Single-Particle Energy Levels Based on the Velocity-Dependent Potential*

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An effective velocity-dependent nucleon-nuclear potential based upon a relatively realistic N-N interaction has been used in this work. This effective potential has been approximated by the analytically solvable Morse function in a simple way. The neutron energy states characterized by the Morse parameters have been calculated for nuclides with $2\leq A^{1/3}\leq 7$ using the independent-particle model. The Morse parameters have been expressed as functions of A and l. The single-particle energy levels obtained from this work have been compared with the available data and other theoretical and numerical results.

1. INTRODUCTION

It is well understood from recent studies of nucleon-nucleon interactions based on meson theory[1] that the fundamental nuclear force has a significant amount of dependence on the relative velocities of the particles involved. It is, therefore, expected that the over-all nucleon-nuclear potential should exhibit some dependence on the velocities of the individual particles. Consequently, velocity-dependent potentials have appeared in many recent studies pertaining to nuclear and particle physics. Most of the workers who have dealt with velocity-dependent potentials have used numerical techniques[2] in solving the Schrödinger equation. There is nothing wrong with numerical methods; in fact, the results thus obtained have proved quite satisfactory. They tend to favor the proposition that velocity-dependent terms should be included in the nuclear potentials for finite nuclei calculation. However, an analytic method for treating velocity-dependent potentials is more desirable. There is an inherently greater insight into a problem offered by explicit analytic solutions, and the analytic expressions for the wave functions and eigenvalues are generally easier to handle in applications of radial wave functions than their corresponding counterparts numerically obtained. In principle, one can approximate a potential containing velocity-dependent terms by an analytic potential so that the Schrödinger equation can be solved in closed form giving analytic wave function and eigenvalue formulas.

The main purpose of these studies is to give a simple method for using a velocity-dependent nucleon-nuclear potential in nuclear problems. The nucleon-nuclear potential, studied in this work, is based on a relatively realistic nucleon-nucleon interaction and is presented in Sec. 2. In Sec. 3 an effective potential is obtained with which to replace the original interaction operator, and in Sec. 4 a simple method for approximating the so-called effective potential by an analytically solvable potential is given. In Sec. 5, using the independent-particle model, this approximation is applied to neutron states for nuclides with $2\leq A^{1/3}\leq 7$. The results thus obtained are then compared with other results and experimental data.

2. VELOCITY-DEPENDENT POTENTIAL

In finite uniform nuclear matter the properties are the same at all points of space, and the energy of a particle cannot depend on its location, but may depend on its velocity. If this is so, then the potential can be considered as a nonlocal potential in ordinary space. This means that the energy of a particle at a point $\mathbf{r}$ depends not only on the wave function at $\mathbf{r}$, but also on the wave function at other points $\mathbf{r}'$ near $\mathbf{r}$. More precisely, the energy operator is not diagonal in coordinate space. Such a nonlocal potential can be introduced into the Schrödinger equation in the form

$$-\frac{\hbar^2}{2M}\Delta_r\psi(\mathbf{r})+\int K(\mathbf{r},\mathbf{r}')\psi(\mathbf{r}')d\mathbf{r}'=E\psi(\mathbf{r}), \quad (1)$$

where the convenient kernel, given by[3]

$$K(\mathbf{r},\mathbf{r}')=-V_0 f(\tfrac{1}{2}(\mathbf{r}+\mathbf{r}'))\delta_b(|\mathbf{r}-\mathbf{r}'|), \quad (2)$$

may be used in (1). The last factor, δ_b, represents the nonlocal character of f; it is any kind of normalized

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[1] See Rev. Mod. Phys. **39**, (1967), Session D.

[2] O. Rojo and L. M. Simmons, Phys. Rev. **125**, 273 (1962) in N-N potential study; P. J. Wyatt *et al.*, *ibid.* **119**, 1031 (1960); B. Rozsnyai, *ibid.* **124**, 860 (1961) in nuclear models study.

[3] W. E. Frahn and R. H. Lemmer, Nuovo Cimento **5**, 1564 (1957); **6**, 664 (1957). The kernel (2) may nevertheless be a reasonable approximation, and it can certainly be handled with relative ease. The authors have shown that Eq. (2) leads to the single-particle operator of the form

$$\frac{1}{4}\left(\frac{1}{2m}\mathbf{p}^2+\mathbf{p}\frac{1}{m}\mathbf{p}+\mathbf{p}^2\frac{1}{2m}\right)+V(\mathbf{r}),$$

where

$$\frac{1}{m}=\frac{1}{M}\left(1-\frac{MV(\mathbf{r})b^2}{2\hbar^2}\right).$$

In the central portion of the nucleus, where $V(\mathbf{r})$ is presumably constant, the effect is simply the introduction of effective mass. In the nuclear surface, where $V(\mathbf{r})$ varies, the effect is more complicated, but, for practical purposes, simple potentials can be empirically adjusted to give reasonable single-particle functions.

distribution of short range b. Besides the velocity-dependent contribution, an over-all potential of the form (1) can be understood as reflecting the nucleon correlations existing in nuclear matter, whereby the presence of a particle at position $\mathbf{r}$ influences the probability of finding another particle at point $\mathbf{r}'$ in the neighborhood of $\mathbf{r}$. This in turn affects the energy of the particle at $\mathbf{r}$ and leads to a potential energy of the form of the integral in (1).

Consider a nucleus of mass number $A+1$ as a system consisting of a nucleon of mass m moving in an average field due to the remaining part of the given system—say, a nucleus of mass number A. Let this average field be represented by the nonlocal potential. Frahn and Lemmer[3] have shown that in the so-called effective-mass approximation this nonlocal potential can be represented by a single-particle potential operator. The form of this potential operator is equivalent to one which has been studied extensively by several workers,[3–13] both in the nucleon-nucleon and nucleon-nuclear contexts. The potential that we consider is of the form

$$V(r)=-V_0 f(r)-(\delta\hbar^2/8\mu)\times[\Delta f(r)+2\nabla\cdot f(r)\nabla+f(r)\Delta], \quad (3)$$

where μ is the reduced mass of the system and δ is a parameter characterizing the degree of velocity dependence of the potential and given by[3]

$$\delta=V_0\mu b^2/2\hbar^2,$$

with

$$b^2=\tfrac{2}{3}\text{ fm}^2, \qquad V_0=70\text{ MeV}. \quad (4)$$

3. FORM FACTOR AND EFFECTIVE POTENTIAL

The form factor $f(r)$ in (3) is the real part of the average nuclear potential. Its form is best described as having an approximately uniform interior region and a diffuse surface which falls off rapidly within 2–4 fm beyond the rms radius. A great variety of potential shapes satisfying these criteria are readily available and have been used in the past. For computational simplicity and because of its continuity properties, we have used the well-known Saxon-Wood form factor

$$f(r)=1/(1+e^{(r-R)/d}), \quad (5)$$

where[14] $R=1.2A^{1/3}$ fm and $d=0.6485$ fm.

It can easily be shown on solving the Schrödinger equation of a system with potential (3) that the radial part of the wave function may be put into the form[15]

$$\chi''(x)-V(x,\epsilon_E)\chi(x)-[\epsilon_E^2/(1+\delta)]\chi(x)=0, \quad (6)$$

where $V(x,\epsilon_E)$ is the energy- and state-dependent potential equivalent to (3) and is given by

$$V(x,\epsilon_E)=-\frac{\epsilon_0^2 f}{1+\delta f}+\frac{1}{4}\frac{\delta f''}{(1+\delta f)}-\frac{1}{4}\frac{\delta^2 f'^2}{(1+\delta f)^2}+\frac{1}{2}\frac{\delta f'}{(1+\delta f)x}+\frac{l(l+1)}{x^2}-\frac{\delta\epsilon_E^2}{1+\delta}\frac{f-1}{1+\delta f}. \quad (7)$$

Equations (6) and (7) introduce $x=r/a$ as the distance variable where a is a convenient unit of length ($=1$ fm). $E_0=\hbar^2/2\mu a^2$ is used as the corresponding unit of energy. The function $\chi(x)=[1+\delta f(x)]^{1/2}G(x)$, where $G(x)/x$ is the usual radial wave function. The dimensionless energy parameters are defined as

$$\epsilon_E^2=-E/E_0, \qquad \epsilon_0^2=V_0/E_0,$$

where E is the eigenvalue which will be characterized below as the particle separation energy. Using these definitions, the parameter δ becomes $V_0/6E_0a^2$. Although the potential (7) has a direct dependence on the energy E, it is found upon generating this potential for nuclei with $2\leq A^{1/3}\leq 7$ in $l=1$ and 2 states that this dependence is rather insensitive to E. This is what was expected since the factor $f-1$ is usually a relatively small residual quantity in bound-state problems.

4. MORSE FUNCTION AND APPROXIMATION

The form of potential (3) used in this work has the same velocity dependence which appears to characterize the fundamental nucleon-nucleon interaction.[4–7] However, the over-all effective potential (7) which was derived from (3) is a simple form of the nucleon-nuclear potential. For finite-nuclei calculations, there are several important factors not included in (7) which should be considered in predicting nuclear properties, e.g., spin-orbit coupling, the Coulomb effect, the asymmetry term (which contributes largely to the line of β stability), the pairing energy, the shell effect, etc. These studies are exploratory in nature. Since at the outset of this work the primary concern is with qualitative results only and with keeping the calculations as simple as possible, the aforesaid terms have been ignored. Actually, aside from the gross effect due to the spin-orbit term the absence of these terms from (7) should be of little consequence. The Coulomb term is eliminated by itself since only the neutron states are being considered, and the remaining terms may be regarded as being taken care of in some way by the form factor (5).

The complicated effective potential given by (7) is approximated by an analytically solvable function, so

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[13] F. Tabakin and K. T. R. Davies, Phys. Rev. **150**, 793 (1966).
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that the eigenfunctions $\chi(x)$ and the eigenvalues E can be obtained in closed forms. A class of simple potentials is available for which the Schrödinger equation can be solved in terms of special functions.[16] In order to replace the potential (7) by some analytic function, the well-known Morse function is used as an illustrative example:

$$V_M = D\{1-\exp[-\beta(x-x_0)]\}^2 - D_0, \quad (8)$$

where D, D_0, β, and x_0 are the adjustable parameters. Since almost any reasonable matching procedure gives about the same parameters of the Morse function.[15] a relatively simple device has been employed to obtain them. The least-squares and χ^2 curve-fitting methods used for matching (7) and (8) give almost the same parameters for finite nuclei such as ^{17}O and ^{41}Ca. As expected, the minima of the two potentials are not significantly different and both are, roughly speaking, around the same point, x_0. This tendency is somewhat distorted because l and A vary. However, the coincidence of x_0 seems to favor slightly smaller values of l and A, whereas the minima tend to favor larger l and A. Since the purpose of this work is to develop the nuclear systematics with the simplest possible methods without involving much sophisticated calculations and programs, a simple procedure in replacing (7) by (8) has been adopted. Essentially, the minimum of the Morse potential (8) was matched to the minimum of the original effective interaction (7) at x_0. In order to obtain more reasonable expressions for the Morse parameters, a little matching was then allowed in the surface region. These parameters are expressed as empirical functions of A and l as follows:

$$D = E_0[9-13.8A^{-1/3}+0.2l(l+1)A^{-1/3}-0.12l(l+1)], \quad (9)$$

$$D_0 = E_0[2.72-1.5A^{-1/3}-0.3l(l+1)A^{-1/3}+0.012l(l+1)], \quad (10)$$

$$\beta = 1/a[3.8+0.3A^{1/3}+0.15l(l+1)]A^{-2/3}. \quad (11)$$

5. SINGLE-PARTICLE ENERGY

Using the Morse potential (8) the wave equation (6) can easily be solved for χ as long as $\exp(\beta x_0)$ is considerably larger than unity.[17] Thus only nuclides with $A \gtrsim 8$ are considered in this work. The allowed eigenvalues and eigenfunctions of (6) are obtained approximately for the first principal quantum number as[15,17]

$$E = -(1+\delta)[D_0 - \beta(DE_0)^{1/2} + \tfrac{1}{4}a^2\beta^2 E_0] \quad (12)$$

and

$$\chi = z^k e^{-z/2}, \quad (13)$$

where $z = 2(D/E_0)^{1/2}\exp[-\beta(x-x_0)]/\beta\hbar$ and $k^2 = [D-D_0-E/(1+\delta)]/E_0a^2\beta^2$.

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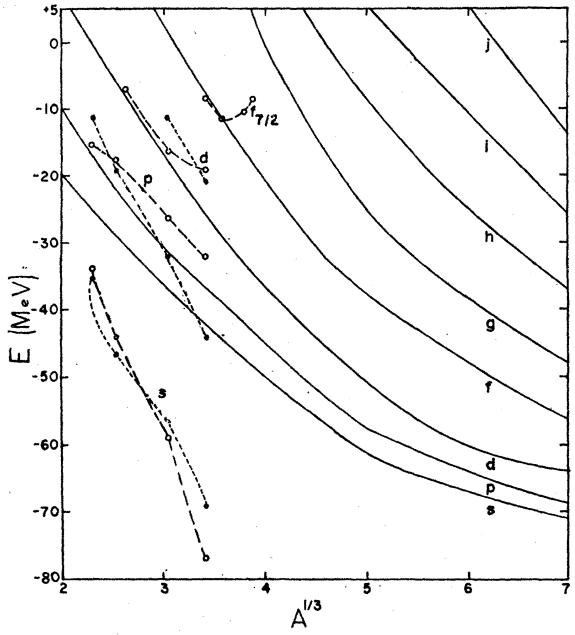

FIG. 1. Solid curves are the single-particle energy levels as functions of $A^{1/3}$ calculated from the velocity-dependent potential. Closed circles are also the single-particle energies for some finite nuclei but from Hartree-Fock calculations (Ref. 22). Open circles indicate the mean values of the observed data (Ref. 18). Dashed lines are merely to guide the eyes. The results of Hartree-Fock correspond to ^{12}C, ^{16}O, ^{28}Si, and ^{40}Ca. The observed values belong to ^{12}C, ^{16}O, ^{19}F, ^{28}Si, ^{40}Ca, ^{48}Ti, ^{56}Fe, and ^{58}Ni.

Figure 1 shows the single-particle energies for the neutron states $0 \leq l \leq 7$ in nuclei having $2 \leq A^{1/3} \leq 7$. These energies have been calculated from the eigenvalue formula (12) using the empirical expressions (9)–(11) for D, D_0, and β, respectively. Also, Fig. 1 shows the mean separation energies (without showing errors bars) experimentally determined[18–23] for some nuclei and energies calculated by other authors[22] using the Hartree-Fock method. Table I lists results for the neutron energy states without spin-orbit splitting for some of the finite nuclei. Although these energies have been calculated for all nuclei with $2 \leq A^{1/3} \leq 7$, Tables I and II show only the results for those nuclei for which the same results are available elsewhere either from other calculations using the velocity-dependent potential or from experiments. The velocity-dependent nucleon-nucleon interaction with a Gaussian form factor was used in the Hartree-Fock calculations,[22] but the spin-orbit term was not included. That is why no spin-orbit splitting occurs in their results.[22] The other numerical and semianalytical

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[22] S. J. Krieger *et al.*, Phys. Letters **22**, 607 (1966).
[23] P. C. Sood, Ph.D. thesis, Florida State University, Tallahassee, Florida, 1957 (unpublished).

TABLE I. Comparison of experimental and calculated separation energies of neutron (MeV) in nuclides ^{12}C, ^{16}O, ^{28}Si, and ^{40}Ca. The two numbers, whenever they occur in the same row, correspond to spin-orbit splitting. The upper number belongs to $j=l+\frac{1}{2}$ values and the lower one to $j=l-\frac{1}{2}$ values.

	This work	HF[a]	GDB[b]	Num[c]	Expt.
^{12}C					
s	25.0	35.2			34.0[d]
p	16.0	11.0			15.0[d]
^{16}O					
s	30.3	46.6	41.0	39.0	44.0±9[e]
p	22.0	19.0	21.6	20.9	19.0±1[e] 17[d]
			14.6	13.7	12.4±1[f]
d	4.0		5.4	5.1	4.1
			−1.9		0.9[f]
^{28}Si					
s	36.5	57.0			59.0[d]
p	30.5	31.7			26.0[d]
d	16.5	10.2			16.0[d]
^{40}Ca					
s	42.0	69.0	52.3	50.9	77.0±14[d]
p	37.5	44.0	34.3	36.3	32.0±4[d]
			31.1	32.0	
d	21.0	20.6	19.1	21.7	15.5[e] 19[d]
			11.3	13.7	8.3
	8.0		5.6	8.1	8.36[g]
				3.6	

[a] Reference 22.
[b] Reference 15.
[c] Quoted in Ref. 15.
[d] Reference 18.
[e] Reference 19.
[f] Reference 20.
[g] Reference 21.

results[15] obtained using a velocity-dependent potential with a polynomial-type form factor have spin-orbit splitting. Their spin-orbit interaction is of the Thomas type, and the polynomial form factor is given by[24]

$$\begin{aligned} f(t) &= 1 && \text{for } r<a_1 \\ &= \tfrac{1}{2}-\tfrac{15}{16}t+\tfrac{10}{16}t^3-\tfrac{3}{16}t^5 && \text{for } a_1<r<a_2 \\ &= 0 && \text{for } r>a_2, \end{aligned} \quad (14)$$

where $t=2[r-\frac{1}{2}(a_1+a_2)]/(a_2-a_1)$. The constants a_1 and a_2 were defined in terms the surface thickness T and half-falloff radius $R(-r_0A^{1/3})$ as $a_1\approx R-T$ and $a_2\approx R+T$. This form factor is somewhat similar to the Saxon-Wood type (6). Experimental values for the separation energies as functions of the mass number are available for the last particles but not for those in inner states. They are imprecise though for lower mass numbers. Unlike the other states, a discrepancy in the energy values for the s state was essentially expected due to the presence of the centrifugal term in potential (7). In order to eliminate or minimize this discrepancy, the energy values in the s state were extrapolated from higher states. While this discrepancy could not be completely removed this way, it was substantially reduced. From Fig. 1 it is apparent that the energy values for the s state obtained from these calculations differ consistently from other calculated or experimental values for nuclei in the same state. Some higher levels were also projected by extrapolating, and they are in good agreement with the previous numerical results.[23]

6. DISCUSSION AND CONCLUSIONS

In this work the single-particle energies calculated for neutrons are in rough agreement with the available data and with the results obtained from other calculations. The exception for the s level was explained in Sec. 5. Nevertheless, it seems appropriate to make a few remarks in general on the relatively small disagreements that one observes between this work and that compared with it. These discrepancies may be caused partly by the uncertainties associated with the previous results used in this comparison. In this regard one must bear in mind the following. (i) The separation energies of the particles in strongly bound states are imprecise. The inner particle data for $2s$-$1d$ shell nuclei have particularly very large uncertainties. (ii) The nucleon-nucleon force used in the Hartree-Fock calculations[22] was not very accurate and was oversimplified. The primary purpose of those calculations[22] was to study the method itself and to apply it to a few sample nuclei as illustrative examples and not to give precise energy levels. Accordingly, they should not be regarded as exact. (iii) The numerical and semianalytical results[15] inherited several uncertainties. Similar uncertainties are

TABLE II. Neutron separation energies (MeV) in the f states of nuclides. The experimental values have spin-orbit splitting corresponding to the $f_{7/2}$ state.

Nuclide	This work f state	Expt (Ref. 18) $1f_{7/2}$ State
^{45}Sc	10.5	6.9
^{48}Ti	12.0	11.4
^{51}V	13.0	8.0
^{52}Cr	13.1	10.5
^{56}Fe	16.1	10.2

[24] P. J. Wyatt *et al.*, Phys. Rev. **119**, 1031 (1960).

involved in the present work. Hence they are explained in connection with the possible source of errors noted in these calculations. In viewing these small disagreements, one must bear in mind the particular way this problem was treated considering the following facts. (a) The potential (7) used in these calculations was over simplified and did not include some of the important terms discussed earlier. (b) Although the criterion for matching the Morse potential to the effective potential (7) is fairly workable for the bound-state problem, one must not overlook the fact that the very method of approximation is rather cursory and does not have a strong mathematical foundation. (c) The form factor (6) is too flat [so is (14), used in Ref. 15] at the origin to make the effective potential look Morse-like, for the s state at least. The same form is used for static as well as velocity-dependent components of (7). (d) The values of the parameters (4) used in these calculations (and also in Ref. 15) are based upon the bound-state and scattering data of the last decade. While determining those values of the parameters (4) no consideration was given then to the data on the tightly bound states. (e) The differences of the order of a few MeV should be viewed in relationship to the value of the depth parameter V_0, which is 70 MeV.

The motivation for these studies has been to provide a simple technique for producing single-particle energy levels based on the velocity-dependent nucleon-nuclear potential and to maintain the beauty and simplicity of the independent-particle model. To sum up, a method of treating a velocity-dependent potential analytically has been presented here. This potential has its foundation in the basic nucleon-nucleon interaction.[3–13] A simple procedure of replacing the effective velocity-dependent potential by approximating Morse functions for nuclei of mass number $A=2^3$ to 7^3 in various states has been prescribed and used. The Morse parameters are given as explicit functions of A and l. The approximation method in this work makes the application of a velocity-dependent potential to nuclear problems as simple as the frequently used harmonic-oscillator potential, which is static and unfortunately highly unrealistic. The eigenvalue formula (12) is almost as convenient as the one belonging to harmonic-oscillator potential. Unlike the parameters used in calculating the energy in the harmonic-oscillator well, the parameters involved in the energy formula (12) have a direct dependence on A and l given by (9)–(11). Equation (12) together with (9)–(11) provides a simple means of calculating single-particle energy states analytically based on the velocity-dependent potential, thus avoiding the use of large electronic computers used previously for numerical solutions.[2] The choice of the velocity dependence of interaction (2) plays a significant part in determining the trend of energy levels. For example, comparing these results with the Hartree-Fock calculations,[22] fortuitously one finds that these calculations tend to produce the falloff of the energy levels with A whereas the Hartree-Fock calculations do not (see Fig. 1). This trend is in the right direction, and one would expect it because of the saturation property of nuclear forces.

Finally, it may be remarked that further definite many-body calculations based upon a rather more realistic nucleon-nucleon interaction for finite nuclei need to be performed and more reliable data taken. Then further studies such as the present work, but incorporating the aforesaid terms not included in potential (7), may be done which should predict the single-particle energy levels more confidently and be applicable to other nuclear problems.

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