2\rangle=\frac{1}{2}kT$, and introducing the classical electromagnetic calculations for $\langle\Delta^2\rangle$ and P, then (20) leads directly to the Rayleigh-Jeans radiation law.

VI. ELECTROMAGNETIC ZERO-POINT ENERGY AND STATISTICAL THERMODYNAMICS: SOME BASIC CONSIDERATIONS

The introduction of electromagnetic zero-point energy seems to require a fundamental reanalysis of the statistical equilibrium between a set of particles and the surrounding electromagnetic radiation.[10] In classical theory, the interaction of the particles and the enclosing walls goes unmentioned. Sometimes one speaks of the walls as being perfectly elastic reflectors of the gas molecules. Perhaps more accurately from the point of view of entropy considerations, the molecules should be thought of as being absorbed by the walls, and then emitted by the walls with the same velocity distribution as that which holds for the gas molecules contained in the box because the walls are assumed at the same temperature T as the gas. We may note, in connection with Sec. V, that when Einstein and Hopf describe the interaction of gas molecules and thermal radiation in the context of classical theory, they neglect the further interaction with the box. Thus equilibrium between particles and radiation is maintained by the combination of forces (a) and (b). If the random interaction (a) of the oscillator with the radiation extracts energy out of the radiation field converting it into kinetic energy of the particle, then the velocity-dependent force (b) slows the particle down and returns the energy to the radiation field. Classical equilibrium corresponds to a balancing of these two energies, and as Einstein and Hopf showed, leads to the Rayleigh-Jeans distribution of thermal radiation.

At the absolute zero of temperature, traditional theory predicts that there is no electromagnetic radiation present, and no particle motion for the ideal gas. However, the introduction of electromagnetic zero-point

[10] The ideas presented in this section represent only the basic considerations required for the derivation of the blackbody spectrum. In a future publication, the author hopes to treat the thermodynamic aspects of the problem, including implications for superfluid helium. It is interesting that a number of phenomena connected with helium II allow a qualitative explanation in terms of the ideas of electromagnetic zero-point energy presented here.

energy alters the entire situation. Now there is fluctuating radiation present even at zero temperature. Hence the particle is subjected to the fluctuating force (a), and exacts energy from the radiation field. But the Lorentz invariance of electromagnetic zero-point radiation means that no velocity-dependent force is possible. Hence the energy is not returned to the radiation field. The particles diffuse to increasingly high velocities. For particles in free space, the velocity distribution will gradually acquire a Lorentz-invariant character, and the space and velocity properties will be quite reminiscent of Milne's expanding universe.[11] However, in a container, the gas molecules will strike the walls and give up their kinetic energy by dipole radiation and by transfer of mechanical energy to the solid walls at absolute zero. The mechanical energy thus spread through the surrounding medium is also eventually returned to the radiation field. When the gas molecule is now emitted from the wall, it will be emitted at a lower velocity. Thus the molecules continue to absorb energy from the radiation field until they strike one of the walls. We see that, at the absolute zero of temperature, the surrounding enclosure plays a crucial role for an ideal gas, and we must introduce the effects of the walls into the equilibrium equations corresponding to (17) derived for the classical case.

We consider first a particle moving with momentum mv_t at time t in a container at temperature $T=0$. After a time interval τ, the momentum is

$$mv_{t+\tau}=mv_t+\Delta+J\,, \tag{21}$$

where Δ is the impulse given to the particle through the fluctuating radiation field, and J is the impulse given to the particle by any wall. Of course, if the particle does not strike a wall during the time interval τ, then for this particle $J=0$. We now square and average over all particles. If the distribution of particle velocities represents an equilibrium, then the average particle momentum squared will not have changed during τ,

$$\langle (mv_t)^2\rangle=\langle (mv_{t+\tau})^2\rangle=\langle (mv_t+\Delta+J)^2\rangle. \tag{22}$$

Carrying out the square in (22), this implies

$$0=2m\langle v_t\Delta\rangle+2m\langle v_tJ\rangle+\langle J^2\rangle+2\langle \Delta J\rangle+\langle \Delta^2\rangle. \tag{23}$$

Now the impulse Δ may be in the plus or minus x direction during the time interval t to $t+\tau$, and hence the averages $\langle v_t\Delta\rangle=0$ and $\langle \Delta J\rangle=0$. On the other hand, if the particle strikes a wall, then v_t will be just opposite in direction to the impulse of the wall which slows the particle down so that $\langle v_tJ\rangle$ is not zero, but is negative definite. The term $\langle J^2\rangle$ may be neglected compared to $\langle \Delta^2\rangle$, giving

$$0=2m\langle vJ\rangle+\langle \Delta^2\rangle \tag{24}$$

at the absolute zero of temperature. One way to understand the neglect of $\langle J^2\rangle$ is to note that the particle momentum $m\,v_t$ will in general be much larger than the fluctuating impulse Δ received during the short time interval τ. Hence, in order for Eq. (23) to hold, J must be much smaller than Δ and so $\langle J^2\rangle$ may be neglected compared to $\langle \Delta^2\rangle$. Equation (24) expresses the fact that during the time interval τ, the particles absorb energy from the zero-point radiation and give up the kinetic energy when striking the walls.

Having obtained this result, we now turn back to the situation for finite temperature when zero-point radiation is also present. In this case, the impulse given by the walls must be included, giving

$$mv_{t+\tau}=m\,v_t+\Delta-Pv_t\tau+J\,, \tag{25}$$

instead of Eq. (17). Again, assuming equilibrium conditions so that $\langle (mv_t)^2\rangle=\langle (mv_{t+\tau})^2\rangle$ and expanding the expression, we have corresponding to (19)

$$\begin{aligned}0=2m\langle v\Delta\rangle+2m\langle vJ\rangle+\langle J^2\rangle+2\langle \Delta J\rangle+\langle \Delta^2\rangle\\ -2mP\tau\langle v^2\rangle+P^2\tau^2\langle v^2\rangle-2P\tau\langle v\Delta\rangle-2P\tau\langle vJ\rangle.\end{aligned} \tag{26}$$

By the random character of Δ, we have $\langle v\Delta\rangle=0$, $\langle \Delta J\rangle=0$. Again noting that the mass m of the particle is at our disposal, we see that the term $-2P\tau\langle vJ\rangle$ may be neglected compared to $2m\langle vJ\rangle$, and $P^2\tau^2\langle v^2\rangle$ neglected compared to $-2mP\tau\langle v^2\rangle$. Now the term $\langle vJ\rangle$ is related to the average kinetic energy loss of the particle on hitting a wall. However, we have seen that this energy from strictly thermal considerations is as often positive as negative and may even be excluded from consideration entirely as in all treatments of classical theory. Rather, it is only the contribution from the zero-point energy in $\langle \Delta^2\rangle$ which must be removed by collisions with the walls as indicated in Eq. (24). Thus we have

$$2m\langle vJ\rangle_T=2m\langle vJ\rangle_{T=0}=-\langle \Delta^2\rangle_{T=0}\,, \tag{27}$$

again neglecting $\langle J^2\rangle$. Thus our final impulse-squared equation becomes

$$\langle \Delta^2\rangle-\langle \Delta^2\rangle_{T=0}=2mP\tau\langle v^2\rangle. \tag{28}$$

This equation differs from the Einstein-Hopf equation (20) by exactly the zero-point term $\langle \Delta^2\rangle_{T=0}$.

VII. DERIVATION OF BLACKBODY SPECTRUM

A. Use of Classical Statistical Mechanics

We now turn to the evaluation of Eq. (28). The average velocity squared $\langle v^2\rangle$ may be evaluated by the equipartition of energy for a free particle due to classical statistical mechanics. Since there is only one degree of freedom considered,

$$\tfrac{1}{2}m\langle v^2\rangle=\tfrac{1}{2}kT. \tag{29}$$

By taking the particle mass m sufficiently large, we find that any contribution to $\langle v^2\rangle$ from zero-point oscillations at $T=0$ is negligible when introduced in Eq. (28); the argument follows from the mass independence of the impulse from the electromagnetic fluctuations. Thus

[11] E. A. Milne, Z. Astrophys. 6, 1 (1933).

our basic equation becomes

$$\langle\Delta^2\rangle-\langle\Delta^2\rangle_{T=0}=2P\tau kT, \qquad (30)$$

where the values of Δ and of P are to be determined by classical electromagnetic theory applied to the interaction of the oscillator and the electromagnetic radiation ncluding zero-point radiation.

B. Calculations Follow Einstein and Hopf

At this point, the appropriate calculations for $\langle\Delta^2\rangle$ and P are precisely those of Einstein and Hopf. The calculations are sketched in the Appendix in more recent notation for relativistic transformations and Fourier decompositions.[12]

C. Results—Planck's Radiation Law

Inserting the results of $\langle\Delta^2\rangle$ and P obtained by Einstein and Hopf [or of Eqs. (51) and (63) of the Appendix] into the relation (30), we have

$$\frac{1}{3}\frac{c^3\pi^2}{kT\omega^2}[\rho^2(\omega,T)-\rho^2(\omega)]=\rho(\omega,T)-\tfrac{1}{3}\omega\frac{d}{d\omega}\rho(\omega,T), \qquad (31)$$

where we have used the notation $\rho(\omega)\equiv\rho(\omega,T=0)$. In Sec. IV, we derived the spectrum of electromagnetic zero-point energy from Lorentz invariance as

$$\rho(\omega)=\hbar\omega^3/2\pi^2c^3. \qquad (32)$$

Consistent with our ideas of the invariant appearance of the electromagnetic zero-point energy spectrum, it may be noticed that the contribution from the zero-point energy does not enter on the right-hand side of Eq. (31), which is derived from the velocity-dependent force $F_x=-Pv$, with

$$P=c\pi^2(6/5)\Gamma[\rho(\omega,T)-\tfrac{1}{3}\omega d\rho(\omega,T)/d\omega]. \qquad (33)$$

Thus the contribution to P from Eq. (32) is

$$\left[\rho(\omega)-\tfrac{1}{3}\omega\frac{d}{d\omega}\rho(\omega)\right]=\frac{\hbar}{2\pi^2c^3}(\omega^3-\tfrac{1}{3}\omega 3\omega^2)=0. \qquad (34)$$

Substituting the zero-point energy spectrum (32) into the left-hand side of (31), we have the differential equation for $\rho(\omega,T)$

$$\frac{1}{3}\frac{c^3\pi^2}{kT\omega^2}\left[\rho^2(\omega,T)-\left(\frac{\hbar\omega^3}{2\pi^2c^3}\right)^2\right]=\rho(\omega,T)-\tfrac{1}{3}\omega\frac{d}{d\omega}\rho(\omega,T). \qquad (35)$$

The solution of the differential equation is

$$\rho(\omega,T)=\frac{\omega^2}{\pi^2c^3}\left(\frac{\hbar\omega}{e^{\hbar\omega/kT}-1}+\tfrac{1}{2}\hbar\omega\right), \qquad (36)$$

which is exactly Planck's distribution[2] for the full blackbody energy spectrum.

The crucial role played by zero-point energy is strikingly clear in the differential equation (35). If the electromagnetic zero-point energy is set equal to zero, then we have the differential equation of Einstein and Hopf corresponding to (20),

$$\frac{1}{3}\frac{c^3\pi^2}{kT\omega^2}\rho^2(\omega,T)=\rho(\omega,T)-\tfrac{1}{3}\omega\frac{d}{d\omega}\rho(\omega,T), \qquad (37)$$

whose solution[13] is the Rayleigh-Jeans law

$$\rho(\omega,T)=(\omega^2/\pi^2c^3)kT. \qquad (38)$$

Only with the introduction of the electromagnetic zero-point energy (32) corresponding to a Lorentz-invariant spectrum does the differential equation give the experimentally observed energy spectrum (36).

D. Energy of an Oscillator in the Electromagnetic Field

In the chain of argument presented here, we have derived the blackbody radiation spectrum without first obtaining the average energy of a quantum-mechanical oscillator. Thus we may reverse the order of argument traditionally followed in quantum theory and may now *derive* the average energy of an oscillator in equilibrium with the blackbody radiation field. Hence, starting from the hypothesis of a Lorentz-invariant spectrum of electromagnetic zero-point energy, we may proceed through a derivation of blackbody radiation and then find the average energy of a mechanical oscillator which is in agreement with quantum theory.

Once again, the mathematics of the argument has been carried out before by writers[14] viewing the problem from a different point of view. Using classical electrodynamics for a nonrelativistic oscillator including radiation damping situated in a fluctuating electromagnetic field, we find that the average energy $\langle\epsilon\rangle$ of the oscillator is

$$\langle\epsilon\rangle=(\pi^2c^3/\omega^2)\rho(\omega,T). \qquad (39)$$

Substituting the result for blackbody radiation $\rho(\omega,T)$ from Eq. (36), we have

$$\langle\epsilon\rangle=\hbar\omega/(e^{\hbar\omega/kT}-1)+\tfrac{1}{2}\hbar\omega, \qquad (40)$$

[12] Notations involving Lorentz transformations have changed so much in the half-century since Einstein and Hopf's work that the present author found their manipulations on the surface incomprehensible. Repeating the full calculations, the author arrived at values identical with their results.

[13] Differential equations (35) and (37) may be written in the form $\partial\rho/\partial\omega=F(\omega,\rho)$, where F is an analytic function of ω and of ρ. Trying the undetermined coefficient form of a power-series solution, and requiring that $\rho(\omega,T)$ be regular in ω at the origin and a continuous function of T, we find that the Planck and Rayleigh-Jeans laws are the unique solutions of (35) and (37), respectively. See G. Birkhoff and G. Rota, *Ordinary Differential Equations* (Ginn and Co., Boston, Mass., 1962).

[14] M. Abraham and R. Becker, *Theorie der Elekrizitat* (B. G. Teubner Verlag, Leipzig, 1933), Vol. II, 6th ed., pp. 373–375.

which is just the usual result of nonrelativistic quantum theory.

We note that when the electromagnetic zero-point energy is included in the calculation for (39), the nonrelativistic kinetic energy is actually divergent. The value given in (39) is, however, twice the (convergent) nonrelativistic potential energy even in this case. The divergence of the kinetic energy expression can not be considered as significant for any physical system, inasmuch as the dipole and other nonrelativistic approximations used in the differential equation for the oscillator will break down at the high frequencies where the kinetic energy divergence occurs.

E. New Understanding of Quantum Theory

The developments presented here, relating electromagnetic zero-point energy to the blackbody spectrum and to the energies of a quantum harmonic oscillator, strongly suggest the possibility of a new interpretation of quantum theory. Flutuating electromagnetic zero-point energy will lead to fluctuations in the positions of electrons. Using the classical Abraham-Lorentz differential equation of motion including radiation damping, it is easy to show that the diffusion coefficient for the particle position must be a constant times $\hbar/m$, where m is the mass of the charged particle. The charge does not enter the diffusion to first order in time. But Nelson[8] has shown that nonrelativistic classical mechanics on which is superimposed a Brownian motion with diffusion coefficient $\hbar/2m$ is fully equivalent to nonrelativistic quantum theory. Thus starting from the assumption of electromagnetic zero-point energy, it may be possible to use nonrelativistic classical mechanics to obtain all of nonrelativistic quantum theory. Moreover, the use of relativistic classical particle dynamics might give a form of relativistic quantum theory.

The present stumbling point in the analysis outlined is the Abraham-Lorentz equation of classical electrodynamics, an equation which is also a sore point in purely classical theory because of the run-away solutions. Thus far, the author has been unable to evaluate the numerical coefficient in the diffusion of a charged particle due to electromagnetic zero-point energy. It may well be that a proper understanding of this point will solve a problem of classical electromagnetism while joining classical theory naturally onto what is now regarded as a separate theory of quanta.

VIII. CLOSING SUMMARY

We have seen that the assumption of electromagnetic zero-point energy can be added to classical theory to obtain the full blackbody radiation spectrum. The hypothesis is somewhat more economical than quantum theory in that the association with each normal mode of zero-point energy $\frac{1}{2}\hbar\omega$ linear in frequency is derived from Lorentz invariance, rather than being postulated in a quantum energy unit $\hbar\omega$. At this point, it is natural to ask further about the role of electromagnetic zero-point energy in thermodynamics, and also to enquire into the equivalence between classical electromagnetism with fluctuating zero-point energy and quantum electrodynamics.

APPENDIX: SUMMARY OF CALCULATIONS OF EINSTEIN AND HOPF

In the following calculations, we are summarizing in more recent notation the work of Einstein and Hopf[3] required in Sec. VII B. The analysis uses random phases to express the fluctuations of thermal radiation.

A. Calculation of the Velocity-Dependent Average Force

We are considering a nonrelativistic dipole oscillator of frequency ω_0 oriented along the z axis in an electromagnetic field $\mathbf{E}(\mathbf{x},t)$,

$$\frac{d^2p}{dt^2}-\Gamma\frac{d^3p}{dt^3}+\omega_0^2p=\tfrac{3}{2}\Gamma c^3E_z, \tag{41}$$

where p is the oscillator dipole moment, and the radiation damping constant for a physical oscillator corresponds to

$$\Gamma=\tfrac{2}{3}e^2/mc^3. \tag{42}$$

The zero-point and thermal electromagnetic radiation is assumed to exist in the form of transverse plane waves as in Eqs. (4)–(9), except that now h and ρ must be regarded as functions of the temperature $h(\omega,T)$, $\rho(\omega,T)$, reducing to $h(\omega)$, $\rho(\omega)$ at $T=0$.

In obtaining the velocity-dependent average force, we are using the fact that the spectrum of thermal radiation loses its isotropy when viewed from the moving particle. Specifically, we make a Lorentz transformation to a frame of reference moving with the particle. Then new electromagnetic fields are experienced by the oscillator, and hence there arises a velocity-dependent force. The transformed fields are exactly as given in Eqs. (10)–(11), with $h(\omega,T)$ replacing $h(\omega)$. In the particle frame of reference, Eq. (41) holds with the differentiation now with respect to t', and the driving field is given by $E_z'(\mathbf{x}',t')$. The frequency will be indicated by ω_0'. Then the dipole moment is

$$p'=\sum_{\lambda=1}^{2}\int d^3k\frac{3c^3}{2\omega'^3}\gamma\left(\epsilon_z+\frac{v}{c}\frac{(\mathbf{k}\times\boldsymbol{\varepsilon})_y}{k}\right)h(\omega_{\mathbf{k}},T)\sin\alpha(\omega')$$
$$\times\cos[\omega't'-\mathbf{k}'\cdot\mathbf{x}'-\alpha(\omega')-\theta(\mathbf{k},\lambda)], \tag{43}$$

where we have defined

$$\cot\alpha(\omega')\equiv(\omega_0'^2-\omega'^2)/\Gamma\omega'^3, \tag{44}$$

and always the primed and unprimed quantities are related as in (11).

The force on the particle due to the interaction of the dipole with radiation is

$$F_x' = \frac{\partial E_x'}{\partial z'} p' - \frac{1}{c} B_y' \frac{dp'}{dt'}. \quad (45)$$

Thus differentiating $E_x'(\mathbf{x}',t')$ in Eq. (10) with respect to z', and $B_y'(\mathbf{x},t)$ in (10) with respect to t', we have

$$F_x' = \sum_{\lambda=1}^{2} \int d^3k\, \epsilon_x k_z' h^2(\omega,T) \frac{3c^3}{2\omega'^3} \times \gamma\left(\epsilon_z + \frac{v}{c}\frac{(\mathbf{k}\times\boldsymbol{\epsilon})_y}{k}\right) \sin^2\alpha(\omega') \times \tfrac{1}{2} - \frac{1}{c}\sum_{\lambda=1}^{2} \int d^3k\, \gamma\left(\frac{(\mathbf{k}\times\boldsymbol{\epsilon})_y}{k} + \frac{v}{c}\epsilon_z\right) h^2(\omega,T) \frac{3c^3}{2\omega'^3} \times \gamma\left(\epsilon_z + \frac{v}{c}\frac{(\mathbf{k}\times\boldsymbol{\epsilon})_y}{k}\right) \sin^2\alpha(\omega') \times \tfrac{1}{2}, \quad (46)$$

where we have noted that the random phases give

$$\langle \sin[\omega_1' t - \mathbf{k}_1'\cdot\mathbf{x}' - \theta(\mathbf{k}_1,\lambda_1)] \times \cos[\omega_2' t' - \mathbf{k}_2'\cdot\mathbf{x}' - \alpha(\omega_2') - \theta(\mathbf{k}_2,\lambda)]\rangle = \tfrac{1}{2} \sin\alpha(\omega_2')\, \delta^3(\mathbf{k}_1 - \mathbf{k}_2)\delta_{\lambda_1\lambda_2}, \quad (47)$$

primed and unprimed variables, as always, related by (11).

In order to simplify the expression, we sum over polarizations, and change all variables to primes using (11). In order to change $\rho(\omega,T) = (\omega^2/c^3)h^2(\omega,T)$ over to a function of ω_2', we expand

$$\rho(\omega,T) \cong \rho(\omega',T) + \frac{\partial\rho(\omega',T)}{\partial\omega'}(\omega-\omega') \cong \rho(\omega',T) + \frac{v}{c}\frac{k_x'}{k'}\omega' \frac{\partial\rho(\omega',T)}{\partial\omega'} \quad (48)$$

to first order in v/c. We then find that

$$F_x' = \int d^3k' \frac{3}{4} \frac{\Gamma^2 c^3 \omega'^3}{(\omega_0'^2 - \omega'^2)^2 + \Gamma^2\omega'^6} \frac{c^3}{\omega'^2} \times \left(\rho(\omega',T) + \frac{v}{c}\frac{k_x'}{k'}\omega' \frac{\partial\rho(\omega',T)}{\partial\omega'}\right) \times \left(k_x' - \frac{k_x' k_z'^2}{k'^2} - 3\frac{v}{c}\frac{k_x'^2}{k'} + 3\frac{v}{c}\frac{k_x'^2 k_z'^2}{k'^3}\right). \quad (49)$$

The change to polar coordinates and angular integrations is straightforward. However, in evaluating the integral over frequency, we assume that $\Gamma\omega_0 \ll 1$, so that the integrand involving $\sin^2\alpha(\omega')$ is sharply peaked at ω_0'. Changing the variable of integration from ω' to $x = \omega' - \omega_0'$, replacing all terms in ω' not involving $\omega' - \omega_0'$ by ω_0', and extending the lower limit of integration to $-\infty$, we finally need to evaluate an integral of the form

$$\int_{-\infty}^{\infty} \frac{dz}{z^2+1} = \pi. \quad (50)$$

Thus

$$F_x' = -vc\pi^2(6/5)\Gamma(\rho(\omega_0',T) - \tfrac{1}{3}\omega_0' \partial\rho(\omega_0',T)/\partial\omega_0'). \quad (51)$$

The relativistic transformations between the stationary and particle frame for the forces and frequencies in (51) involve completely negligible higher powers of v/c and hence we may drop all primes in Eq. (51) when abstracting the value for P in $F_x = -Pv$.

B. Calculation of the Fluctuating Impulse

The fluctuating impulse given to the particle during a time τ due to interaction of the oscillator vibrations with fluctuations of the electromagnetic field can be computed for an oscillator at rest as

$$\Delta = \int_{t=0}^{t=\tau} F_x dt = \int_{t=0}^{t=\tau} \left(\frac{\partial E_x}{\partial z} p - \frac{1}{c} B_y \frac{dp}{dt}\right) dt. \quad (52)$$

The expression can be simplified by partial integrations on the second term:

$$\int_{t=0}^{t=\tau} B_y \frac{dp}{dt} dt = [B_y p]_0^\tau - \int_{t=0}^{t=\tau} \frac{\partial B_y}{\partial t} p dt. \quad (53)$$

The first term on the right does not depend upon the time interval τ and may be neglected. The last term may be transformed using Maxwell's equation

$$\nabla \times \mathbf{E} = c^{-1}\partial\mathbf{B}/\partial t \quad (54)$$

to give

$$\Delta = \int_{t=0}^{t=\tau} \frac{\partial E_z}{\partial x} p dt. \quad (55)$$

Again we take the radiation field as composed of fluctuating transverse plane waves given by (4), with $h(\omega_\mathbf{k})$ replaced by $h(\omega_\mathbf{k},T)$. It follows that the dipole moment given by the differential equation (41) satisfies

$$p = \sum_{\lambda=1}^{2} \int d^3k \frac{3c^3}{2\omega^3} \epsilon_z(\mathbf{k},\lambda) h(\omega_\mathbf{k},T) \sin\alpha(\omega) \times \cos[\omega t - \mathbf{k}\cdot\mathbf{x} - \alpha(\omega) - \theta(\mathbf{k},\lambda)], \quad (56)$$

with $\alpha(\omega)$ the same function as in (44), but now without primes.

Formally, $\partial E_z(\mathbf{x},t)/\partial x$ may be obtained by differentiating in Eq. (4). However, the evaluation of the impulse due to the fluctuating fields calls for an integral over time which is not appropriately averaged with the same random phase appearing in $E_z(\mathbf{x},t)$ and $\partial E_z(\mathbf{x},t)/\partial x$. In a separate article which forms a necessary preliminary to their work, Einstein and Hopf[15] derive a general result of probability theory which shows that $E_z(\mathbf{x},t)$ and $\partial E_z(\mathbf{x},t)/\partial x$ must be regarded as independent quantities in this time integral. Hence we will write $\partial E_z/\partial x$ with a second, independent phase,

$$\frac{\partial E_z(\mathbf{x},t)}{\partial x}=\sum_{\lambda=1}^{2}\int d^3k\epsilon_z(\mathbf{k},\lambda)k_x \times\cos[\omega t-\mathbf{k}\cdot\mathbf{x}-\xi(\mathbf{k},\lambda)]. \quad (57)$$

Qualitatively, we may say that because of the random fluctuations we expect the average of Δ to vanish; it is only the average value of Δ^2 which will increase with time.

Thus with these considerations in mind, we write

$$\Delta=\int_{t=0}^{t=\tau}dt\sum_{\lambda_1=1}^{2}\sum_{\lambda_2=1}^{2}\int d^3k_1\int d^3k_2\,\epsilon_z(\mathbf{k}_1,\lambda_1)h(\omega_1,T)k_{1x} \times\frac{3c^3}{2\omega^3}\frac{\epsilon_z(\mathbf{k}_2,\lambda_2)}{(\omega_0^2-\omega^2)^2-\Gamma^2\omega^6}h(\omega_2,T)\sin\alpha(\omega_2) \times\cos[\omega_1 t-\mathbf{k}_1\cdot\mathbf{x}-\xi(\mathbf{k}_1,\lambda_1)] \times\cos[\omega_2 t-\mathbf{k}_2\cdot\mathbf{x}-\alpha(\omega_2)-\theta(\mathbf{k}_2,\lambda_2)]. \quad (58)$$

Choosing the origin of coordinates at the location of the particle, we may drop the $\mathbf{k}_1\cdot\mathbf{x}$ and $\mathbf{k}_2\cdot\mathbf{x}$ contributions. The time integral gives

$$\int_{t=0}^{t=\tau}dt\cos(\omega_1 t-\xi_1)\cos(\omega_2 t-\theta_2-\alpha_2) =\left[\frac{1}{\omega_1+\omega_2}\sin\left(\frac{\omega_1+\omega_2}{2}\tau\right)\cos\left(\frac{\omega_1+\omega_2}{2}\tau-\xi_1-\theta_2-\alpha_2\right) +\frac{1}{\omega_2-\omega_1}\sin\left(\frac{\omega_2-\omega_1}{2}\tau\right)\cos\left(\frac{\omega_2-\omega_1}{2}\tau+\xi_1-\theta_2-\alpha_2\right)\right], \quad (59)$$

where $\xi_1\equiv\xi(\mathbf{k}_1,\lambda_1)$, $\theta_2\equiv\theta(\mathbf{k}_2,\lambda_2)$, $\alpha_2\equiv\alpha(\omega_{\mathbf{k}_2})$.

Next we square and evaluate the mean value $\langle\Delta^2\rangle$. Now the phases $\xi_2+\theta_1$ and $\xi_2'-\theta_1'$ are not correlated and hence the cross terms from Eq. (59) vanish, leaving only the squared terms from

$$\left\langle\cos\left(\frac{\omega_2\pm\omega_1}{2}\tau\mp\xi(\mathbf{k}_1,\lambda_1)-\theta(\mathbf{k}_2,\lambda_2)-\alpha_2\right) \times\cos\left(\frac{\omega_2'\pm\omega_1'}{2}\tau\mp\xi(\mathbf{k}_1',\lambda_1')-\theta(\mathbf{k}_2',\lambda_2')-\alpha_2\right)\right\rangle =\tfrac{1}{2}\delta^3(\mathbf{k}_1-\mathbf{k}_1')\delta_{\lambda_1\lambda_1'}\delta^3(\mathbf{k}_2-\mathbf{k}_2')\delta_{\lambda_2\lambda_2'}. \quad (60)$$

Summing over polarizations, we find that

$$\langle\Delta^2\rangle=\sum_{\lambda_1=1}^{2}\sum_{\lambda_2=1}^{2}\int d^3k_1\int d^3k_2\left(1-\frac{k_{1z}^2}{k_1^2}\right)k_{1x}^2h^2(\omega_1,T) \times\left(1-\frac{k_{2z}^2}{k_2^2}\right)h^2(\omega_2,T)\frac{3c^3}{2\omega_2^3}\sin^2\alpha_2 \times\left[\frac{1}{(\omega_1+\omega_2)^2}\sin^2\left(\frac{\omega_1+\omega_2}{2}\tau\right) +\frac{1}{(\omega_2-\omega_1)^2}\sin^2\left(\frac{\omega_2-\omega_1}{2}\tau\right)\right]\times\tfrac{1}{2}. \quad (61)$$

Again it is convenient to change to polar coordinates. Only the integrations over ω_1 and ω_2 require comment. The term involving $\omega_1+\omega_2$ may be neglected in comparison with the $\omega_2-\omega_1$ term, which may be regarded as a sharply peaked (δ functionlike) integrand, requiring us to set $\omega_1=\omega_2$. The basic integral here is

$$\int_{-\infty}^{\infty}\frac{\sin^2\tau x}{x^2}dx=\pi\tau. \quad (62)$$

The function $\sin^2\alpha(\omega_2)$ is similarly regarded as sharply peaked for $\Gamma\omega_0\ll1$, and the evaluation of the integral in ω_2 is carried out in the same style as indicated in part A of the Appendix. Then, recalling Eq. (9) connecting $h(\omega,T)$ and $\rho(\omega,T)$, we obtain

$$\langle\Delta^2\rangle=(4\Gamma\pi^4c^4\tau/5\omega^2)\rho^2(\omega,T). \quad (63)$$

C. Added Note: Earlier Literature on Electromagnetic Zero-Point Radiation

During the research continued after submitting this paper for publication, the author became aware of several further papers on electromagnetic zero-point radiation. In 1916, Nernst[16] suggested that the universe might contain zero-point radiation in agreement, except for a factor of 2, with the $\frac{1}{2}\hbar\omega$ per normal mode proposed by Planck. While apparently not realizing the Lorentz invariance implicit in this spectrum, Nernst does comment, essentially on the basis of our Eq. (15), that the assumed spectrum will not give rise to frictional forces on objects moving at constant velocity. Nernst is concerned with the energy divergence implicit in the zero-point spectrum and speculates on an appropriate high-frequency cutoff. By contrast, we note that it is precisely this divergence which allows us to escape the usual dictum that the energy (if finite) of a Lorentz-invariant vacuum state must vanish.

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[15] A. Einstein and L. Hopf, Ann. Physik 33, 1096 (1910).

[16] W. Nernst, Verhandl. Deut. Phys. Ges. 18, 83 (1916).

There also is interesting work by Marshall[17] published during 1963–65 proposing essentially the same view as that arrived at by the author, that electromagnetic zero-point radiation can be regarded as the cause of particle quantum motion. In this sense, quantum motions are experimental evidence for zero-point radiation. However, Marshall does not touch the question treated in this paper, that of deriving Planck's radiation law from electromagnetic zero-point radiation and classical theory. It is interesting that Marshall came upon the Lorentz invariance of the zero-point spectrum as an afterthought. It was this Lorentz invariance of the electromagnetic zero-point spectrum, and the consequent absence of velocity-dependent damping forces, which struck the author as so fundamental that it prompted the line of development carried out in the present paper.

[17] T. W. Marshall, Proc. Roy. Soc. (London) **A276**, 475 (1963); Proc. Cambridge Phil. Soc. **61**, 537 (1965); Nuovo Cimento **38**, 206 (1965). I wish to thank Dr. B. Robertson for bringing this work to my attention.

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Generalization of the Reissner-Nordström Solution to the Einstein Field Equations

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The field equations for coupled gravitational and zero-mass scalar fields in the presence of a point charge are solved in the spherically symmetric static case. The resulting solution is the generalization of the Reissner-Nordström solution in the presence of a zero-mass meson field.

I. INTRODUCTION

The well-known Reissner-Nordström solution of Einstein's equations corresponds to the situation with a charged mass-point at the origin of spherical coordinates. That solution plays a justifiably important role in studies of the interaction of electromagnetic and gravitational fields.

Recently,[1] we had occasion to treat the problem of interacting gravitational and zero-mass meson fields in the axially symmetric, static situation, and obtained the (spherically symmetric) generalization of the Schwarzschild solution in the presence of a zero-mass meson field. The form of that solution was considerably simpler than the form originally obtained by Janis *et al.*[2]

Indeed, the form of the generalized Schwarzschild solution is simple enough to give hope that the Reissner-Nordström solution could also be generalized in the presence of a zero-mass meson field. In this paper we obtain the desired generalization.

The main reason for presenting the solution is that exact solutions of Einstein's gravitational equations are scarce. Thus any such exact solutions may be useful even though somewhat unrealistic.

II. FIELD EQUATIONS

We wish to solve the field equations

$$G_{\mu\nu}=-KT_{\mu\nu}-KE_{\mu\nu}\,, \tag{1}$$

$$T_{\mu\nu}=\varphi_\mu\varphi_\nu-\tfrac{1}{2}g_{\mu\nu}\varphi^\alpha\varphi_\alpha\,, \tag{2}$$

$$E_{\mu\nu}=g^{ab}F_{a\mu}F_{b\nu}-\tfrac{1}{4}g_{\mu\nu}F_{ab}F^{ab}\,, \tag{3}$$

where φ_μ is a gradient and $F_{\mu\nu}$ is the Maxwell tensor.

We are interested in the spherically symmetric, static situation, in which case the line element is[3]

$$ds^2=e^\alpha dR^2+e^\beta d\Omega^2-e^\gamma dt^2\,, \tag{4}$$

and we choose coordinates such that $\alpha+\gamma$ vanishes.

We assume that only φ_1 and F_{14} are nonzero functions of R. We further assume, as in the Reissner-Nordström solution, that J_μ, the current, vanishes except for a point-charge singularity at the origin of coordinates. (Because of our choice of coordinates, the "origin" will not be at $R=0$.)

To proceed, we first look at the Maxwell equation

$$F^{\mu\nu}{}_{|\nu}=0 \tag{5}$$

and obtain immediately that

$$F_{14}=\epsilon e^{-\beta}\,, \tag{6}$$

where ϵ is the charge.

We next look at the wave equation for φ, viz.,

$$\varphi^\mu{}_{|\mu}=0\,, \tag{7}$$

and obtain directly that

$$\varphi_1=ce^{\alpha-\beta} \tag{8}$$

for some constant c.

It is then a simple matter to calculate the components of $T_{\mu\nu}$ and $E_{\mu\nu}$, and one finds that the nonzero

[1] R. Penney, Phys. Rev. **174**, 1578 (1968).

[2] A. I. Janis, E. T. Newman, and J. Winicour, Phys. Rev. Letters **20**, 878 (1968).

[3] J. L. Synge, *Relativity: The General Theory* (Wiley-Interscience, Inc., New York, 1960), p. 270.