THE PHYSICAL REVIEW

A journal of experimental and theoretical physics established by E. L. Nichols in 1893

SECOND SERIES, VOL. 182, NO. 5 — 25 JUNE 1969

Stationary Axially Symmetric Generalizations of the Weyl Solutions in General Relativity*

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(Received 8 July 1968; revised manuscript received 29 January 1969)

It is shown that a necessary condition that normal-hyperbolic solutions of the Einstein vacuum field equations for the metric tensor defined by the quadratic differential form $ds^2=fdu^2-2mdudv-ldv^2-e^{2\gamma}(dx^2+dz^2)$ (where f, l, m, and γ are functions of x and z, and $fl+m^2=x^2$) be of type III or N is that $x^{-1}f$, $x^{-1}l$, and $x^{-1}m$ be functions of a single function μ; it is further shown that no such nonflat solutions exist. Solutions having this functional dependence are found to belong to one of three classes: the Weyl class and two classes which may be obtained from it. One of these classes is characterized by Sachs-Penrose type-I stationary solutions having one real and two distinct complex-conjugate eigenvalues. The other class is characterized by Sachs-Penrose type-II stationary solutions admitting a single shear-, twist-, and expansion-free doubly degenerate geodesic ray which is also a null, hypersurface-orthogonal Killing vector. Further invariant properties of these classes are discussed, as well as the special case where μ depends only upon x.

I. INTRODUCTION

THE study of exact solutions of the general-relativistic field equations for empty space was originally of some interest to physicists because these solutions are thought to correspond to gravitational fields external to matter distributions which are in some sense localized. The fact that intrinsic singularities are often present in these solutions is of no particular concern since one eventually hopes to join to these solutions interior solutions, or solutions with a nonvanishing energy-momentum tensor, in the region in which the exterior solution exhibits these intrinsic singularities.

On the other hand, it was realized that one could instead treat the singularities in some cases as idealized matter distributions, such as point particles in Newtonian mechanics, and develop the study of empty-space solutions without any particular regard to the associated interior solutions, if indeed such solutions exist. It soon became evident that this study was mathematically very rich, and at the same time one was freed from the difficulty of proposing physical models from which the matter tensor must be constructed. The result is that many distinct classes of empty-space solutions[1] have been discovered since the discovery of the first non-trivial solution by Schwarzschild and Droste[2]; few of these solutions have been joined to exact interior solutions.

Considerable interest has been shown[3–11] in such solutions for the quadratic differential form (QDF)

$$\begin{aligned} ds^2 &= g_{\alpha\beta}dx^\alpha dx^\beta \\ &= fdu^2-2mdudv-ldv^2-e^{2\gamma}(dx^2+dz^2), \quad (1.1) \\ g_{\alpha\beta} &= g_{\alpha\beta}(x,z); \end{aligned}$$

we shall use the summation convention with Greek

* This research was completed at Temple University, where it was supported in part by the Aerospace Laboratories of the Office of Aerospace Research, U. S. Air Force.

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[1] A review of exact solutions is given by J. Ehlers and W. Kundt, in *Gravitation: An Introduction to Current Research*, edited by L. Witten (John Wiley & Sons, Inc., New York, 1962), p. 49.
[2] K. Schwarzschild, Sitzber. Preuss. Akad. Wiss. 7, 189 (1916); J. Droste, Versl. k. Akad. v. Weten. 25, 460 (1916).
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indices taking the values 0, 1, 2, 3, and make the following identification:

$$x^0=u,\quad x^1=x,\quad x^2=v,\quad x^3=z.$$

Since this metric is independent of u, v and the QDF (1.1) is form-invariant under the coordinate transformation

$$u, v \rightarrow -u, -v,$$

it seems a reasonable candidate for the exterior fields of rotating matter distributions having axial symmetry. The best known example of such a solution is that found by Kerr,[12,13] which is a special case of the general class of solutions discovered by Kerr and Schild[14]; it is frequently considered to be the external solution for a rotating mass distribution.

This QDF has the property that it is form-invariant under the coordinate transformation

$$\bar{x}=\bar{x}(x,z),\quad \bar{z}=\bar{z}(x,z), \tag{1.2}$$

where $\bar{x}$ and $\bar{z}$ are restricted by

$$\partial^2\bar{x}/\partial x^2+\partial^2\bar{x}/\partial z^2=0 \tag{1.3}$$

and

$$\frac{\partial\bar{x}}{\partial x}=\frac{\partial\bar{z}}{\partial z},\quad \frac{\partial\bar{x}}{\partial z}=-\frac{\partial\bar{z}}{\partial x}. \tag{1.4}$$

It is a consequence of the field equations that Δ, defined by

$$\Delta^2\equiv fl+m^2, \tag{1.5}$$

satisfies Eq. (1.3); it is customary to transform to new variables, the so-called canonical coordinates $\bar{x}$ and $\bar{z}$, where $\bar{x}=\Delta$ and $\bar{z}$ is a function conjugate to Δ obtained by solving Eqs. (1.4). This has the effect of reducing the number of independent metric components to three, thereby simplifying the field equations. These equations and the Christoffel symbols are listed in Appendix A. By defining Δ as in Eq. (1.5) and demanding that the components of the metric tensor be real functions of real variables, we have limited ourselves to normal-hyperbolic solutions, since the determinant $g\equiv\|g_{\alpha\beta}\|$ is given by

$$g=-(fl+m^2)e^{4\gamma}.$$

In this paper we consider those normal-hyperbolic solutions of the field equations for which the corresponding QDF takes the form of the QDF (1.1) in canonical coordinates showing that a necessary condition that such a solution be of algebraic type III or N is that $x^{-1}f$, $x^{-1}m$, and $x^{-1}l$ be pairwise functionally dependent. Although it is found that no type III or N solutions exist in this case, the solutions obtained from the above assumption of functional dependence are analyzed, showing that they may be classified into one of three classes distinguished by invariant properties such as algebraic type and Killing structure. One of these classes is the Weyl class; the remaining two classes, one of which is the Lewis[4] class and the other a generalization of a solution obtained from the Lewis class by van Stockum,[5] are simply related to the Weyl class. Although the solutions discovered by Lewis and van Stockum have been known for some time, the invariant mathematical properties of these solutions have not been discussed in detail in the literature. Because of the importance of the knowledge of such properties for the physical interpretation of solutions, the mathematical properties of these generalized Weyl solutions are discussed in some detail.

[12] R. P. Kerr, Phys. Rev. Letters **11**, 237 (1963).

[13] For a detailed study of this solution see R. H. Boyer and R. W. Lindquist, J. Math. Phys. 8, 265 (1967).

[14] R. P. Kerr and A. Schild, in *Pubblicazioni del comitato naizonale per le manifestazione celebrative del IV centenario della nascita de Galileo Galilei* (G. Barbèra, Florence, 1965), Vol. II, tomo 1, p. 222.

II. SOLUTIONS WITH REAL EIGENVALUES

From the Christoffel symbols in Appendix A and the fact that $g_{\alpha\beta}=g_{\alpha\beta}(x,z)$, it is easily seen that the curvature tensor for the QDF (1.1) will have the form

$$R_{AB}=\begin{bmatrix} R_{11} & 0 & R_{13} & R_{14} & 0 & R_{16} \\ 0 & R_{22} & 0 & 0 & R_{25} & 0 \\ R_{13} & 0 & R_{33} & R_{34} & 0 & R_{36} \\ R_{14} & 0 & R_{34} & R_{44} & 0 & R_{46} \\ 0 & R_{25} & 0 & 0 & R_{55} & 0 \\ R_{16} & 0 & R_{36} & R_{46} & 0 & R_{66} \end{bmatrix}, \tag{2.1}$$

where R_{AB} is related to the curvature tensor $R_{\alpha\beta\gamma\delta}$ as usual by establishing a correspondence between the single indices A and B and the pairs of indices $\alpha\beta$ and $\gamma\delta$, respectively, such that

$$1\rightarrow 10,\ 2\rightarrow 20,\ 3\rightarrow 30,\ 4\rightarrow 23,\ 5\rightarrow 31,\ 6\rightarrow 12.$$

Defining

$$g_{AB}\rightarrow g_{\alpha\beta\gamma\delta}\equiv g_{\alpha\delta}g_{\beta\gamma}-g_{\alpha\gamma}g_{\beta\delta}$$

we obtain for the QDF (1.1)

$$g_{AB}=\begin{bmatrix} fe^{2\gamma} & 0 & 0 & 0 & 0 & -me^{2\gamma} \\ 0 & \Delta^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & fe^{2\gamma} & me^{2\gamma} & 0 & 0 \\ 0 & 0 & me^{2\gamma} & -le^{2\gamma} & 0 & 0 \\ 0 & 0 & 0 & 0 & -e^{4\gamma} & 0 \\ -me^{2\gamma} & 0 & 0 & 0 & 0 & -le^{2\gamma} \end{bmatrix}. \tag{2.2}$$

Hence, in determining the eigenvalues of R_{AB}, i.e., the solutions λ of

$$\|R_{AB}-\lambda g_{AB}\|=0,$$

we see that the determinant can be factored into the product of two determinants, and the eigenvalue equation takes the form

$$\begin{vmatrix} R_{22}-\lambda g_{22} & R_{25} \\ R_{25} & R_{55}-\lambda g_{55} \end{vmatrix} \times \begin{vmatrix} R_{11}-\lambda g_{11} & R_{13} & R_{14} & R_{16}-\lambda g_{16} \\ R_{13} & R_{33}-\lambda g_{33} & R_{34}-\lambda g_{34} & R_{36} \\ R_{14} & R_{34}-\lambda g_{34} & R_{44}-\lambda g_{44} & R_{46} \\ R_{16}-\lambda g_{16} & R_{36} & R_{46} & R_{66}-\lambda g_{66} \end{vmatrix}=0.$$

For the QDF (1.1) we find

$$-4e^{2\lambda}R_{22}=4\Delta^2e^{-2\lambda}R_{55}=\dot{f}\dot{l}+\dot{m}^2+f'l'm'^2,$$
$$R_{25}=\tfrac{1}{4}\Delta^{-2}[m(f'\dot{l}-\dot{f}l')+\dot{m}(fl'-lf')+m'(l\dot{f}-f\dot{l})], \quad (2.3)$$

where

$$\dot{f}\equiv\partial f/\partial x,\quad f'\equiv\partial f/\partial z.$$

Hence, one pair of complex conjugate eigenvalues is given by the solution of

$$(R_{22}-\lambda g_{22})(R_{55}-\lambda g_{55})-(R_{25})^2=0, \quad (2.4)$$

that is,

$$\lambda=-\tfrac{1}{4}\Delta^{-2}e^{-2\gamma}(\dot{f}\dot{l}+\dot{m}^2+f'l'+m'^2)$$
$$\pm\tfrac{1}{4}i\Delta^{-2}e^{-2\gamma}[m(\dot{f}l'-f'\dot{l})+\dot{m}(fl'-lf')+m'(l\dot{f}-f\dot{l})], \quad (2.5)$$

with $i\equiv\sqrt{(-1)}$.

We begin by investigating the consequences if λ is real, that is,

$$m(\dot{f}l'-f'\dot{l})+\dot{m}(fl'-lf')+m'(l\dot{f}-f\dot{l})=0. \quad (2.6)$$

Of course, if any one of f, l, or m is zero, Eq. (2.6) is identically satisfied. Assume that $f\neq 0$ [a coordinate transformation can always be made in the QDF (1.1) such that $f\neq 0$]. Proceeding in canonical coordinates, i.e., coordinates for which $\Delta^2=x^2$, we solve Eq. (1.5) for l and substitute the result in Eq. (2.6), obtaining

$$\frac{\dot{f}}{f}\frac{m'}{m}-\frac{f'}{f}\frac{\dot{m}}{m}+\frac{1}{x}\left(\frac{f'}{f}-\frac{m'}{m}\right)=0. \quad (2.7)$$

If we now define

$$xF\equiv f,\quad xL\equiv l,\quad xM\equiv m,$$

Eqs. (1.5) and (2.7) take the form

$$FL+M^2=1 \quad (1.5')$$

and

$$\dot{F}M'-F'\dot{M}=0, \quad (2.7')$$

respectively. If F is not constant, the most general solution of Eq. (2.7′) is $M=M(F)$ and, from Eq. (1.5′), $L=L(F)$; if F is constant, we see immediately from Eq. (1.5′) that L and M are functionally related. Thus Eq. (2.6) requires that F, L, and M be functions of a single arbitrary function which we shall denote by μ.

One could now proceed to solve the vacuum field equations, given in Appendix A, to find all solutions satisfying Eq. (2.6), but since these computations are long and tedious, although straightforward, we will show instead how they may be obtained from the Weyl[15] solutions which may be written in the form

$$ds^2=xe^{2\mu}du^2-xe^{-2\mu}dv^2-e^{2\gamma}(dx^2+dz^2). \quad (2.8)$$

We now make the coordinate transformation

$$\sqrt{2}u=a\bar{u}+b\bar{v},\quad \sqrt{2}v=p\bar{u}+q\bar{v},\quad 2\mu=s\bar{\mu},$$

[15] H. Weyl, Ann. Physik 54, 117 (1917).

obtaining from the QDF (2.8)

$$ds^2=\tfrac{1}{2}x(a^2e^{s\bar{\mu}}-p^2e^{-s\bar{\mu}})d\bar{u}^2+x(abe^{s\bar{\mu}}-pqe^{-s\bar{\mu}})d\bar{u}d\bar{v}$$
$$+\tfrac{1}{2}x(b^2e^{s\bar{\mu}}-q^2e^{-s\bar{\mu}})d\bar{v}^2-e^{2\gamma}(dx^2+dz^2); \quad (2.9)$$

this is the first set of solutions obtained by Lewis[4] and is, for real constants, obviously static. If we now set

$$a=q=-1,\quad b=p=\tfrac{1}{2}s=\sqrt{(-1)},$$

the QDF (2.9) takes the form

$$ds^2=x\cos 2\mu\, du^2+2x\sin 2\mu\, dudv-x\cos 2\mu\, dv^2-e^{2\gamma}(dx^2+dz^2), \quad (2.10)$$

when we drop the bars for convenience; this is the second set of solutions discovered by Lewis[4] and later obtained by Tiwari and Misra[7] by the use of a different method. If instead we set

$$a=p=s^{-1/2},\quad b=s^{-1/2}-s^{1/2},\quad q=s^{-1/2}+s^{1/2}$$

in the QDF (2.9) and take the limit as $s\to 0$, we obtain

$$ds^2=x\mu du^2+2x(\mu-1)dudv+x(\mu-2)dv^2-e^{2\gamma}(dx^2+dz^2); \quad (2.11)$$

van Stockum[5] obtained this solution for the particular case where $\mu=c\ln|x|$, with c constant. In general, $\mu=\mu(x,z)$ in the QDF (2.11). We will subsequently refer to the QDF's (2.8), (2.10), and (2.11) as the W class, L class, and S class of solutions, respectively.

We see from the field equations that for all three classes μ is an arbitrary solution of

$$\ddot{\mu}+x^{-1}\dot{\mu}+\mu''=0. \quad (2.12)$$

For the W and L classes of solutions γ must be a solution of

$$\dot{\gamma}=x(\dot{\mu}^2-\mu'^2)-1/4x,\quad \gamma'=-2x\dot{\mu}\mu', \quad (2.13)$$
$$\dot{\gamma}=x(\mu'^2-\dot{\mu}^2)-1/4x,\quad \gamma'=-2x\dot{\mu}\mu', \quad (2.14)$$

respectively; for the S class of solutions the equations can be integrated in general, yielding

$$\gamma=-\tfrac{1}{4}\ln|x|. \quad (2.15)$$

III. TYPE-III AND TYPE-N SOLUTIONS

Since all eigenvalues of R_{AB} are zero for those solutions which are of algebraic type III and N, a necessary condition that a solution of the vacuum field equations for the QDF (1.1) be one of these types is that

$$R_{22}=R_{55}=R_{25}=0.$$

Since $R_{25}=0$ limits the solutions to the W, L, and S classes, we now look for solutions of these classes for which

$$\dot{f}\dot{l}+\dot{m}^2+f'l'+m'^2=0; \quad (3.1)$$

this reduces to the equations

$$\dot{\mu}^2+\mu'^2-1/x^2=0, \quad (3.2)$$
$$\dot{\mu}^2+\mu'^2+1/x^2=0, \quad (3.3)$$

and

$$1=0, \qquad (3.4)$$

for the W, L, and S classes, respectively. There are obviously no real solutions of Eqs. (3.3) and (3.4), and the only solutions of the pair of Eqs. (2.12) and (3.2) are

$$\mu=\pm\ln|x|, \quad \mu=\ln|z\pm(x^2+z^2)^{1/2}|,$$

which are all flat. We thus conclude that there are no solutions of the vacuum field equations of algebraic type III or N for the QDF (1.1) expressed in canonical coordinates. We point out that there exists a class of "noncanonical" solutions of algebraic type N which can be put into the form of the QDF (1.1) but which cannot be expressed in canonical coordinates since $fl+m^2$ is constant.[16]

IV. MATHEMATICAL PROPERTIES OF THE SOLUTIONS

In discussing the matehmatical properties of these solutions it is convenient to introduce differential forms[17] ω^μ such that

$$ds^2=\eta_{\alpha\beta}\omega^\alpha\omega^\beta, \qquad (4.1)$$

where

$$\eta_{\alpha\beta}=\mathrm{diag}(1,\,-1,\,-1,\,-1).$$

Using the exterior differential calculus to determine the Riemann tensor, one automatically obtains

$$R_{AB}=\begin{pmatrix} M & N \\ N & -M \end{pmatrix}, \quad A,\,B=1,\,2,\,\cdots,\,6, \qquad (4.2)$$

where M, N are traceless, symmetric 3×3 matrices. The calculations are outlined in Appendix B following Misner,[17] and the exterior derivatives $d\omega^\alpha$ of the differential forms ω^α as well as the connection forms $\omega^\alpha{}_\beta$ are given for the three classes in terms of the differential forms given below. Because of the simple form of R_{AB} the eigenvalues are easily computed as solutions of

$$\|M+iN-\lambda I\|=0, \qquad (4.3)$$

where I denotes the unit $3+3$ matrix.[18] The second-order differential invariants may then be read from the eigenvalues.

Debever[19] has shown that in every empty space-time there exists at least one and not more than four null vectors k^μ satisfying

$$k_{[\mu}R_{\epsilon]\alpha\beta[\gamma}k_{\delta]}k^\alpha k^\beta=0. \qquad (4.4)$$

Sachs[20] has given a classification of the Riemann tensor based upon Eq. (4.4) and the following equations:

$$R_{\epsilon\alpha\beta\gamma}k^\gamma=0, \qquad (4.5)$$

$$R_{\epsilon\alpha\beta[\gamma}k_{\delta]}k^\beta=0, \qquad (4.6)$$

$$R_{\epsilon\alpha\beta[\gamma}k_{\delta]}k^\alpha k^\beta=0; \qquad (4.7)$$

if the Riemann tensor admits one solution of Eq. (4.5) it is said to be of algebraic type N, it if admits one solution of Eq. (4.6) it is type III, if it admits two solutions of Eq. (4.7) it is type D, if it admits one solution of Eq. (4.7) it is type II, and if it admits four solutions of Eq. (4.4) it is type I. All Riemann tensors of type other than type I are said to be algebraically special; also any solution of one of Eqs. (4.5)–(4.7) is also a solution of the subsequent equations as well as Eq. (4.4).

If one introduces a complex null tetrad

$$[k_\mu,m_\mu,t_\mu,\bar{t}_\mu]$$

such that k_μ and m_μ are real and t_μ is complex, $\bar{t}_\mu$ being the complex conjugate of t_μ, and all products are zero except

$$k^\mu m_\mu=-t^\mu\bar{t}_\mu=-1,$$

we can write

$$k_{\alpha;\beta}=(\theta+i\omega)t_\alpha\bar{t}_\beta+\sigma t_\alpha t_\beta+\Omega t_\alpha k_\alpha+Ak_\alpha t_\beta+Bk_\alpha k_\beta+\text{c.c.};$$

the optical parameters θ, ω, $|\sigma|$, and $|\Omega|$ are called the expansion, twist, shear or distortion, and rotation, respectively.[21]

A. W Class

Since the mathematical properties of these solutions are carefully and thoroughly discussed by Ehlers and Kundt,[1] we only list a few of these properties for completeness. Setting

$$\omega^0=(\surd x)e^\mu du, \quad \omega^1=e^\gamma dx, \quad \omega^2=(\surd x)e^{-\mu}dv, \quad \text{and } \omega^3=e^\gamma dz, \qquad (4.8)$$

we obtain

$$M=\begin{bmatrix} -\frac{1}{2}f_1+f_2 & 0 & f_3 \\ 0 & f_1 & 0 \\ f_3 & 0 & -\frac{1}{2}f_1-f_2 \end{bmatrix}, \qquad (4.9)$$

$$N=0, \qquad (4.10)$$

$$\lambda_1=f_1, \quad \lambda_{2,3}=-\tfrac{1}{2}f_1\pm(f_2{}^2+f_3{}^2)^{1/2}, \qquad (4.11)$$

where

$$f_1=e^{-2\gamma}(\dot\mu^2+\mu'^2-1/4x^2),$$
$$f_2=e^{-2\gamma}[x\dot\mu(\dot\mu^2-3\mu'^2-1/4x^2)-\ddot\mu-\dot\mu/x],$$
$$f_3=e^{-2\gamma}[x\mu'(3\dot\mu^2-\mu'^2-1/4x^2)-\dot\mu'-\mu'/2x]. \qquad (4.12)$$

There are two, and only two, independent second-order

[16] R. B. Hoffman, J. Math. Phys. **10**, 953 (1969).
[17] C. W. Misner, J. Math. Phys. **4**, 924 (1963), Appendix A.
[18] See A. S. Petrow, *Einstein-Räume*, translated by H. Treder (Akademie-Verlag, Berlin, 1964), p. 90.
[19] R. Debever, Compt. Rend. **249**, 1324 (1959).
[20] R. K. Sachs, Proc. Roy. Soc. (London) **A264**, 309 (1961). An equivalent formulation in the spinor formalism is given by R. Penrose, Ann. Phys. (N. Y.) **10**, 171 (1960).
[21] These scalars are discussed in some detail by Sachs in Ref. 20, but the notation used here is that given by W. Kundt, Z. Physik **163**, 77 (1961).

differential invariants

$$f_1, \quad f_2^2+f_3^2. \tag{4.13}$$

All eigenvalues are necessarily real.

In the case where $\mu=\mu(x)$ we obtain the most general solution

$$ds^2=x^{2c}du^2-x^{2-2c}dv^2-x^{2c(c-1)}(dx^2+dz^2). \tag{4.14}$$

This solution is flat for $c=0$, 1 and of type D for $c=\frac{1}{2}$, 2, -1. The solutions of Eq. (4.7) as well as the optical parameters for these algebraically special solutions are given in Table I.

B. L Class

For the L class we introduce the differential forms

$$\omega^0=(\sqrt{x})(\cos\mu\, du+\sin\mu\, dv), \quad \omega^1=e^\gamma dx,$$
$$\omega^2=(\sqrt{x})(\sin\mu\, du-\cos\mu\, dv), \quad \omega^3=e^\gamma dz,$$

with the condition $\mu\neq 0$, obtaining

$$M=\begin{bmatrix} -\frac{1}{2}f_1 & 0 & 0 \\ 0 & f_1 & 0 \\ 0 & 0 & -\frac{1}{2}f_1 \end{bmatrix}, \quad N=\begin{bmatrix} -f_2 & 0 & f_3 \\ 0 & 0 & 0 \\ f_3 & 0 & f_2 \end{bmatrix}, \tag{4.15}$$

$$\lambda_1=f_1, \quad \lambda_{2,3}=-\tfrac{1}{2}f_1\pm i(f_2^2+f_3^2)^{1/2}, \tag{4.16}$$

where

$$f_1=-e^{-2\gamma}(\dot{\mu}^2+\mu'^2+1/4x^2),$$
$$f_2=e^{-2\gamma}[x\mu'(\mu'^2-3\dot{\mu}^2)-\dot{\mu}'-3\mu'/4x], \tag{4.17}$$
$$f_3=e^{-2\gamma}[x\dot{\mu}(\dot{\mu}^2-3\mu'^2)+\ddot{\mu}+5\dot{\mu}/4x].$$

The only L solution with real eigenvalues, i.e., that for which $f_2=f_3=0$, is that with μ constant, and that particular solution is a Weyl solution. In general we expect two independent second-order differential invariants

$$f_1, \quad f_2^2+f_3^2. \tag{4.18}$$

Since the eigenvalues are all different, we conclude that the L solutions are of algebraic type I.

Although the Killing equations

$$l_{\alpha;\beta}+l_{\beta;\alpha}=0$$

have not been solved in general for the L class, one can solve these equations simultaneously with

$$\epsilon^{\alpha\beta\gamma\delta}l_\beta l_{\gamma;\delta}=0, \tag{4.19}$$

where $\epsilon^{\alpha\beta\gamma\delta}$ is the Levi-Civita tensor density; Eq. (4.19) is the necessary and sufficient condition that l_α be hypersurface orthogonal. When $\mu=\mu(x)$ we have three and only three independent Killing vectors

$$l^\mu=\delta_0{}^\mu, \quad \delta_2{}^\mu, \quad \text{and} \quad \delta_3{}^\mu, \tag{4.20}$$

the latter being spacelike and hypersurface-orthogonal. In all other cases there are no hypersurface-orthogonal Killing vectors, so that all L solutions must be stationary. If $\mu=$const, the corresponding QDF can be transformed to the QDF (4.14) with $c=\frac{1}{2}$; this solution admits four independent Killing vectors including a timelike and a spacelike hypersurface-orthogonal vector.[1]

TABLE I. Null rays and optical parameters for the algebraically special solutions for the QDF (4.14).

c	k^μ	$\lvert\sigma\rvert$	θ	ω	$\lvert\Omega\rvert$
2	$(1,0,0,\pm1)$	0	0	0	$\frac{1}{2}x^{-3}$
$\frac{1}{2}$	$(1,0,\pm1,0)$	0	0	0	$(4x)^{-3/4}$
-1	$(1,\pm x^{-3},0,0)$	0	0	0	0

Returning to the case where $\mu'=0$,[22] we have

$$2\mu=c\ln|x|,$$

in which c is constant, and

$$\lambda_1=-\tfrac{1}{4}(c^2+1)x^{(c^2-3)/2},$$

$$\begin{pmatrix}\lambda_2\\ \lambda_3\end{pmatrix}=\tfrac{1}{8}(c^2+1)(1\pm ic)x^{(c^2-3)/2}.$$

In this case we find one and only one independent second-order differential invariant $x^{(c^2-3)/2}$. Note that for $c^2<3$ the associated space is asymptotically flat in the sense that this invariant approaches zero as $x\to\infty$ and is singular at $x=0$; for $c^2>3$ the roles of $x=0$ and ∞ are reversed. In the case where $c^2=3$ the differential invariant reduces to a constant.

If we now let τ denote an affine parameter and set

$$\hat{h}=dh/d\tau, \quad 2\mathfrak{L}=ds^2, \quad 2\mu=c\ln|x|,$$

we obtain upon dividing the QDF (2.10) by $d\tau^2$

$$2\mathfrak{L}=x[\cos(c\ln|x|)](\hat{u}^2-\hat{v}^2)+2x[\sin(c\ln|x|)]\hat{u}\hat{v}-e^{2\gamma}(\hat{x}^2+\hat{z}^2)\equiv\epsilon,$$

where ϵ is a constant of the motion; τ is chosen such that ϵ is $+1$, 0, or -1 according as ds^2 is timelike, null, or spacelike, respectively. The geodesic equations are obtained from $\mathfrak{L}$ by the variational principle

$$\delta\int\mathfrak{L}(x^\mu,\hat{x}^\mu)d\tau=0.$$

Since the coordinates u, v, and z are cyclic, we have immediately three additional constants of the motion:

$$P_u\equiv\partial\mathfrak{L}/\partial\hat{u}=x\cos(c\ln|x|)\,\hat{u}+x\sin(c\ln|x|)\,\hat{v},$$
$$P_v\equiv\partial\mathfrak{L}/\partial\hat{v}=x\sin(c\ln|x|)\,\hat{u}-x\cos(c\ln|x|)\,\hat{v},$$

and

$$P_z=\partial\mathfrak{L}/\partial\hat{z}=x^{(c^2+1)/2}\hat{z}.$$

[22] Mathematical properties of the W-, L-, and S-class solutions with $\mu'=0$ are discussed in detail in R. Hoffman, Ph.D. thesis, Lehigh University, 1967 (unpublished).

Solving for $\hat{u}$, $\hat{v}$, $\hat{z}$, and $\hat{x}^2$, we find

$$\hat{u}=x^{-1}[P_u\cos(c\ln|x|)+P_v\sin(c\ln|x|)],$$
$$\hat{v}=x^{-1}[P_u\sin(c\ln|x|)-P_v\cos(c\ln|x|)],$$
$$\hat{z}=x^{-(c^2+1)/2},$$

and

$$\hat{x}^2=x^{-(c^2+1)/2}\{-\epsilon+x^{-1}[\cos(c\ln|x|)(P_u^2-P_v^2)+2P_uP_v\sin(c\ln|x|)]+P_z^2x^{-(c^2+1)/2}\}.$$

We see, therefore, $\hat{u}$, $\hat{v}$, $\hat{z}$, and $\hat{x}$ are finite everywhere except at $x=0$; since, except for $c^2=3$, the associated space is either singular or flat at $x=0$ we conclude that these spaces are geodesically complete in the sense that the geodesics can be continued for arbitrarily large values of the affine parameter provided they do not strike the singularity. If $c^2=3$ the space is not geodesically complete.

C. S Class

Finally, for this class we introduce the differential forms

$$\omega^0=(\sqrt{x})[(u+\tfrac{1}{4})du+(\mu-7/4)dv],\quad \omega^1=x^{-1/4}dx,$$
$$\omega^2(\sqrt{x})=[(\mu-\tfrac{1}{4})du+(\mu-9/4)dv],\quad \omega^3=x^{-1/4}dz,$$

obtaining

$$M=x^{1/2}\begin{pmatrix}\tfrac{1}{2}f_1-f_2 & 0 & -f_3\\ 0 & -f_1 & 0\\ -f_3 & 0 & \tfrac{1}{2}f_1+f_2\end{pmatrix},\tag{4.21}$$

$$N=x^{1/2}\begin{pmatrix}-f_3 & 0 & f_2\\ 0 & 0 & 0\\ f_2 & 0 & f_3\end{pmatrix},$$

$$\lambda_1=-\tfrac{1}{4}x^{-3/2},\quad \lambda_2=\lambda_3=\tfrac{1}{8}x^{-3/2},\tag{4.22}$$

where

$$f_1=1/4x^2,\quad f_2=2\ddot{\mu}+5\dot{\mu}/2x,\quad f_3=2\dot{\mu}'+3\mu'/2x.\tag{4.23}$$

We have one, and only one, independent second-order differential invariant

$$x^{-3/2}.\tag{4.24}$$

Since there are only two distinct eigenvalues, these solutions must be of algebraic type II or D; we find one, and only one, solution

$$k^\mu=(1,0,-1,0)\tag{4.25}$$

of Eq. (4.7), and the solutions must be type II. We also find k^μ to be shear-, twist-, and expansion-free with a rotation

$$|\Omega|=(4x)^{-3/2}.\tag{4.26}$$

The Killing equations can be solved in general for this class of solutions, yielding two, and only two, independent solutions

$$l^\mu=(1,0,-1,0)\tag{4.27}$$

and

$$l^\mu=(1,0,1,0),\tag{4.28}$$

except when

$$\mu=mz+A\ln|x|+B,\quad A,m\text{ not both zero},\tag{4.29}$$

or

$$\mu=[AJ_0(mx)+BH_0(mx)]e^{mz},\quad A,B\text{ not both zero},\tag{4.30}$$

where A, N, and m are constants and J_0 and H_0 are the zero-order Bessel and Hankel functions, respectively. In this case we find the additional Killing vectors

$$l^\mu=[m(u+v),0,-m(u+v),-2]\tag{4.31}$$

and

$$l^\mu=[m(u+2v),0,-mv,-2],\tag{4.32}$$

respectively. The Killing vector in Eq. (4.27) is null and hypersurface-orthogonal for all solutions of this class; the only other hypersurface-orthogonal Killing vector occurs when $\mu'=0$ [i.e., $m=0$ in Eq. (4.29)], and is the spacelike vector in Eq. (4.31) with $m=0$. We thus conclude that the S solutions are all stationary.

If μ is a solution of Eq. (2.12) the solution $c^2\mu$ (where $c=\text{const}\neq0$) is equivalent to the original solution, for if we make the coordinate transformation

$$u=c\bar{u}+(c-1/c)\bar{v},\quad v=(1/c)\bar{v}$$

in the QDF (2.11), we obtain

$$ds^2=xc^2\mu d\bar{u}^2+2x(c^2\mu-1)d\bar{u}d\bar{v}+x(c^2\mu-2)d\bar{v}^2-e^{2\gamma}(dx^2+dz^2).$$

Specializing to the case where $\mu'=0$,[22] we have

$$\mu=c\ln|x|,\quad c=\pm1,0,$$

and

$$2\mathfrak{L}=xc\ln|x|\hat{u}^2+2x(c\ln|x|-1)\hat{u}\hat{v}+x(c\ln|x|-2)\hat{v}^2-e^{2\gamma}(\hat{x}^2+\hat{z}^2)\equiv\epsilon,$$
$$\hat{u}=x^{-1}[-(c\ln|x|-2)P_u+(c\ln|x|-1)P_v],$$
$$\hat{v}=x^{-1}[(c\ln|x|-1)P_u-c\ln|x|P_v],$$
$$\hat{z}=x^{1/2}P_z,$$
$$\hat{x}^2=x^{1/2}\{-\epsilon+x^{-1}[-(c\ln|x|-2)P_u^2+2(c\ln|x|-1)\times P_uP_v-c\ln|x|P_v^2]+x^{1/2}P_z^2\}.$$

Since all solutions of the S class are singular at $x=0$ and asymptoically flat as $x\to\infty$, we conclude that these solutions are geodesically complete in the same sense as that stated previously since $\hat{u}$, $\hat{v}$, $\hat{z}$, and $\hat{x}$ are bounded everywhere except at $x=0$, ∞.

V. CONCLUSIONS

A necessary condition that a solution of the vacuum field equations for the QDF (1.1) be of type III or N is that $x^{-1}f$, $x^{-1}l$, and $x^{-1}m$ be arbitrary functions of the single function μ, which may be any solution of Eq.

(2.12). Finding that there exist no such solutions subject to the condition that $x^2=fl+m^2$ for which the eigenvalues vanish, we conclude that solutions of type III or N are possible only if Δ^2 is constant. In Ref. 16 it was shown that, in this case, the solutions form a subclass of the so-called plane-fronted gravitational waves which are necessarily type N. Investigation of solutions for which f, l, and m have the previously stated functional dependence shows that all such solutions for which $fl+m^2=x^2$ are either of the Weyl class or one of two classes, designated by L and S, respectively, obtainable from the W class.

The L class was first discovered by Lewis[4] and has the QDF (2.10) with γ a solution of Eq. (2.14). These solutions are stationary type-I solutions having no hypersurface-orthogonal Killing vectors except for a spacelike vector when $\mu'=0$. The eigenvalues of the Riemann tensor are all distinct, one being real and the other two complex conjugates; in general we expect two independent second-order differential invariants [see Eqs. (4.17) and (4.18)], but when $\mu'=0$ we obtain only one.

Van Stockum[5] obtained the first solution of the S class in the particular case where $\mu=-\ln|x|$. This class, having the QDF (2.11) with $\gamma=-\frac{1}{4}\ln|x|$, is characterized by type-II stationary solutions with a single shear-, twist-, expansion-free double-degenerate geodesic ray which is also a null hypersurface-orthogonal Killing vector. The rotation is proportional to the single second-order differential invariant, which is $x^{-3/2}$ for all S class solutions, and the two distinct eigenvalues are real. All solutions of this class are asymptotically flat in the sense that this differential invariant vanishes as $x\rightarrow\infty$, and all of these solutions are singular at $x=0$. In general there are two, and only two, Killing vectors except when μ has the form in Eq. (4.29) or (4.30), in which case there is one additional Killing vector given by Eq. (4.31) or (4.32), respectively. The only additional hypersurface-orthogonal Killing vector is the spacelike vector obtained when $\mu'=0$.

All of these solutions have been obtained and discussed in canonical coordinates. The S-class solutions, for example, are asymptotically flat in the sense that the Riemann tensor vanishes as $x\rightarrow\infty$, but they are not asymptotically Minkowskian. Thus, it is not clear whether any theorems derived under the assumption that the solutions for the QDF (1.1) in canonical coordinates be asymptotically Minkowskian still hold for these solutions, and the question naturally arises as to whether these two assumptions are so restrictive as to exclude physically interesting solutions. These solutions, when viewed strictly in terms of the metric tensor itself, are not at all well behaved; on the other hand, when analyzed in terms of the various intrinsic properties, most of these difficulties are seen to arise simply because of the choice of coordinates. One is led, in fact, to the conjecture that canonical coordinates, although suited to the discussion of the various mathematical properties of a given metric tensor, are not suited to the physical interpretation of the results; however, attempts to find a coordinate system suited to a physical interpretation of the above solutions have been thus far unsuccessful. Finally, one has the usual problem of deciding upon the range of coordinates, a problem which is especially difficult when the metric is not asymptotically Minkowskian.

ACKNOWLEDGMENTS

The author wishes to express his sincere appreciation to Professor P. Havas for his encouragement and guidance during the course of this investigation; he is also indebted to Dr. G. Roemhild and Dr. H. Goenner for the many helpful discussions of this topic. The hospitality of Temple University, where this work was completed, is gratefully acknowledged.

APPENDIX A

For the metric tensor in the QDF (1.1) we find

$$g^{00}=l/\Delta^2,\quad g^{02}=-m/\Delta^2,\quad g^{22}=-f/\Delta^2,\quad g^{11}=g^{33}=e^{-2\gamma},$$

and all other $g^{\alpha\beta}=0$. The affine connection defined by

$$\Gamma_{\alpha}{}^{\gamma}{}_{\beta}=\tfrac{1}{2}g^{\gamma\sigma}(g_{\alpha\sigma,\beta}+g_{\beta\sigma,\alpha}-g_{\alpha\beta,\sigma})$$

is given in Table II.

The field equations defined by

$$R_{\alpha\beta}=\Gamma_{\alpha}{}^{\gamma}{}_{\gamma,\beta}-\Gamma_{\alpha}{}^{\gamma}{}_{\beta,\gamma}+\Gamma_{\alpha}{}^{\mu}{}_{\gamma}\Gamma_{\beta}{}^{\gamma}{}_{\mu}-\Gamma_{\alpha}{}^{\gamma}{}_{\beta}\Gamma_{\gamma}{}^{\mu}{}_{\mu}$$

reduce to

$$R_{00}=-\tfrac{1}{2}[\ddot{f}+f''-x^{-1}\dot{f}+f(h_1+h_2)]=0,$$
$$R_{02}=\tfrac{1}{2}[\ddot{m}+m''-x^{-1}\dot{m}+m(h_1+h_2)]=0,$$
$$R_{22}=\tfrac{1}{2}[\ddot{l}+l''-x^{-1}\dot{l}+l(h_1+h_2)]=0,$$
$$R_{11}=\ddot{\gamma}+\gamma''-x^{-1}\dot{\gamma}-\tfrac{1}{2}h_1=0,$$
$$R_{33}=\ddot{\gamma}+\gamma''+x^{-1}\dot{\gamma}-\tfrac{1}{2}h_2=0,$$
$$R_{13}=x^{-1}\gamma'+h_3=0,$$
$$R_{01}=R_{03}=R_{12}=R_{23}=0,$$

TABLE II. Components of the affine connection for the QDF (1.1).

γ	0	1	2	3
α β				
0 0	0	$\frac{1}{2}\dot{f}e^{2\gamma}$	0	$\frac{1}{2}f'e^{2\gamma}$
0 1	$(l\dot{f}+m\dot{m})/2\Delta^2$	0	$(f\dot{m}-m\dot{f})/2\Delta^2$	0
0 2	0	$-\frac{1}{2}\dot{m}e^{2\gamma}$	0	$-\frac{1}{2}m'e^{2\gamma}$
0 3	$(lf'+mm')/2\Delta^2$	0	$(fm'-mf')/2\Delta^2$	0
1 1	0	$\dot{\gamma}$	0	$-\gamma'$
1 2	$(m\dot{l}-l\dot{m})/2\Delta^2$	0	$(f\dot{l}+m\dot{m})/2\Delta^2$	0
1 3	0	γ'	0	$\dot{\gamma}$
2 2	0	$-\frac{1}{2}\dot{l}e^{2\gamma}$	0	$-\frac{1}{2}l'e^{2\gamma}$
2 3	$(ml'-lm')/2\Delta^2$	0	$(fl'+mm')/2\Delta^2$	0
3 3	0	$-\dot{\gamma}$	0	γ'

where

$$h_1=\Delta^{-2}(\dot{f}\dot{l}+\dot{m}^2),\quad h_2=\Delta^{-2}(f'l'+m'^2),$$
$$h_3=\tfrac{1}{4}\Delta^{-2}(\dot{f}l'+f'\dot{l}+2\dot{m}m').$$

APPENDIX B

To compute the curvature tensor using the differential forms, we use the procedure outlined by Misner.[17] If we write

$$ds^2=g_{\alpha\beta}\omega^\alpha\omega^\beta,$$

the connection forms are completely determined by

$$dg_{\alpha\beta}=\omega_{\alpha\beta}+\omega_{\beta\alpha},$$
$$d\omega^\alpha=-\omega^\alpha{}_\beta\wedge\omega^\beta,$$

where $\wedge$ denotes the exterior product. From Eq. (4.1) we see that the first of these yields

$$\omega_{(\alpha\beta)}=0.$$

We then solve the second for $\omega^\alpha{}_\beta$. The curvature forms $\theta^\alpha{}_\beta$ are then obtained from

$$\theta^\alpha{}_\beta=d\omega^\alpha{}_\beta+\omega^\alpha{}_\mu\wedge\omega^\mu{}_\beta.$$

From the connection forms we obtain the affine connection since

$$\omega^\alpha{}_\beta=\Gamma^\alpha{}_{\beta\mu}\omega^\mu,$$

and from the curvature forms we obtain the curvature tensor since

$$\theta^\alpha{}_\beta=\tfrac{1}{2}R^\alpha{}_{\beta\gamma\mu}\omega^\gamma\wedge\omega^\mu;$$

the geometrical objects $\Gamma^\gamma{}_{\alpha\beta}$ and $R^\alpha{}_{\beta\gamma\mu}$ are of course the components of the affine connection and the curvature tensor with respect to the basis ω^α, and their indices, as well as those of ω^α, $\omega^\alpha{}_\beta$, and $\theta^\alpha{}_\beta$, are raised and lowered with $\eta_{\alpha\beta}$. Thus we see that $R_{\alpha\beta\gamma\delta}$ has the symmetries indicated in Sec. II and $\Gamma_{\alpha\beta\gamma}$ has, in general, the single symmetry

$$\Gamma_{(\alpha\beta)\gamma}=0.$$

We now list the forms $d\omega^\alpha$ and $\omega^\alpha{}_\beta$ for the three classes of solutions; the forms $d\omega^\alpha{}_\beta$ and $\theta^\alpha{}_\beta$ are easily obtained from the definition of $\theta^\alpha{}_\beta$ in terms of $R^\alpha{}_{\beta\gamma\delta}$ and $d\omega^\alpha{}_\beta$.

A. *W* Class

$$d\omega^0=e^{-\gamma}[(1/2x-\dot{\mu})\omega^1\wedge\omega^2+\mu'\omega^2\wedge\omega^3],$$
$$d\omega^1=-e^{-\gamma}\gamma'\omega'\wedge\omega^3,$$
$$d\omega^2=e^{-\gamma}[(1/2x-\dot{\mu})\omega^1\wedge\omega^2+\mu'\omega^2\wedge\omega^3],$$
$$d\omega^3=e^{-\gamma}\dot{\gamma}\omega^1\wedge\omega^3,$$
$$\omega^0{}_1=e^{-\gamma}(1/2x+\dot{\mu})\omega^0,$$
$$\omega^0{}_2=0,$$
$$\omega^0{}_3=e^{-\gamma}\mu'\omega^0,$$
$$\omega^1{}_2=e^{-\gamma}(\dot{\mu}-1/2x)\omega^2,$$
$$\omega^1{}_3=e^{-\gamma}(\gamma'\omega^1-\dot{\gamma}\omega^2),$$
$$\omega^2{}_3=-e^{-\gamma}\mu'\omega^2.$$

B. *L* Class

$$d\omega^0=e^{-\gamma}[-(1/2x)\omega^0\wedge\omega^1-\dot{\mu}\omega^1\wedge\omega^2+\mu'\omega^2\wedge\omega^3],$$
$$d\omega^1=-e^{-\gamma}\gamma'\omega^1\wedge\omega^3,$$
$$d\omega^2=e^{-\gamma}[(1/2x)\omega^1\wedge\omega^2-\dot{\mu}\omega^0\wedge\omega^1-\mu'\omega^0\wedge\omega^3],$$
$$d\omega^3=e^{-\gamma}\dot{\gamma}\omega^1\wedge\omega^3,$$
$$\omega^0{}_1=(1/2x)e^{-\gamma}\omega^0-\dot{\mu}\omega^2,$$
$$\omega^0{}_2=0,$$
$$\omega^0{}_3=-\mu'e^{-\gamma}\omega^2,$$
$$\omega^1{}_2=-\dot{\mu}e^{-\gamma}\omega^0-(1/2x)e^{-\gamma}\omega^2,$$
$$\omega^1{}_3=\gamma'e^{-\gamma}\omega^1-\dot{\gamma}e^{-\gamma}\omega^3,$$
$$\omega^2{}_3=\mu'e^{-\gamma}\omega^0.$$

C. *S* Class

$$d\omega^0=x^{1/4}[-(1/2x+2\dot{\mu})\omega^0\wedge\omega^1-2\mu'\omega^0\wedge\omega^3-2\dot{\mu}\omega^1\wedge\omega^2+2\mu'\omega^2\wedge\omega^3],$$
$$d\omega^1=0,$$
$$d\omega^2=x^{1/4}[-2\dot{\mu}\omega^0\wedge\omega^1-2\mu'\omega^0\wedge\omega^3+(1/2x-2\dot{\mu})\omega^1\wedge\omega^2+2\mu'\omega^2\wedge\omega^3],$$
$$d\omega^3=-\tfrac{1}{4}x^{-3/4}\omega^1\wedge\omega^3,$$
$$\omega^0{}_1=x^{1/4}[(1/2x+2\dot{\mu})\omega^0-2\dot{\mu}\omega^2],$$
$$\omega^0{}_2=0,$$
$$\omega^0{}_3=x^{1/4}(2\mu'\omega^0-2\mu'\omega^2),$$
$$\omega^1{}_2=x^{1/4}[-2\dot{\mu}\omega^0+(2\dot{\mu}-1/2x)\omega^2],$$
$$\omega^1{}_3=\tfrac{1}{4}x^{-3/4}\omega^3,$$
$$\omega^2{}_3=x^{1/4}(2\mu'\omega^0-2\mu'\omega^2).$$