Nuclear Q Values in a Dense Stellar Plasma

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Corrections to nuclear Q values in a dense stellar plasma are calculated by means of a modified ionic cluster expansion in which the electrons are treated as a dielectric medium. Corrections beyond the weak-screening (Debye) approximation are given in tabular form for purposes of interpolation. A comparison between the Debye correction and the double-cluster correction is made for the chemical potential of a one-component plasma; the latter correction is ~20% of the former for a plasma parameter as large as unity.

I. INTRODUCTION

IN nuclear astrophysics there occur situations where the interior of an evolved star may be considered to be in local nuclear equilibrium for at least some of the nuclear species present. Examples of such situations are the red-giant stage with the equilibrium reactions

$$\begin{aligned} He^4+He^4 &\leftrightarrow Be^8\,, \\ He^4+Be^8 &\leftrightarrow C^{12*} \end{aligned} \tag{1}$$

and the more advanced stage of stellar evolution for which the silicon-burning reactions are applicable.[1] If nuclear equilibrium has in fact been established, detailed knowledge of nuclear reaction rates is not needed to determine the concentrations of the equilibrium constituents, but rather these concentrations are related to one another in a relatively simple manner determined by statistical mechanics. The dominant feature of the relationship between the constituent densities is the presence of a Boltzmann factor involving the Q value (energy release) of the particular equilibrium reaction taking place. The Q values, however, are not those measured in the laboratory. At the high temperatures and densities for which nuclear equilibrium holds, matter is essentially fully ionized, and the Coulomb energy of an ionized nucleus in the ambient plasma is more negative[1] than the Coulomb energy of the bound atomic electrons in the laboratory case. The problem of calculating the Coulomb energy of a nucleus in a plasma has been considered by Salpeter[2] in general for the two cases in which the Coulomb corrections are either small (weak screening) or large (strong screening) compared to the thermal energy kT. It turns out, however, that in many instances[1] the Coulomb energy is comparable to the thermal energy and it is of interest in such cases to obtain corrections beyond that valid for weak screening.

It is the purpose of this paper to develop these additional corrections by the method of a modified cluster expansion that has been applied to classical plasmas by a number of authors.[3–5] In Sec. II, an effective ion-ion interaction is constructed in the approximation that the electrons are treated as a linear dielectric medium. In an Appendix, a general formulation of the effective ion-ion interaction is given which implicitly incorporates the electron-ion and electron-electron interaction in a relativistic manner and which leads to the interaction of Sec. II in the linear dielectric approximation. In Sec. III this effective interaction is used to obtain the Q value for a general nuclear reaction, and corrections beyond those for weak screening are tabulated as a basis for interpolation. Finally, in Sec. IV a comparison is made between the weak-screening corrections and the additional corrections calculated in this paper.

II. EFFECTIVE ION-ION INTERACTION

Consider first a plasma containing fully ionized nuclei and a *uniform* neutralizing electron background. The assumption of a uniform background is a reasonable first approximation at high density where the electron gas is highly degenerate. The ion Hamiltonian for such a system is

$$H_I=\sum_\alpha\left(\frac{p_\alpha^2}{2M_\alpha}+M_\alpha c^2\right)+\tfrac{1}{2}\sum_{\alpha\neq\alpha'}q_\alpha q_{\alpha'}\frac{1}{|\mathbf{r}_\alpha-\mathbf{r}_{\alpha'}|}\,, \tag{2}$$

where Greek indices refer to the individual nuclei, and $\mathbf{p}_\alpha$, $\mathbf{r}_\alpha$, M_α, and q_α are the momentum, position, mass, and charge of the αth nucleus. The nuclear rest energy has been included since some of this energy is converted into (or from) kinetic energy of the various participants in the nuclear reactions. We have explicitly left out the *nuclear* interaction part of the Hamiltonian since, although it is essential for the *maintenance* of nuclear equilibrium, it nevertheless has a negligible effect on the equilibrium properties themselves.

The Hamiltonian (2) can be modified to take the electrons into account indirectly by treating them as a dielectric medium in which the nuclei are embedded. With the assumptions that (a) the electrons move at much higher velocities than do the ions and (b) the electron-ion interaction is small on the average compared to the electron kinetic energy, it can be shown that the ionic Hamiltonian is effectively modified to

[1] D. Bodansky, D. D. Clayton, and W. A. Fowler, Astrophys. J. Suppl. 148, 299 (1968).
[2] E. E. Salpeter, Australian J. Phys. 7, 373 (1954).
[3] R. Abe, Progr. Theoret. Phys. (Kyoto) 22, 213 (1959).
[4] I. R. Iukhnovskii, Zh. Eksperim. i Teor. Fiz. 34, 379 (1958) [English transl.: Soviet Phys.—JETP 7, 263 (1958)].
[5] S. G. Brush, H. E. DeWitt, and J. G. Trulio, Nucl. Fusion 3, 5 (1963).

the form

$$H_I'=\sum_\alpha\left(\frac{p_\alpha^2}{2M_\alpha}+M_\alpha c^2+\delta M_\alpha c^2\right)+\tfrac{1}{2}\sum_{\alpha\neq\alpha'}q_\alpha q_{\alpha'}\phi(|\mathbf{r}_\alpha-\mathbf{r}_{\alpha'}|),\quad(3)$$

with

$$\phi(r)=4\pi\int\frac{d^3k}{(2\pi)^3}\frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{k^2\epsilon(\mathbf{k},0)},\quad(4)$$

where $\epsilon(\mathbf{k},0)$ is the exact static longitudinal dielectric constant for the interacting electron gas in the presence of a uniform neutralizing background of positive charge. In an Appendix, this result is derived as the weak-coupling limit of an effective ion Hamiltonian which itself requires only the validity of the "static" ion approximation [assumption (a) above].

The expression $q_\alpha q_{\alpha'}\phi(|\mathbf{r}_\alpha-\mathbf{r}_{\alpha'}|)$ in Eq. (3) is the electron-screened (but not ion-screened) ion-ion interaction, while the quantity $\delta M_\alpha c^2$ is the interaction energy between the αth ion and the induced electron charge density. This self-energy can be obtained from the electron-screened ion-ion interaction with $\alpha=\alpha'$ provided the (infinite) self-energy associated with the point nucleus is first removed. Using Eq. (4), we then obtain

$$\delta M_\alpha c^2=\tfrac{1}{2}q_\alpha^2\lim_{\mathbf{r}_\alpha\to\mathbf{r}_{\alpha'}}\left(\phi(|\mathbf{r}_\alpha-\mathbf{r}_{\alpha'}|)-\frac{1}{|\mathbf{r}_\alpha-\mathbf{r}_{\alpha'}|}\right)=\tfrac{1}{2}q_\alpha^2 4\pi\int\frac{d^3k}{(2\pi)^3}\frac{1}{k^2}\left(\frac{1}{\epsilon(\mathbf{k},0)}-1\right).\quad(5)$$

The electron dielectric constant $\epsilon(\mathbf{k},0)$ has been calculated in random-phase approximation (including relativistic effects) by several authors.[6,7] To sufficient approximation, we then have

$$\epsilon(\mathbf{k},0)=1+\kappa_e^2/k^2,\quad(6)$$

where

$$\kappa_e^2=4\pi e^2\int\frac{d^3p}{(2\pi\hbar)^3}\frac{1}{E_p}\left(1+\frac{E_p^2}{p^2c^2}\right)[f_-(E_p)+f_+(E_p)],\quad(7)$$

with $E_p=(p^2c^2+m^2c^4)^{1/2}$ and $f_\pm(E_p)=[\exp\beta(E_p\pm\mu)+1]^{-1}$. Here m is the electron mass and μ is the electron chemical potential. Equation (7) includes positrons as well as electrons and is valid relativistically. At low temperatures and densities where positrons are not in equilibrium with electrons and the radiation field, the f_+ term should be dropped. With the approximation of Eq. (6), we then find

$$\phi(r)=e^{-\kappa_e r}/r\quad\text{and}\quad\delta M_\alpha c^2=-\tfrac{1}{2}q_\alpha^2\kappa_e.\quad(8)$$

[6] B. Jancovici, Nuovo Cimento **25**, 428 (1962).
[7] V. N. Tsytovich, Zh. Eksperim. i Teor. Fiz. **40**, 1775 (1961) [English transl.: Soviet Phys.—JETP **13**, 1249 (1961)].

III. PLASMA CORRECTIONS TO NUCLEAR Q VALUES

Consider a system of nuclei undergoing a set of nuclear reactions of the form

$$\sum_i \nu_{ir}a_i=0,\quad(9)$$

where the Latin index i labels the nuclear species; a_i is the symbol designating the ith species and ν_{ir} is the number of nuclei of the ith species undergoing the rth reaction. The integers ν_{ir} are taken as positive for nuclei initiating the direct reaction and negative for nuclei initiating the inverse reaction. From thermodynamics,[8] the conditions for nuclear equilibrium are given by the set of equations

$$\sum_i \nu_{ir}\mu_i=0,\quad(10)$$

where μ_i is the chemical potential of the ith species and is related to the Helmholtz free energy F by

$$\mu_i=(\partial F/\partial N_i)_{T,V}.\quad(11)$$

The free energy is then related to the microscopic physics through the canonical ionic partition function Z_I by

$$F=F_p-kT\ln Z_I,\quad(12)$$

where F_p is the free energy of a system of interacting electrons, positrons, and radiation with a uniform neutralizing background of positive charge, and

$$Z_I=\prod_i\frac{1}{N_i!}\int\prod_\alpha\frac{d^3p_\alpha d^3r_\alpha}{h^3}e^{-\beta H_I'}.\quad(13)$$

Here $\beta=1/kT$, N_i is the number of particles in the ith nuclear species, and H_I' is the effective ion Hamiltonian and is given by Eq. (3) in the approximation that the electrons are treated as a linear dielectric medium. After carrying out the momentum integrations, we obtain

$$Z_I=\prod_i\left(\frac{\exp(-\beta N_iM_i'c^2)}{N_i!\Omega_i^{N_i}}\right)\bar{Z}_I,\quad(14)$$

where

$$\Omega_i=(h^2/2\pi M_ikT)^{3/2},\quad M_i'=M_i+\delta M_i,$$

and

$$\bar{Z}_I=\int\prod_\alpha d^3r_\alpha e^{-\beta U},\quad(15)$$

with

$$U=\tfrac{1}{2}\sum_{\alpha\neq\alpha'}q_\alpha q_{\alpha'}\phi(|\mathbf{r}_\alpha-\mathbf{r}_{\alpha'}|).\quad(16)$$

The configurational partition function with a general two-body interaction has been expressed as a modified cluster expansion by Iukhnovskii,[4] in which clusters of increasingly larger numbers of ions become more important with increasing density. Using these results

[8] P. Morse, *Thermal Physics* (W. A. Benjamin, Inc., New York, 1965), p. 137.

and keeping only terms through the double clusters, we have from Ref. 4

$$\bar{Z}_i = V^N \exp\left(\frac{V}{4\pi^2}\int_0^\infty \{w(\mathbf{k})-\ln[1+w(\mathbf{k})]\}k^2dk + 2\pi V\sum_{ij} n_in_j\int_0^\infty \{e^{-\beta\Phi_{ij}(r)}-1+\beta\Phi_{ij}(r) - \tfrac{1}{2}[\beta\Phi_{ij}(r)]^2\}r^2dr\right), \quad (17)$$

where V is the system volume,

$$w(\mathbf{k}) = \kappa_I^2\int_0^\infty \phi(r)\frac{\sin kr}{k} rdr, \quad (18)$$

and

$$\Phi_{ij}(r) = \frac{2}{\pi}\frac{q_iq_j}{r}\frac{1}{\kappa_I^2}\int_0^\infty \frac{w(\mathbf{k})}{1+w(\mathbf{k})}\sin kr\, kdk, \quad (19)$$

with

$$\kappa_I^2 = 4\pi\beta\sum_i q_i^2n_i \quad \text{and} \quad n_i = N_i/V.$$

Using the expression for $\phi(r)$ as given in Eq. (8), $w(\mathbf{k})$ and $\Phi_{ij}(r)$ become

$$w(\mathbf{k}) = \kappa_I^2/(k^2+\kappa_e^2) \quad (20)$$

and

$$\Phi_{ij}(r) = q_iq_je^{-\kappa r}/r, \quad (21)$$

where $\kappa^2 = \kappa_e^2+\kappa_I^2$. In most cases for which the present work is applicable, κ_e^2 is actually small compared to κ_I^2. The ratio κ_e^2/κ_I^2 is given by λ/Z_{av}, where Z_{av} is the average nuclear atomic number and λ varies from unity for a Maxwellian electron gas to $kT/\epsilon_F \ll 1$ for a degenerate electron gas with Fermi energy ϵ_F. The present formulation, however, is valid when κ_e^2/κ_I^2 is not completely negligible. Finally, then, $\bar{Z}_i$ of Eq. (17) can be evaluated and from this expression the free energy takes the form

$$F = F_p' + F_0 + F_D + F_2, \quad (22)$$

where

$$\begin{aligned} F_p' &= F_p + kTV\kappa_e^3/12\pi, \\ F_0 &= \textstyle\sum_i N_i\{M_ic^2 + kT[\ln(n_i\Omega_i)-1]\}, \\ F_D &= -kTV\kappa^3/12\pi, \\ F_2 &= 2\pi VkT\sum_{ij} n_in_j\int_0^\infty \{\exp[-\beta\Phi_{ij}(r)]-1 + \beta\Phi_{ij}(r) - \tfrac{1}{2}[\beta\Phi_{ij}(r)]^2\}r^2dr. \end{aligned} \quad (23)$$

The term F_p' is independent of the nuclear species numbers N_i and will not affect the determination of the Q values; From Eq. (23) it contains the term F_p which is the free energy of the system with the ions replaced by a uniform, neutralizing background of positive charge. The term F_0 is the free energy of an ideal gas of nuclei. The term F_D gives the Coulomb free energy of the plasma in the Debye-Hückel approximation and leads to the weak-screening result of Salpeter.[2] Finally, the term F_2 gives the double-cluster contribution to the Coulomb free energy and leads to additional Q-value corrections.

The condition for nuclear equilibrium, Eq. (10), may then be written as

$$\prod_i (n_i\Omega_i)^{\nu_{ir}} = e^{-\beta(Q_{0r}+Q_{Dr}+Q_{2r})}, \quad (24)$$

where

$$Q_{0r} = \sum_i \nu_{ir}M_ic^2, \quad (25)$$

$$Q_{Dr} = \sum_i \nu_{ir}\mu_{iD}, \quad (26)$$

and

$$Q_{2r} = \sum_i \nu_{ir}\mu_{i2}, \quad (27)$$

with

$$\mu_{iD} = (\partial F_D/\partial N_i)_{T,V} \quad \text{and} \quad \mu_{i2} = (\partial F_2/\partial N_i)_{T,V}. \quad (28)$$

The quantities Q_{0r}, Q_{Dr}, and Q_{2r} are the contributions to the energy release for the rth reaction from free ionized nuclei, from the Debye approximation to the Coulomb energy, and from the double-cluster terms, respectively. Carrying out the differentiation indicated in Eq. (28), we then obtain

$$\mu_{iD} = -\tfrac{1}{2}kT\xi_{ii} \quad (29)$$

and

$$\mu_{i2} = -kT[\sum_j A(\xi_{ij})u_j + \xi_{ii}\sum_{j,k} B(\xi_{jk})u_ju_k], \quad (30)$$

where $u_i = 4\pi n_i/\kappa^3$ and $\xi_{ij} = q_iq_j\beta\kappa$. The functions $A(y)$ and $B(y)$ are given by

$$\begin{aligned} A(y) &= \tfrac{1}{3}yh_1(y) + \tfrac{2}{3}yh_2(y) - \tfrac{1}{4}y^2, \\ B(y) &= \tfrac{1}{4}yh_1(y) - \tfrac{1}{8}y^2 - \tfrac{3}{4}A(y), \end{aligned} \quad (31)$$

where

$$h_p(y) = \frac{1}{p!}\int_0^\infty x^pe^{-x}(1-e^{-yf(x)})dx, \quad (32)$$

with $f(x) = e^{-x}/x$. It is evident that to find the actual values of the various energy releases, the complicated set of nonlinear equations (24)–(30) must be solved for those n_i that cannot be independently specified. We do not deal with that problem in this paper. The functions $A(y)$ and $B(y)$ have been tabulated in Table I for values of y suitable for interpolation out to $y=2$. This value is actually beyond the range of validity for the double cluster. This range should be somewhat less than unity, although a more quantitative check on the convergence of the cluster expansion would be obtained by looking at the triple-cluster contributions to the plasma thermodynamics.

IV. COMPARISON BETWEEN DEBYE AND DOUBLE-CLUSTER CORRECTIONS

Most of the thermonuclear reations for which the results of this paper are applicable are of the form

$$a_1 + a_2 \leftrightarrow a_3, \quad (33)$$

TABLE I. Functions $A(y)$ and $B(y)$. Numbers in parentheses are powers of 10 by which the indicated value of $A(y)$ and $B(y)$ should be multiplied.

y	$A(y)$	$B(y)$	y	$A(y)$	$B(y)$
0.00	0.000	0.000	0.75	−5.861 (−2)	8.626 (−2)
0.05	−5.835 (−5)	4.863 (−6)	0.80	−6.846 (−2)	1.021 (−2)
0.10	−3.645 (−4)	3.483 (−5)	0.85	−7.918 (−2)	1.195 (−2)
0.15	−1.047 (−3)	1.090 (−4)	0.90	−9.079 (−2)	1.385 (−2)
0.20	−2.192 (−3)	2.432 (−4)	0.95	−1.033 (−1)	1.593 (−2)
0.25	−3.868 (−3)	4.507 (−4)	1.00	−1.167 (−1)	1.818 (−2)
0.30	−6.129 (−3)	7.432 (−4)	1.10	−1.463 (−1)	2.320 (−2)
0.35	−9.020 (−3)	1.131 (−3)	1.20	−1.797 (−1)	2.896 (−2)
0.40	−1.258 (−2)	1.624 (−3)	1.30	−2.168 (−1)	3.547 (−2)
0.45	−1.684 (−2)	2.229 (−3)	1.40	−2.579 (−1)	4.276 (−2)
0.50	−2.184 (−2)	2.956 (−3)	1.50	−3.030 (−1)	5.085 (−2)
0.55	−2.759 (−2)	3.810 (−3)	1.60	−3.520 (−1)	5.975 (−2)
0.60	−3.412 (−2)	4.798 (−3)	1.70	−4.051 (−1)	6.949 (−2)
0.65	−4.146 (−2)	5.926 (−3)	1.80	−4.623 (−1)	8.007 (−2)
0.70	−4.961 (−2)	7.201 (−3)	1.90	−5.236 (−1)	9.152 (−2)
			2.00	−5.893 (−1)	1.038 (−1)

in which case the Debye contribution takes on the simple form

$$Q_D=-q_1q_2\kappa\,, \tag{34}$$

which is the weak-screening result obtained by Salpeter[2] from a somewhat different point of view. The double-cluster corrections, however, do not lead to any simple form but, as is apparent from Eqs. (27) and (30), depend in general on all the nuclear species explicitly as well as implicitly through the inverse Debye radius κ. Since it is not our intent to discuss specific astrophysical situations but rather to simply make available Coulomb corrections beyond the Debye approximation, we shall limit the discussion of μ_{i2} to that of a one-component plasma and compare it to μ_{iD}. This comparison should then give some indication as to the general size of the double-cluster correction to the Q value as compared to the Debye correction.

For a one-component plasma the parameters u and ξ are related by $u^{-1}=\xi=q^2\beta\kappa$, and from Eqs. (29) and (30) we obtain

$$\mu_2/\mu_D=2[A(\xi)+B(\xi)]/\xi^2. \tag{35}$$

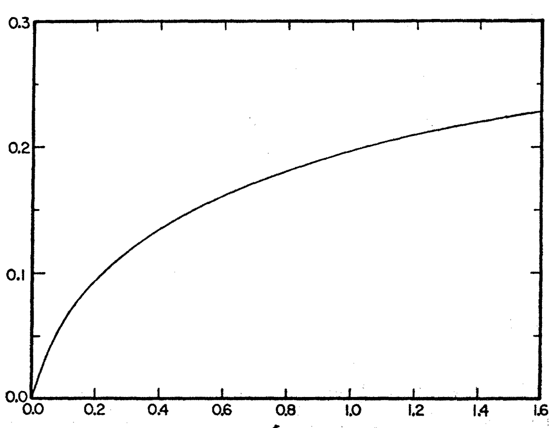

FIG. 1. Ratio $-\mu_2/\mu_D$ as a function of the plasma parameter ξ.

This ratio is plotted in Fig. 1 as a function of the plasma parameter ξ. It is interesting to note that even at $\xi=1$ the double-cluster corrections are only 20% of the Debye corrections. This result is also indicated in the analysis of the equation of state for a one-component plasma.[9,10] This apparent accuracy of the Debye approximation could be better verified by including higher-order clusters to determine the convergence of the cluster expansion.

APPENDIX

In this Appendix we derive an expression for the effective ion Hamiltonian which, in the limit of weak electron-ion interaction, reduces to Eq. (3). In the approximation that the ionic motion contributes only to the ideal part of the internal energy and does not affect the Coulomb energy of the plasma, the canonical partition function for the system is given by

$$Z=(\prod_i N_i!)^{-1}\int\prod_\alpha \frac{d^3r_\alpha d^3p_\alpha}{h^3}e^{-\beta H_I}\,\mathrm{Tr}e^{-\beta(H_p+H_{eI})}, \tag{A1}$$

where Tr stands for the trace over the appropriate complete set of plasma states with static ions, H_I is given by Eq. (2), H_p is Hamiltonian for an interacting system of electrons and photons with a uniform neutralizing background of positive charge, and H_{eI} is the electron-ion interaction and is given by

$$H_{eI}=-\sum_\alpha q_\alpha e\int\frac{\rho(\mathbf{x})}{|\mathbf{x}-\mathbf{r}_\alpha|}d^3x. \tag{A2}$$

The detailed nature of H_p will not be needed for the derivation given below. We note that in its full generality it is relativistic and automatically includes electrons and positrons as well. The quantity $\rho(\mathbf{x})$ is the electron charge density and is given by

$$\rho(\mathbf{x})=\psi^\dagger(\mathbf{x})\psi(\mathbf{x})\,, \tag{A3}$$

where $\psi(\mathbf{x})$ is the relativistic electron field operator and $\psi^\dagger(\mathbf{x})$ the Hermitian adjoint field operator. These operators satisfy the canonical anticommutation rules

$$\psi(\mathbf{x})\psi^\dagger(\mathbf{x}')+\psi^\dagger(\mathbf{x}')\psi(\mathbf{x})=\delta^3(\mathbf{x}-\mathbf{x}')\,, \tag{A4}$$

with all other anticommutators vanishing.

We now replace H_{eI} by λH_{eI}, where λ is a measure of the coupling between ions and electrons. For $\lambda=1$ we have the full coupling that corresponds to the actual plasma, while for $\lambda=0$ the electrons and ions are not coupled. Next we define

$$Z_p^\lambda(\mathbf{r}_\alpha)=\mathrm{Tr}e^{-\beta(H_p+\lambda H_{eI})}. \tag{A5}$$

With the standard technique[11] of taking the logarithmic

[9] D. L. Bowers and E. E. Salpeter, Phys. Rev. **119**, 1180 (1960).
[10] J. G. Trulio and S. G. Brush, Phys. Rev. **121**, 940 (1961).
[11] L. P. Kadanoff and G. Baym, *Quantum Statistical Mechanics* (W. A. Benjamin, Inc., New York, 1963), Chap. 2.

derivative of $Z_p^\lambda(\mathbf{r}_\alpha)$ with respect to λ and then integrating from $\lambda=0$ to $\lambda=1$, we obtain

$$\ln Z_p(\mathbf{r}_\alpha)=\ln Z_p-\beta\int_0^1 \frac{d\lambda}{\lambda}\langle\lambda H_{eI}\rangle', \tag{A6}$$

where

$$\langle X\rangle'=\mathrm{Tr}(\tilde{\rho}X)/\mathrm{Tr}(\tilde{\rho}), \tag{A7}$$

with

$$\tilde{\rho}=e^{-\beta(H_p+\lambda H_{eI})}. \tag{A8}$$

Here the quantity $Z_p(\mathbf{r}_\alpha)$ is $Z_p^\lambda(\mathbf{r}_\alpha)$ for $\lambda=1$. The quantity Z_p is $Z_p^\lambda(\mathbf{r}_\alpha)$ for $\lambda=0$ and is independent of the ionic coordinates; it is the canonical partition function for the electron-positron-radiation system with uniform neutralizing background of positive charge. The interaction term in Eq. (A6) can now be expanded as a perturbation series[12] in the electron-ion interaction to give

$$Z_p(\mathbf{r}_\alpha)=Z_p e^{-\beta\Delta(\mathbf{r}_\alpha)}, \tag{A9}$$

where

$$\Delta(\mathbf{r}_\alpha)=\int_0^1 \frac{d\lambda}{\lambda}\langle S(\beta)\lambda H_{eI}\rangle/\langle S(\beta)\rangle. \tag{A10}$$

Here

$$\langle X\rangle=\mathrm{Tr}(\tilde{\rho}_0 X)/\mathrm{Tr}(\tilde{\rho}_0), \tag{A11}$$

with

$$\tilde{\rho}_0=e^{-\beta H_p}, \tag{A12}$$

and the quantity $S(\beta)$ is given by

$$S(\beta)=\sum_{n=0}^{\infty}\frac{1}{n!}(-\lambda)^n\int_0^\beta\cdots\int_0^\beta d\beta_1\cdots d\beta_n \times T[H_{eI}(\beta_1)\cdots H_{eI}(\beta_n)]. \tag{A13}$$

In Eq. (A13) the symbol T arranges the operator within the square brackets with successively larger values of β_i standing to the left. The operators $H_{Ie}(\beta_i)$ themselves are related to H_{eI} by

$$H_{eI}(\beta_i)=e^{\beta_i H_p}H_{eI}e^{-\beta_i H_p}. \tag{A14}$$

Equation (A9) may now be inserted back into Eq. (A1) to obtain

$$Z=Z_p Z_I, \tag{A15}$$

where

$$Z_I=(\prod_i N_i!)^{-1}\int\prod_\alpha \frac{d^3r_\alpha d^3p_\alpha}{h^3}e^{-\beta H_I'}, \tag{A16}$$

and the effective ion Hamiltonian H_I' is given by

$$H_I'=H_I+\Delta(\mathbf{r}_\alpha). \tag{A17}$$

The effective interaction $\Delta(\mathbf{r}_\alpha)$ given in Eq. (A10) may be written as a sum of N-body interactions

$$\Delta(\mathbf{r}_\alpha)=\sum_{l=0}^{N_I}\Delta_l(\mathbf{r}_\alpha), \tag{A18}$$

where N_I is the total number of ions in the system. Since in the present paper we are including only double clusters in the analysis, it is then consistent to keep only the one- and two-body effective interactions Δ_1 and Δ_2. We note here, however, that all effects of the electron-ion interaction have now been incorporated implicitly in an effective ion-ion interaction. Physically, the nonionic part of the plasma has been replaced by a (generally nonlinear) dielectric medium. If the further approximation of keeping only the first-order electron-ion interaction term in $\Delta^{(1)}$ and $\Delta^{(2)}$ is made, these two quantities reduce to

$$\Delta_1=-2\pi\sum_\alpha q_\alpha^2\int\frac{d^3k}{(2\pi)^3}\left(1-\frac{1}{\epsilon(\mathbf{k},0)}\right) \tag{A19}$$

and

$$\Delta_2=2\pi\sum_{\alpha\neq\alpha'}q_\alpha q_{\alpha'}\int\frac{d^3k}{(2\pi)^3}e^{i\mathbf{k}\cdot(\mathbf{r}_\alpha-\mathbf{r}_{\alpha'})}\frac{1}{k^2}\times\left(\frac{1}{\epsilon(\mathbf{k},0)}-1\right), \tag{A20}$$

where $\epsilon(\mathbf{k},0)$ is the exact static, wave-vector-dependent longitudinal dielectric constant for the electron plasma with uniform background. The quantities Δ_1 and Δ_2 are connected to the dielectric constant through the relationship[6]

$$\frac{k^2}{4\pi e^2}\left(1-\frac{1}{\epsilon(\mathbf{k},0)}\right)=\frac{1}{V}\sum_{nm}p_n\frac{\langle n|\rho(\mathbf{k})|m\rangle\langle m|\rho(-\mathbf{k})|n\rangle+\langle n|\rho(-\mathbf{k})|m\rangle\langle m|\rho(\mathbf{k})|n\rangle}{E_m-E_n}, \tag{A21}$$

where $\rho(\mathbf{k})$ is the Fourier transform of $\rho(\mathbf{x})$, $|m\rangle$ is an eigenstate with energy E_m of the electron plasma, $p_n=e^{-\beta E_n}/\sum_m e^{-\beta E_m}$, and V is the system volume. Equations (A19) and (A20) then combine with H_I to give the effective ion Hamiltonian (3).

[12] R. Brout and P. Carruthers, *Lectures on the Many-Electron Problem* (Wiley-Interscience, Inc., New York, 1963), Chap. 2.