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Analytically Solvable Problems in Radiative Transfer. I

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As is well known, the transfer of radiation in a medium is described by an integral equation, first given by Biberman and Holstein. They assumed that the emission coefficient is proportional to the absorption coefficient. After a discussion of the relation of this type of radiative transfer to Brownian motion, we solve the integral equation for a slab and for all line shapes of interest with and without hyperfine structure in the limit of high optical depth.

INTRODUCTION

The theory of imprisonment of resonance radiation is of fundamental importance for many problems in low-density plasmas. Compton[1] tried to use it to explain the behavior of low-voltage arcs.[2] He seems to have been the first who noticed a certain, perhaps only formal, analogy between the phenomenon of repeated absorption and re-emission, from which the effect stems, and Brownian Motion, and it was his suggestion that the phenomenon could be described by a diffusion equation. A few years later the diffusion equation was derived rigorously by Milne[3] from the basic equations of radiative transfer with the tacit assumption that the frequencies of the absorbed and re-emitted quanta are the same. It was shown that the predictions of the theory were not in agreement with the experiments.[4] Attention, therefore, remained focused on the problem. The notion that the phenomenon should be described by a diffusion equation even misled an author who had written down the correct initial equations.[5] Kenty[6] seems to have been the first who succeeded in solving the discrepancy between theory and experiment by taking into account the shape of the spectral line. He calculated an effective diffusion constant for the Doppler profile that was substantially in agreement with the experiment. The formulation of the problem in terms of an integral equation by Biberman[7] and Holstein[8] about 15 years later was physically much simpler and led to an expression for the Doppler profile that was in fair agreement with the one found by Kenty. Holstein[8] calculated the lowest eigenvalue of the integral equation for a number of line shapes and volumes by a variational procedure. Later on the Russian literature[9] showed progress towards an analytical approach. Hearn, Hummer and others,[10] following a different, but equivalent formulation of the problem common among astrophysicists,[11] calculated some interesting quantities numerically for a slab and Doppler or Voigt profiles, for instance, the line shape of the radiation emitted by a slab. In the meantime many experiments[12] were performed, mostly with the purpose of verifying the dependence of the lowest eigenvalue on the number of absorbing particles when the line shape is the Doppler profile.

Our aims in this paper are as follows. In the formulation of the problem by Biberman and Holstein it is clear that a diffusion equation is impossible. It is, however, not clear if this is due to the assumption that the frequency of the emitted quantum has no correlation with the frequency of the absorbed one, an assumption that often is fulfilled only partially (Doppler broadening, for instance). In other words, it is not clear if a diffusion equation will appear when we have only partial correlation so that the case treated by Biberman and Holstein is really a singular case. In Sec. I, therefore, we shall treat the problem from the point of view of the theory of random flights which led so many authors to believe that the phenomenon could be described by a diffusion equation. Our result is that the description of the phenomenon by a diffusion equation is a singular case, since it is possible only when the Fourier transform of the probability kernel, which we shall use, is Gaussian. The results obtained in Sec. I suggest a possibility of solving the Biberman-Holstein integral equation analytically.

We shall be concerned with this question, the *pièce de résistance* of this paper, in Sec. II. Eigenvalues and eigenfunctions are given for a slab, the line shapes being the Doppler, Voigt, and Lorentz profiles. We shall also deal with the important case of hyperfine structure. In a subsequent paper we shall apply the results to calculating the line shape of the radiation emerging from a slab, etc. The reader who is primarily interested in the results of Sec. II may immediately step over to this section and does not need to read Sec. I.

I. IMPRISONMENT OF RESONANCE RADIATION AND THE PROBLEM OF RANDOM FLIGHTS

Let us consider an infinite medium. The chance for a quantum emitted at a point with radiusvector $\vec{r}$ at frequency ν (we neglect natural line breadth) to be absorbed in a volume element $d\vec{r}\,'$ with radiusvector $\vec{r}\,'$ is given by[7,8]

$$d\vec{r}\,'\tau(\nu,\vec{r}-\vec{r}\,')\equiv d\vec{r}\,'k(\nu)\exp[-k(\nu)|\vec{r}-\vec{r}\,'|]/4\pi|\vec{r}-\vec{r}\,'|^2 \quad . \tag{1}$$

Stimulated emission is not taken into account. $k(\nu)$ is the absorption coefficient. It has been assumed that the number of ground-state atoms is constant. As can be readily verified

$$\int\int\int d\vec{r}\,'\tau(\nu,\vec{r}-\vec{r}\,')=1 \quad ,$$

giving the probability of being absorbed anywhere in space as unity. We introduce the conditional probability function $P(\nu_{j+1}|\nu_j)$ such that $P(\nu_{j+1}|\nu_j)d\nu_{j+1}$ gives the probability that a quantum absorbed at ν_j is emitted at ν_{j+1}. We now calculate the probability that a quantum emitted at $\vec{r}_0$ at frequency ν_0 in m steps ($m \gg 1$) reaches $\vec{r}_m$ with $\vec{r}_m-\vec{r}_0=\vec{R}$ irrespective its final frequency. This problem has been treated in detail by Chandrasekhar.[13] It is called the problem of random flights. The chance is

$$W(\vec{r}_m-\vec{r}_0=\vec{R},m,\nu_0)d\vec{R}d\nu_0=d\vec{R}d\nu_0\int d\nu_1\cdots d\nu_m\int d\vec{r}_1\cdots d\vec{r}_m\,\delta(\vec{r}_m-\vec{r}_0-\vec{R})\tau(\nu_0,\vec{r}_1-\vec{r}_0)$$
$$\times P(\nu_1|\nu_0)\,\tau(\nu_1,\vec{r}_2-\vec{r}_1)P(\nu_2|\nu_1)\cdots\tau(\nu_{m-1},\vec{r}_m-\vec{r}_{m-1})P(\nu_m,\nu_{m-1}) . \tag{2}$$

We note that the integrals over the space coordinates are now of the convolution type if P contains no space variables.[14] Since the Fourier transform of a convolution is the product of the Fourier transforms of both functions, it is convenient to calculate the Fourier transform of W. It will be shown that we can extract all needed quantities from it. Therefore we introduce the Fourier representation of the δ function.

$$\delta(\vec{r}_m-\vec{r}_0-\vec{R})=(2\pi)^{-3}\int\exp[i(\vec{r}_m-\vec{r}_0-\vec{R})\cdot\vec{\sigma}\,]\,d\vec{\sigma} .$$

With the aid of this representation of the δ function, the integral in Eq. (2) becomes a product of Fourier transforms of τ. The calculation of this transform is, fortunately, straightforward

$$\int(4\pi\rho^2)^{-1}\exp[-\rho k(\nu)+i\vec{\sigma}\cdot\vec{\rho}\,]\,\rho^2\sin\vartheta\, d\varphi d\vartheta d\rho=\sigma^{-1}\arctan[\sigma/k(\nu)] \tag{3}$$

with $\sigma=(\sigma_x{}^2+\sigma_y{}^2+\sigma_z{}^2)^{1/2}$. We insert Eq. (3) in Eq. (2) and readily find

$$W d\vec{R} d\nu_0 = \frac{d\vec{R} d\nu_0}{(2\pi)^3} \int d\vec{\sigma} \frac{e^{-i\vec{R}\cdot\vec{\sigma}}}{\sigma^m} \int \prod_{j=1}^{m} d\nu_j k(\nu_{j-1}) P(\nu_j | \nu_{j-1}) \arctan \frac{\sigma}{k(\nu_{j-1})} . \tag{4}$$

For the further calculations it is necessary to specify the functions $k(\nu)$ and $P(\nu_j | \nu_{j-1})$. $k(\nu)$ can be written as $k(\nu) = k_0 \mathfrak{L}(\nu)$, with $\mathfrak{L}(\nu)$ the line shape of the resonance radiation absorbed by the atoms normalized such that $\int \mathfrak{L}(\nu) d\nu = 1$. We consider two cases for $P(\nu_j | \nu_{j-1})$. (1) $P(\nu_j | \nu_{j-1}) = \delta(\nu_j - \nu_{j-1})$ [the frequency of the emitted quantum is the same as the frequency of the absorbed one, the natural line breadth considered as negligibly small compared with the breadth of $\mathfrak{L}(\nu)$]. (2) $P(\nu_j | \nu_{j-1}) = \mathfrak{L}(\nu_j)$ (the frequency of the emitted quantum is entirely uncorrelated with the absorbed one within the spectral line). For $\mathfrak{L}(\nu)$ we again consider two cases:

(a) $d\nu \mathfrak{L}(\nu) = \frac{1}{\pi} \frac{dx}{x^2+1}$, $x = \frac{2(\nu - \nu_0)}{\Delta\nu_L}$, $k_0 = \frac{2\pi e^2}{mc} \frac{Nf}{\Delta\nu_L}$, (Lorentz profile);

(b) $d\nu \mathfrak{L}(\nu) = \frac{dx}{\sqrt{\pi}} e^{-x^2}$; $x = \frac{2(\nu - \nu_0)}{\Delta\nu_D} (\ln 2)^{1/2}$, $k_0 = \frac{2\pi e^2}{mc} \frac{Nf}{\Delta\nu_D} (\ln 2)^{1/2}$, (Doppler profile).

ν_0 is the resonance frequency, $\Delta\nu_L$ and $\Delta\nu_D$ the Lorentz and Doppler breadths.[15] In case (1) the integrals are readily evaluated and Eq. (4), after multiplication with $\mathfrak{L}(\nu_0)$, becomes

$$\mathfrak{L}(\nu_0) W d\vec{R} d\nu_0 = d\vec{R} d\nu_0 \, \mathfrak{L}(\nu_0) (2\pi)^{-3} \int d\sigma \, e^{i\vec{\sigma}\cdot\vec{R}} \{[k(\nu_0)/\sigma] \arctan[\sigma/k(\nu_0)]\}^m . \tag{4a}$$

As is shown in the exposition given by Chandrasekhar, it is sufficient to retain only the first two terms of the expansion of arctan.[13] We have

$$\{[k(\nu_0)/\sigma] \arctan[\sigma/k(\nu_0)]\}^m \sim \{1 - \tfrac{1}{3}(\sigma/k(\nu_0))^2\}^m \sim \exp\{-\tfrac{1}{3} m [\sigma/k(\nu_0)]^2\}$$

and obtain, therefore, for Eq. (4) in case (1) the following expression:

$$W(\vec{R}, m) \, \mathfrak{L}(\nu_0) d\vec{R} d\nu_0 = d\vec{R} d\nu_0 \, \mathfrak{L}(\nu_0) (2\pi)^{-3} \int d\sigma e^{-i\vec{\sigma}\cdot\vec{R}} \exp\{-\tfrac{1}{3} m [\sigma/k(\nu_0)]^2\} . \tag{5}$$

We notice that the integral is the Fourier transform of a Gaussian distribution. The consequences of this will be discussed below. We first proceed to the calculation of the integral in Eq. (4) in the case that $P(\nu_j | \nu_{j-1}) = \mathfrak{L}(\nu_j)$. It is convenient to multiply Eq. (4) by $\mathfrak{L}(\nu_0)$ to perform the integration with respect to ν_0 and to omit the one with respect to ν_n. Equation (4) becomes now the Fourier transform of a product of integrals, namely,

$$W(\vec{R}, m, \nu_m) d\nu_m d\vec{R} = \mathfrak{L}(\nu_m) d\nu_m d\vec{R} (2\pi)^{-3} \int d\sigma \, e^{-\vec{R}\cdot\vec{\sigma}} \left(\int_0^\infty \mathfrak{L}(\nu)[k(\nu)/\sigma] \arctan[\sigma/k(\nu)] \, d\nu\right)^m . \tag{6}$$

Let us define a function $f(\sigma)$ by

$$f(\sigma) \equiv \int_{-\infty}^{+\infty} dx \, \mathfrak{L}(x) k(x) \arctan[\sigma/k(x)] .$$

In this equation x is the frequency measured from the line center as introduced before. Moreover $k(x) = k_0 \mathfrak{L}(x)$ and $\int_{-\infty}^{+\infty} \mathfrak{L}(x) dx = 1$. It is easily verified that

$$df/d\sigma = 1 - \int_{-\infty}^{+\infty} dx \; \mathfrak{L}(x) / \{[(k_0/\sigma) \mathfrak{L}(x)]^2 + 1\} . \tag{7}$$

Now suppose that $\mathfrak{L}(x)$ is symmetric: $\mathfrak{L}(x) = \mathfrak{L}(-x)$. Then

$$\int_{-\infty}^{+\infty} \frac{dx \, \mathfrak{L}(x)}{1 + [(k_0/\sigma) \mathfrak{L}(x)]^2} = \frac{2d}{d(k_0/\sigma)} \int_0^\infty \arctan\left(\frac{k_0}{\sigma} \mathfrak{L}(x)\right) dx .$$

We define a positive quantity Δ by $(k_0/\sigma) \mathfrak{L}(\Delta) = 1$ for $(k_0/\sigma) \to \infty$ and introduce the new variable $y = x/\Delta$. (Δ is half the breadth of the profile at σ/k_0.) Equation (7) becomes

$$\frac{df}{d\sigma} = 1 - \frac{2d}{d(k_0/\sigma)}\, \Delta \int_0^\infty dy \arctan\left(\frac{k_0}{\sigma}\, \mathfrak{L}(y\Delta)\right). \tag{8}$$

If $k_0/\sigma \gg 1$, then $\Delta \gg 1$ and we need for the evaluation of the integral in Eq. (8) only the asymptotic behavior of $\mathfrak{L}$. If further

$$\arctan[k_0\, \mathfrak{L}(y\Delta)/\sigma] = \arctan[\, \mathfrak{L}(y\Delta)/\, \mathfrak{L}(\Delta)]$$

is independent of Δ for $\Delta \to \infty$ (and this is the case for all common line shapes), then this integral is a constant $\frac{1}{2}C$ (dependent on $\mathfrak{L}$), and we have

$$\frac{1}{\sigma} f(\sigma) \sim 1 - \frac{C}{\sigma}\int_0^\sigma \frac{d\Delta(\sigma')}{d(k_0/\sigma')}\, d\sigma' = 1 + \frac{C}{\sigma}\int_{\Delta(\sigma)}^\infty \frac{d\sigma'}{d(k_0/\sigma')}\, d(\Delta).$$

Using again the definition of Δ we obtain

$$\sigma^{-1} f(\sigma) \sim 1 - (Ck_0/\sigma)\int_{\Delta(\sigma)}^\infty \mathfrak{L}^2(x)dx. \tag{9}$$

The physical interpretation of this formula will be discussed extensively in Sec. II of this paper.

As has been said, it is sufficient to retain the first two terms in the expansion of the Fourier transform. Equation (9) contains therefore enough information. Let us now consider the two special cases of $\mathfrak{L}(x)$ mentioned before, the Doppler and Lorentz profiles. It requires only some algebra to find for $\sigma/k_0 \ll 1$ or (what amounts to the same) $\Delta \gg 1$ the following asymptotic expressions. For the Doppler profile note that

$$\mathfrak{L}(y\Delta)/\, \mathfrak{L}(\Delta) \to \infty, \text{ for } 0 \le y < 1; \quad = 1, \text{ for } y = 1; \quad \to 0, \text{ for } 1 < y;$$

and therefore C is equal to π and

$$\sigma^{-1} f(\sigma) \sim 1 - \tfrac{1}{4}\pi\, \sigma/k_0 (\ln k_0/\sigma\sqrt{\pi})^{1/2}. \tag{10}$$

For the Lorentz profile C is equal to $\pi\sqrt{2}$ and

$$\sigma^{-1} f(\sigma) \sim 1 - [(2\pi)^{1/2}/3]\,(\sigma/k_0)^{1/2}. \tag{11}$$

Note that in both formulas the definition of k_0 differs due to a different definition of the half-breadths. With the aid of this equation we obtain for Eq. (6) for the Doppler profile

$$W(\vec{R}, m, \nu_m)d\nu_m d\vec{R} = \mathfrak{L}(\nu_m)d\nu_m d\vec{R}(2\pi)^{-3}\int d\vec{\sigma}\, e^{-\vec{R}\cdot\vec{\sigma}} \exp[-\tfrac{1}{4}m\pi\, \sigma/k_0(\ln k_0/\sigma\sqrt{\pi})^{\frac{1}{2}}], \tag{12}$$

and for the Lorentz profile

$$W(\vec{R}, m, \nu_m)d\nu_m d\vec{R} = \mathfrak{L}(\nu_m)d\nu_m d\vec{R}(2\pi)^{-3}\int d\vec{\sigma}\, e^{-i\vec{R}\cdot\vec{\sigma}} \exp[-\tfrac{1}{3}m(2\pi)^{\frac{1}{2}}(\sigma/k_0)^{\frac{1}{2}}]. \tag{13}$$

Note that these equations contain the time elapsed between the emission of the quantum at $\vec{r}_0$ and the absorption at $\vec{r}_m$. Since an atom has been excited in the resonance state m times and every atom has a natural lifetime of $1/\gamma$, the time elapsed is $t = m/\gamma$ or $m = \gamma t$ (in absence of stimulated emission). The passage to a differential equation has been discussed at length by Chandrasekhar.[13] Let us consider this point now. Comparing Eqs. (12) and (13) with Eq. (5) we notice that only the last mentioned is Gaussian. This has important consequences. Following Chandrasekhar[13] we can obtain a differential equation of the diffusion type for W: $\partial W/\partial t = D\nabla^2 W$, D being expressed as (for the notation, see again Ref. 11)

$$D = \lim_{\Delta t \to 0} \frac{1}{3\Delta t}\int d(\Delta\vec{R})W(\Delta\vec{R}, \Delta t)(\Delta\vec{R})^2 = \frac{1}{3(2\pi)^3}\int d(\Delta\vec{R})(\Delta\vec{R})^2 \int d\vec{\sigma}\, e^{-i\vec{\sigma}\cdot\Delta\vec{R}} F(\sigma, \Delta t). \tag{14}$$

$F(\sigma, \Delta t)$ denotes the Fourier transform of W as given in Eqs. (5), (12), and (13). In evaluating this integral we use

$$(\Delta\vec{R})^2 \exp[-i\vec{\sigma}\cdot\Delta\vec{R}]F(\sigma,\Delta t) = -\frac{d^2}{d\vec{\sigma}^2}\exp(-i\vec{\sigma}\circ\Delta\vec{R})F(\sigma,\Delta t)$$

$$+2\frac{d}{d\vec{\sigma}}\exp(-i\sigma\cdot\Delta\mathrm{R})\cdot\frac{d}{d\vec{\sigma}}F(\sigma,\Delta t)+\exp(-i\vec{\sigma}\cdot\Delta\vec{R})\frac{d^2}{d\vec{\sigma}^2}F(\sigma,\Delta t).$$

We substitute this identity in Eq. (14). The first term upon integration with respect to $\vec{\sigma}$ can be put in the form of a surface integral that vanishes if $F(\sigma)$ goes sufficiently fast to zero as is fulfilled here for all cases. The second term is integrated partially and yields a term equal to minus twice the third term. For the third term we perform the integration over $\Delta\vec{R}$ and obtain a δ function. Since $F(\sigma)$ depends, as has been shown, only on the absolute value of σ, Eq. (14) becomes

$$D=\lim_{\Delta t\to 0}\frac{1}{3\Delta t}\int d\vec{\sigma}\,\delta(\vec{\sigma})\,\frac{1}{\sigma^2}\frac{d}{d\sigma}\,\sigma^2\frac{d}{d\sigma}F(\sigma,\Delta t).$$

It is now easily verified that for Fourier transforms of the type $\exp[-\Delta t\sigma^{\alpha}]$, this expression is zero for $\alpha>2$, diverges for $\alpha<2$, and only gives a finite result, different from zero for $\alpha=2$, a Gaussian kernel (Eq. 5). Therefore only for the case described by Eq. (5), where the frequency of the emitted quantum is the same as the frequency of the absorbed one (within the natural line breadth) a diffusion equation is possible. The result that D is infinite for Eqs. (12) and (13) arises from the fact that they are not Gaussian. Therefore, indeed, it is exceptional that a diffusion equation is possible and we may expect that the kernels are not Gaussian either when we take partial correlation into account, i.e., when the conditional probability function $P(\nu_j|\nu_{j-1})$ is intermediate between $\delta(\nu_j-\nu_{j-1})$ and[16] $\mathfrak{L}(\nu_j)$. For a Doppler profile the exact expression for $P(\nu_j|\nu_{j-1})$ is known.[14] However, since the integrals [see Eq. (14)] become difficult, we shall not discuss this here. We shall treat it in a forthcoming paper.

II. SOLUTION OF THE TRANSFER EQUATION FOR A SLAB

The calculations in Sec. I did not result in the derivation of a diffusion equation. It is remarkable, however, that the exponentials of Eqs. (12) and (13) show a great resemblance with the first eigenvalue of the Biberman-Holstein integral equation[7,8]

$$(1-\beta/\gamma)n(\vec{r})=\int_0^\infty d\nu$$

$$\times\int_V\frac{\exp[-k(\nu)|\vec{r}-\vec{r}\,'|]}{4\pi|\vec{r}-\vec{r}\,'|^2}\,\mathfrak{L}(\nu)k(\nu)n(\vec{r}\,')d\vec{r}\,', \qquad (15)$$

the integration being over the volume V. The symbols have their common meaning as defined in Ref. 8. The natural lifetime of the excited atom is γ^{-1}, and the effective lifetime of the excited atoms in the enclosure is β^{-1}. This first eigenvalue has been calculated by Holstein by means of a variational procedure.

This resemblance is not fortuitous. In recent years Widom[17] has proved in a number of papers how the eigenvalues of integral equations with translation kernels like in Eq. (15) defined on a finite volume V are determined by the Fourier transform of the kernel. The result is an asymptotic one. It applies in our case when the smallest length characteristic for the volume V is large in units of the photon mean free path at the central frequency (neglecting natural breadth). In other words, it applies when the optical depth k_0L is large, where L is the characteristic thickness.

We shall now state Widom's result with a few simplifications for a one-dimensional problem, i.e., a slab with thickness L. Consider the following integral equation [note that Eq. (15) is of this type]:

$$\int_{-L/2}^{+L/2}K(x-x')\psi(x')dx'=(1-\beta/\gamma)\psi(x). \qquad (16)$$

Denote the Fourier transform of K by TK and let it be possible to write TK as follows:

$$TK(\sigma_x)\sim 1-\sigma_x^{\ \alpha}F(\sigma_x),\quad \sigma_x\to 0,\quad 0<\alpha\leq 2. \qquad (17)$$

$F(\sigma_x)$ being a non-negative slowly varying function near zero, then for the jth eigenvalue we have

$$\beta_j/\gamma\sim[\lambda_j(\tfrac{1}{2}L)^{\alpha}]^{-1}F(2/L). \qquad (18)$$

The quantities λ_j are eigenvalues of a certain integral equation on $[-1,+1]$, dependent on α, for which expressions have been given explicitly.[17] For the eigenfunctions ψ_j of Eq. (16) a similar result applies. The ψ_j are asymptotically equal (at least they converge in mean square) to the eigenfunctions f_j of the above-mentioned integral equation that yields as well the quantities λ_j. For the following it is important to note that in Eq. (17) $F(\sigma_x)$ may be a positive constant as well as a function like

$$(\ln k_0/\sigma_x\sqrt{\pi})^{1/2}.$$

We have, therefore, to do two things. We have first to calculate the Fourier transform of the kernel in Eq. (15). If it can be cast in the form given by Eq. (17) for a particular value of α, then we have to solve the integral equation given by Widom for that value of α in order to find the eigenvalues λ_j and the eigenfunctions f_j.

The Fourier transform of the kernel has already been calculated in Sec. I. We found

$$\int d\vec{r}\, e^{i\vec{\sigma}\cdot\vec{r}} \int_0^\infty d\nu\, \mathfrak{L}(\nu) k(\nu) \frac{\exp[-k(\nu)r]}{4\pi r^2} \sim 1-(Ck_0/\sigma)\int_{\Delta(\sigma)}^\infty \mathfrak{L}^2(y)dy\,. \tag{19}$$

C is the constant dependent on the line shape as defined in Eq. (8). $\Delta > 0$ is the asymptotic solution of

$$\mathfrak{L}(\Delta)=\sigma/k_0, \quad \text{for } \sigma/k_0 \to 0\,.$$

See under Eq. (7).

We now discuss the physical interpretation of this formula. As is immediately clear from Eq. (1), a photon with frequency ν has a mean free path equal to $[k(\nu)]^{-1}=[k_0\, \mathfrak{L}(x)]^{-1}$. Since σ is always a quantity of the order of L^{-1}, we see that a photon with frequency $y \geqslant \Delta$ has a mean free path of the order of L or larger. Therefore Eq. (19) shows that the behavior of the Fourier transform of the integral kernel at large optical depth is, to first order, exclusively determined by the behavior of the line shape at those frequencies at which a photon, once emitted, immediately escapes from the plasma. The core of the line is unimportant and can be considered as entirely opaque. Since there exists a direct relationship between the Fourier transform given in Eq. (19) and the eigenvalues and eigenfunctions of the integral equation [see Eqs. (17) and (18)], the same conclusion applies for these quantities.

In Sec. I we used Eq. (19) in order to find particular expressions when the line shapes are the Doppler and Lorentz profiles. Because in this paper we restrict the discussion to a slab, we put $\sigma_y=\sigma_z=0$ in Eqs. (10) and (11) (this is equivalent with performing the integrations over y and z) and obtain for a Doppler profile

$$TK(\sigma_x) \sim 1-\tfrac{1}{4}\pi\sigma_x/k_0(\ln k_0/\sigma_x\sqrt{\pi})^{1/2}, \tag{20}$$

and for a Lorentz profile

$$TK(\sigma_x) \sim 1-\tfrac{1}{3}(2\pi)^{1/2}(\sigma_x/k_0)^{1/2}\,. \tag{21}$$

In Appendix A we shall show that Eq. (21) applies as well for a Voigt profile. Furthermore we shall derive there expressions for the Fourier transform when the line shape is a Doppler profile with hfs.

If the different hyperfine components overlap (and this situation prevails), we have (for derivation and conditions, see Appendix A)

$$TK(\sigma_x) \sim 1-\tfrac{1}{8}\pi(\sigma_x/k_0) \times\left[\left(\ln\frac{R_1k_0}{\sigma_x\sqrt{\pi}}\right)^{-\frac{1}{2}}+\left(\ln\frac{R_nk_0}{\sigma_x\sqrt{\pi}}\right)^{-\frac{1}{2}}\right], \tag{22}$$

but if the components can be considered as nonoverlapping[8] (this situation is not consistent with the limit $k_0L\to\infty$; it can nevertheless occur in practice, see Appendix A)

$$TK(\sigma_x) \sim 1-\frac{\pi}{4}\frac{\sigma_x}{k_0}\sum_{j=1}^{n}\frac{1}{(\ln R_jk_0/\sigma_x\sqrt{\pi})^{1/2}}\,, \tag{23}$$

the summation being over all n hyperfine components. R_j is the relative intensity of the jth component, and R_1 and R_n are the relative intensities of the most outward lying ones. The physical interpretation of the formulas will be discussed in Appendix A.

If the line shapes are the Voigt ($a \neq 0$) or Lorentz profiles, hfs is wiped out in first approximation and Eq. (21) remains valid. For a discussion, see Appendix A.

Up to now we have been concerned with the Fourier transforms of the integral kernel of Eq. (15). We have now to solve the integral equations given by Widom for particular values of α. For a Doppler profile with or without hfs, we see, by comparison of Eq. (20), (22), and (23) with Eq. (17), that we must have $\alpha=1$; for the Voigt or Lorentz profiles $\alpha=\frac{1}{2}$. These integral equations will be solved in Appendix B for $\alpha=1$ and in Appendix C for $\alpha=\frac{1}{2}$. In each case an expansion of the eigenfunctions in some suitable set of functions is assumed. The coefficients in these expansions must be calculated. By performing the integrations, the integral equation is reduced to an infinite matrix. The determination of the eigenvalues and eigenvectors is performed numerically, after truncation of the matrix. They appear to converge excellently as a function of the truncation order. This shows that we have chosen, indeed, a good expansion. It will be proven in Appendix B that the eigenfunctions f_j of the integral equation given there can be expressed in the Tschebyscheff polynomials of the second kind U_m.[18] Since the eigenfunctions ψ_j of Eq. (15) for a Doppler profile converge to these eigenfunctions f_j, we have (for $k_0L\to\infty$; we shall omit this henceforth)

$$\psi_j(x) \sim (1-\xi^2)^{1/2}\sum_{m=0}^{\infty} a_{mj}U_m(\xi), \qquad \xi=2x/L, \quad |\xi|\leqslant 1 \tag{24}$$

independent of possible hfs. The eigenfunctions constitute an orthonormal system. For the first three even and the first two odd eigenfunctions the coefficients $a_{m,j}$ are given in Appendix B. In Fig. 1 the first two even and in Fig. 2 the first two odd eigenfunctions have been displayed. The eigenvalues of Eq. (15) for a Doppler profile are obtained from the Fourier transforms Eqs. (20), (22), and (23) and the values of λ_j, found in Appendix B, following the prescription given in Eqs. (17) and (18). We have (the numbers μ_j are given in Table I for $j = 1, 2, 3, 4$, and 5) for a simple line

$$(\beta_j/\gamma)(k_0 L/\sqrt{\pi})(\ln k_0 L/2\sqrt{\pi})^{1/2} \sim \mu_j \tag{25}$$

for a line with overlapping hyperfine components,

$$(\beta_j/\gamma)(k_0 L/\sqrt{\pi}) \sim \frac{\mu_j}{2}\left[\left(\ln \frac{R_1 k_0 L}{2\sqrt{\pi}}\right)^{-\frac{1}{2}} + \left(\ln \frac{R_n k_0 L}{2\sqrt{\pi}}\right)^{-\frac{1}{2}}\right], \tag{26}$$

R_1 and R_n being the relative intensities of the most outside lying components (see Appendix A) and, finally, for a line consisting of independent hyperfine components with relative intensities R_j,

$$\frac{\beta_j}{\gamma}\frac{k_0 L}{\sqrt{\pi}} \sim \mu_j \sum_j \left(\ln \frac{R_j k_0 L}{2\sqrt{\pi}}\right)^{-\frac{1}{2}}. \tag{27}$$

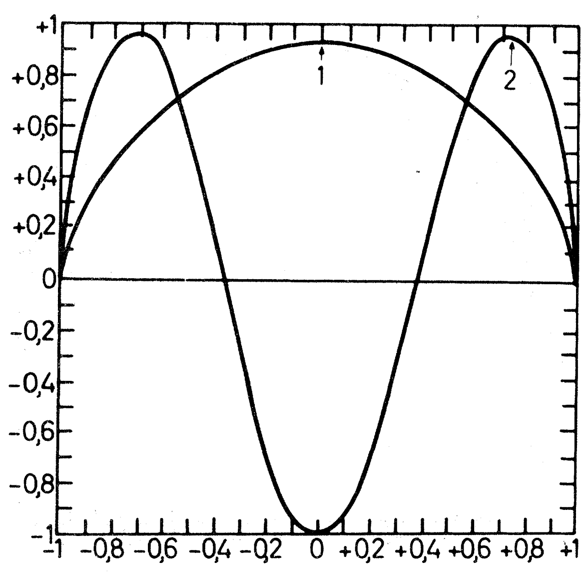

FIG. 1. Eigenfunctions of Eq. (15) at high optical depth when the line shape is a Doppler profile with or without hfs; (1) first even eigenfunction; (2) second even eigenfunction.

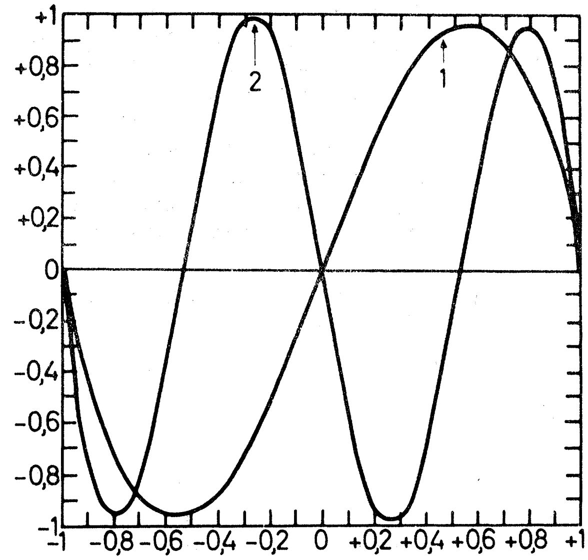

FIG. 2. Eigenfunctions of Eq. (15) at high optical depth when the line shape is a Doppler profile with or without hfs; (1) first odd eigenfunction; (2) second odd eigenfunction.

The value for $j = 1$ corresponds to the first even eigenfunction, and the one for $j = 2$ to the first odd eigenfunction, etc. Note that the quantity $k_0 L/\sqrt{\pi}$ is the same as $k_0 L$ used by Mitchell and Zemansky[15] and Holstein.[8] This is brought about because we have always used a Doppler profile normalized to one, while these authors have not. Compare the first eigenvalue with the result from a variational calculation obtained by Holstein, $1.875/\sqrt{\pi} = 1.058$. The first eigenfunction differs from the one calculated by Holstein. This is not surprising since a variational procedure always yields inaccurate eigenfunctions.

It will be shown in Appendix C that the orthonormal eigenfunctions ψ_j of Eq. (15) for a Voigt ($a \neq 0$) or Lorentz profile can be expressed as

$$\psi_j(x) \sim (1-\xi^2)^{1/4} \sum_{m=0}^{\infty} b_{m,j} U_m(\xi), \qquad \xi = 2x/L, \quad |\xi| \leq 1. \tag{28}$$

TABLE I. Values of the first five coefficients μ_j, Eqs. (25), (26), and (27).

j	μ_j
1	1.026 05
2	2.441 34
3	3.825 67
4	5.221 78
5	6.611 41

For the first three even and the first two odd eigenfunctions the $b_{m,j}$ will be given in Appendix C. In Fig. 3 the first two even eigenfunctions and in Fig. 4 the first two odd eigenfunctions have been given. Moreover in Fig. 5 the first eigenfunction for Doppler and Lorentz profiles are compared with the first eigenfunction of the diffusion equation $\cos\frac{1}{2}\pi\xi$. The differences have a physical meaning. By the mechanism of repeated absorption and emission accompanied by frequency changes, the radiation is transferred more easily than without such changes. In the latter case (that is described by a diffusion equation, as we know), the density of excited atoms is determined by shorter distances than in the former one. Therefore the eigenfunctions given in Eqs. (24) and (28) must be flatter than those of the diffusion equation, as we indeed see from Fig. 5 for $j = 1$. Moreover since a Doppler profile falls off far more sharply in the wings than a Lorentz profile, the transfer of radiation must again be easier when we have a Lorentz profile than when the line shape is given by a Doppler profile. Therefore, again, the eigenfunctions for a Lorentz profile must be flatter than the ones for a Doppler profile as is clearly exhibited by Eqs. (24) and (25) and by Fig. 5. Moreover we expect that the oscillations of the eigenfunctions ($j \geq 1$) are smoother in the first case than in the latter. In Appendix B we shall show that, when the eigenfunctions for a Doppler profile Eq. (24), are expressed in the functions

$$(1 - \xi^2)^{1/2} P_m^{(1,1)}(\xi).$$

$P_m^{(1,1)}(\xi)$ being Jacobi polynomials, the expansion coefficients in this new representation behave in the same way as the expansion coefficients of the eigenfunctions for the Lorentz profile, Eq. (28). The transformation is simple and amounts only to a transformation from the Tschebyscheff to the Jacobi polynomials. The oscillations of the

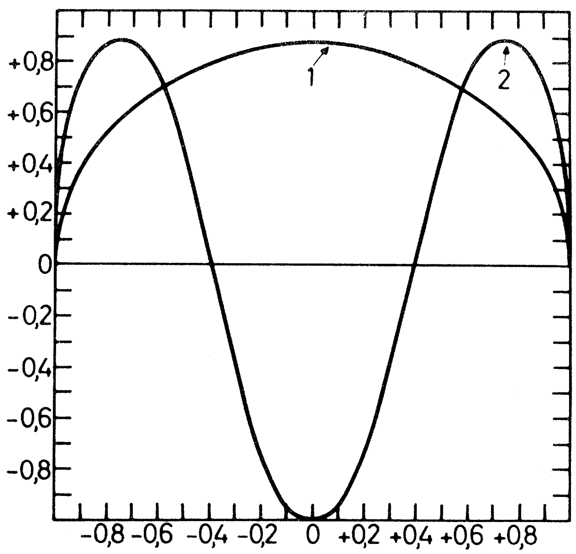

FIG. 3. Eigenfunctions of Eq. (15) at high optical depth when the line shape is a Voigt ($a \neq 0$) or Lorenz profile with or without hfs; (1) first even eigenfunction; (2) second even eigenfunction.

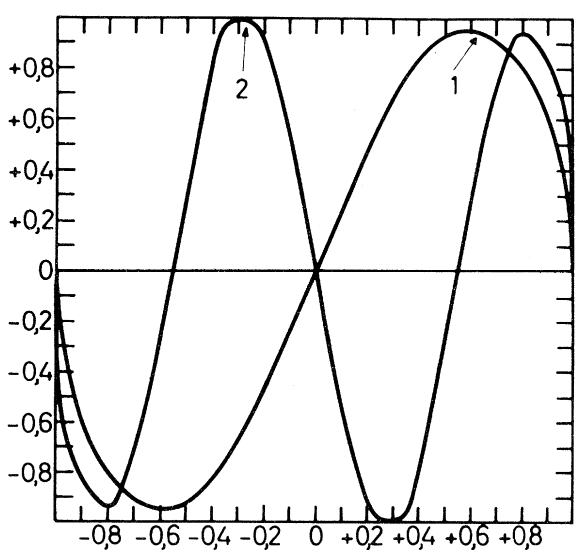

FIG. 4. Eigenfunctions of Eq. (15) at high optical depth when the line shape is a Voigt ($a \neq 0$) or Lorentz profile with or without hfs; (1) first odd eigenfunction; (2) second even eigenfunction.

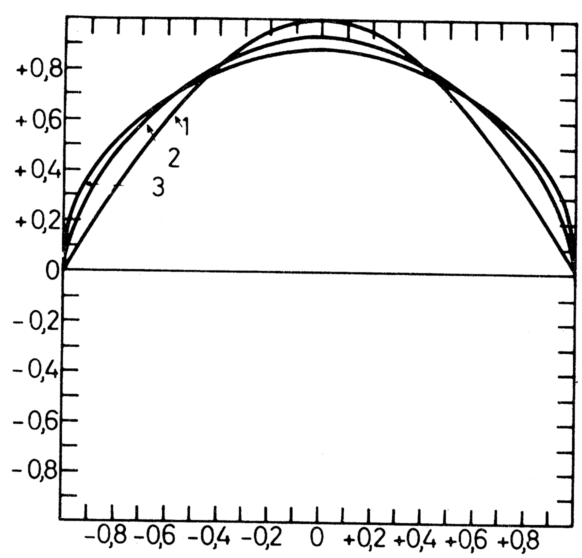

FIG. 5. Comparison of the first even eigenfunctions of the diffusion equation (1), of Eq. (15) when the line shape is a Doppler profile (2), and of Eq. (15) when the line shape is a Voigt ($a \neq 0$) or Lorentz profile (3). Explanation is in the text.

Tschebyscheff polynomials are much smoother than those of the Jacobi polynomials. Therefore the oscillations in the eigenfunctions for the Lorentz profile are smoother than those for the Doppler profile, exactly what we expected.

Now that we have discussed the eigenfunctions, let us turn our attention to the eigenvalues of Eq. (15) for Voigt ($a \neq 0$) and Lorentz profiles. Using the eigenvalues λ_j as found in Appendix C and the Fourier transform Eq. (21), we have according to the prescription given in Eqs. (17) and (18)

$$\beta_j/\gamma \sim \mu_j'/(k_0 L)^{1/2}. \tag{29}$$

The values of μ_j' are given for $j = 1, 2, 3, 4$, and 5 in Table II. The value for $j=1$ corresponds to the first even eigenfunction, and the value for $j=2$ corresponds to the first odd eigenfunction, etc. Compare the first value with the one calculated by Holstein, 1.15. The excellent agreement, however, must be considered as somewhat fortuitous because from a variational calculation with such a simple trial function as Holstein's, one would never expect an accuracy higher than a few percent. If we compare Table I with Table II we see that the numbers in Table I increase at a faster rate than they do in Table II. Therefore determination of the first eigenvalue by a decay experiment will be easier for a Doppler profile than for a Lorentz profile. In the latter case the higher-order modes may not disappear sufficiently rapidly.

We do not discuss extensively the eigenvalues and eigenfunctions when the line shape is that appropriate for statistical broadening. For a short discussion, however, we refer to Appendix C.

Finally, we mention that the eigenfunctions are particularly important for the calculation of stationary solutions of problems in radiative transfer, for instance when atoms are excited and deexcited by electrons. The solutions for the infinite slab can directly be applied to astrophysical problems. (In the laboratory the infinite slab can be simulated by a box with perfectly reflecting walls.) We shall discuss this in a subsequent paper.

TABLE II. Values of the first five coefficients μ_j', Eq. (29).

j	μ_j'
1	1.14638
2	1.89243
3	2.39733
4	2.82075
5	3.18421

III. CONCLUSIONS AND REMARKS

In Sec. II of this paper we have calculated in first-order asymptotic approximation the eigenvalues and eigenfunctions of the Biberman-Holstein integral equation. The question remains how well they approximate the true solutions for a particular value of k_0L. As is well known, for asymptotic expansions the error is always of the order of the last neglected term. Unfortunately mathematical results concerning the higher-order terms are lacking. Only in a very particular case something is known.[19] Let it be possible to write the Fourier transform TK of K Eq. (17) as

$$TK(\sigma_x) \sim 1 - \sigma_x^2, \quad \sigma_x \to 0.$$

This leads to a diffusion equation. For the eigenvalues the following result holds:

$$\beta_j/\gamma \sim (\pi^2 j^2/L^2)[1 + 2\omega/L] ,$$

where

$$\omega = \frac{1}{\pi}\int_{-\infty}^{+\infty}\left(\frac{TK'(\sigma_x)}{TK(\sigma_x)-1} - \frac{2}{\sigma_x}\right)\frac{d\sigma_x}{\sigma_x}.$$

Note that, while the first term only depends on the behavior of $TK(\sigma_x)$ near $\sigma_x = 0$, the second involves the *whole* Fourier transform. The next term in the asymptotic expansions, Eqs. (24)–(29), is therefore *not* afforded by the next term in the asymptotic expansion of the Fourier transforms. A simple order-of-magnitude calculation confirms that this indeed must be so. For convenience, we restrict the discussion to a slab. Notice that, roughly, a slab can lose its radiation in two ways: (a) every volume element loses radiation but only in a small part of the spectral line, namely, that part where the photons have a mean free path of the order of or larger than L (this has been discussed in this paper); (b) a small sheath near the boundaries of the order of k_0^{-1} loses radiation but the whole spectral line contributes. Both terms can be assessed using the expressions for β_1 and the behavior of the eigenfunctions near the boundary. It appears that the term (b) is some power of $(k_0L)^{-1}$ smaller than the first term in the asymptotic expansion of (a). It is, however, in the case of a Lorentz profile larger than the second term in (a). For a Doppler profile things are different since the higher-order terms in (a) involve logarithms. Therefore we can conjecture that for a Doppler profile the higher-order terms are still afforded by (a). If this is true, the eigenfunctions given in Eq. (24) remain the same but the eigenvalues Eq. (25) become as a straightforward expansion of Eq. (7) [*not* Eq. (19)] shows

$$\frac{\beta_j}{\gamma}\frac{k_0 L}{\sqrt{\pi}}\left(\ln\frac{k_0 L}{2\sqrt{\pi}}\right)^{1/2} \sim \mu_j\left(1-\frac{1}{4\ln k_0 L/2\sqrt{\pi}}\right).$$

As this is only a conjecture, we do not want to go into the details of the calculation. As has been said, results are lacking for Fourier transforms of the type we are interested in. Therefore the question raised here cannot be discussed further and we must leave it open.

IV. ACKNOWLEDGMENT

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APPENDIX A

In this Appendix we discuss the asymptotic expansion of the Fourier transform of the Biberman-Holstein integral kernel when the line shapes are the Voigt and the Doppler profile with hfs.

When Doppler and collision broadening are both present and are statistically independent, we have

$$\mathfrak{L}(x)=\frac{a}{\pi\sqrt{\pi}}\int_{-\infty}^{+\infty} dt\,\frac{e^{-t^2}}{(t-x)^2+a^2} \qquad (A.1)$$

with

$$a=\frac{\Delta\nu_L}{\Delta\nu_D}(\ln 2)^{1/2},\quad x=\frac{2(\nu-\nu_0)}{\Delta\nu_D}(\ln 2)^{1/2}.$$

The limit $a=0$ gives the Doppler profile, and the limit $a\to\infty$ the Lorentz profile. The absorption coefficient is again

$$k(x)=k_0\,\mathfrak{L}(x),\text{ with } k=\frac{2\pi e^2}{mc}\,\frac{Nf}{\Delta\nu_D}(\ln 2)^{1/2}.$$

It is known that the following expansion exists.[15,20]

$$\mathfrak{L}(x)=\frac{e^{-x^2}}{\sqrt{\pi}}+\frac{a}{\pi}\frac{1}{x^2}+\cdots. \qquad (A.2)$$

Only the second term contributes in the line wings. It is now easily shown that for the Voigt profile the formula for the Lorentz profile, Eq. (12), applies as well. However, it should always be ascertained if, indeed, the whole frequency range over which is integrated in Eq. (19) (i.e., $y\geq\Delta(k_0L)$) is determined by Lorentz broadening or if the Doppler term in Eq. (A.2) is small in the whole region.

Let us now turn our attention to the case of hfs. It can be shown that for a Voigt profile ($a\neq 0$) with hfs, the second term of the expansion (A.2) still applies.[20] Therefore hfs does not change Eq. (21) nor the solution of the integral equation either. For a Doppler profile, however, things are quite different. As is well known, the nuclear spin $\vec{I}$ couples with the angular momentum $\vec{J}$ to a resultant designated by $\vec{F}$.

$$\vec{F}=\vec{I}+\vec{J}.$$

To a high degree of the approximation the splitting is caused by the magnetic dipole interaction.[21] We therefore discard splitting by electric quadrupole interaction. The perturbing Hamiltonian is of the form

$$H=A\vec{I}\cdot\vec{J}.$$

A is a proportionality constant that has been measured for a number of elements and transitions. For a Doppler line with hfs we have

$$\mathfrak{L}(x)=\sum_{F,F'} R_{F,F'}\,\frac{\exp[-(x-x_{F,F'})^2]}{\sqrt{\pi}}, \qquad (A.3)$$

the summation being over the F values of both the upper and lower levels. The $R_{F,F'}$ are the relative intensities and the $x_{F,F'}$ the shifts from the central frequency in the transition $F'\to F$, $J'\to J$ in units of the Doppler breadth. In first-order perturbation theory the first quantity can be expressed in 6-j symbols.[20,21]

$$R_{F,F'}=\frac{(2F+1)(2F'+1)}{(2I+1)}\begin{Bmatrix} J & F & I\\ F' & J' & 1\end{Bmatrix}^2. \qquad (A.4)$$

For $x_{F,F'}$ we have the expression

$$x_{F,F'}=[(\ln 2)^{1/2}/h\Delta\nu_D]\times\{A'[F'(F'+1)-J'(J'+1)-I(I+1)]-A[F(F+1)-J(J+1)-I(I+1)]\}. \qquad (A.5)$$

It can be shown that, because of the unitary property of the 6-j symbols,

$$\sum R_{F,F'}=1,$$

an obvious requirement for the relative intensities. The derivation of the asymptotic formula proceeds along the same lines as described in Sec. I. However, since $\mathfrak{L}(x)$ is not symmetric from Eq. (7) on, the derivations are somewhat different. We have

$$\int_{-\infty}^{+\infty} \frac{dx\, \mathfrak{L}(x)}{(k_0/\sigma)^2 \mathfrak{L}^2(x)+1} = \frac{d}{d(k_0/\sigma)} \times \int_0^{\infty} dx \{\arctan[(k_0/\sigma)\mathfrak{L}(x)] + \arctan[(k_0/\sigma)\mathfrak{L}(-x)]\}. \quad (A.6)$$

Now introduce quantities $\Delta > 0$ and $\Delta' > 0$ defined by $\mathfrak{L}(\Delta) = \sigma/k_0$, $\mathfrak{L}(-\Delta) = \sigma/k_0$, for $\sigma/k_0 \to 0$.

It is possible that the quantities Δ and Δ' are not uniquely defined by Eq. (A.6). This is in particular the case when the lines are separated so well that they in fact can be considered as independent. It is true that if we choose Δ small enough a unique solution can always be found. However in every physical situation Δ is a small quantity, which cannot become arbitrarily small because below a certain value the applicability of Eq. (A.3) ceases and essentially the Lorentz broadening in the line wings determines the situation. As has been mentioned σ/k_0 is of the order of $(k_0 L)^{-1}$.

Let us denote by $x_{F_0,F_0'}$ and $x_{F_1,F_1'}$ the greatest and the smallest of the quantities $x_{F,F'}$ (that means the greatest in absolute value of the negative $x_{F,F'}$), and the corresponding relative intensities by $R_{F_0,F_0'}$ and $R_{F_1,F_1'}$, then

$$\mathfrak{L}(x\Delta) = \frac{R_{F_0,F_0'}}{\sqrt{\pi}} \exp[-(x\Delta - x_{F_0,F_0'})^2] \times \{1 + {\sum}'(R_{F,F'}/R_{F_0,F_0'}) \times \exp[-(x_{F_0,F_0'} - x_{F,F'}) \times (2x\Delta - x_{F_0,F_0'} - x_{F,F'})]\}.$$

$\sum'$ means that in the summation the term with $F = F_0$ and $F' = F_0'$ should be excluded. Since by the definition of $x_{F_0,F_0'}$ $(x_{F_0,F_0'} - x_{F,F'}) > 0$ and $\Delta \to \infty$, we see that

$$\mathfrak{L}(x\Delta) \sim \frac{R_{F_0,F_0'}}{\sqrt{\pi}} \exp[-(x\Delta - x_{F_0,F_0'})^2]. \quad (A.7)$$

Similarly

$$\mathfrak{L}(-x\Delta') \sim \frac{R_{F_1,F_1'}}{\sqrt{\pi}} \exp[-(x\Delta' + x_{F_1,F_1'})^2]$$

The solution of $\mathfrak{L}(\Delta) = \sigma/k_0$ is

$$\Delta \sim x_{F_0,F_0'} + (\ln R_{F_0,F_0'} k_0/\sigma\sqrt{\pi})^{1/2}. \quad (A.8)$$

Similarly

$$\Delta' \sim -x_{F,F_1'} + (\ln R_{F_1,F_1'} k_0/\sigma\sqrt{\pi})^{1/2}.$$

Note that at every edge only the most outside lying component contributes. The other ones do not because their line shape falls off too rapidly as a function of frequency. For the rest (i.e., for $-\Delta' \leq x \leq \Delta$) the medium is entirely opaque to the line. We can be a little more specific. A Doppler-broadened absorption line of strength R behaves in first-order asymptotic theory as a square of breadth $2(\ln Rk_0/\sigma\sqrt{\pi})^{1/2}$, in which all light is absorbed. Because of the requirement that the Doppler line with hfs is opaque for $-\Delta' \leq x \leq \Delta$, we must have for every two successive components with relative intensities R and R' and displacements from the unperturbed central frequency δx and $\delta x'$ (see Fig. 6)

$$\left(\ln \frac{Rk_0}{\sigma\sqrt{\pi}}\right)^{\frac{1}{2}} + \left(\ln \frac{R'k_0}{\sigma\sqrt{\pi}}\right)^{\frac{1}{2}} \geq (\delta x - \delta x'). \quad (A.9)$$

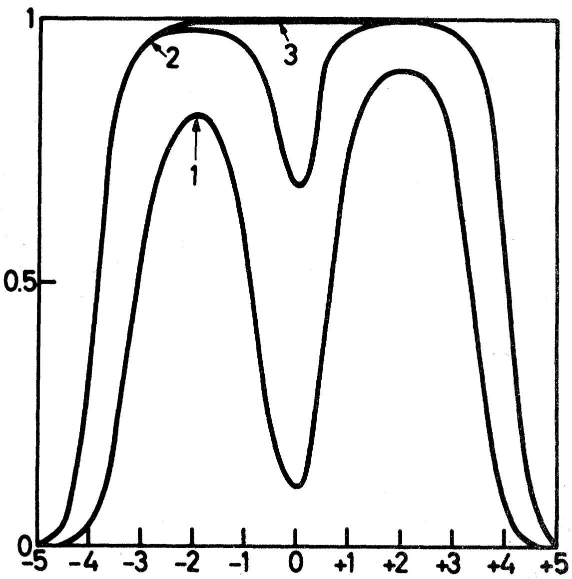

FIG. 6. Comparison of the integrand in Eq. (A.6) multiplied by $2/\pi$ for different values of $k_0/\sigma\sqrt{\pi}$ with the asymptotic expansion. Horizontally the dimensionless frequency x as defined below Eq. (4). The line is assumed to consist of two components with a separation corresponding to $x_1 - x_2 = 4$. The relative intensities are $\frac{1}{3}$ and $\frac{2}{3}$; (1) $k_0/\sigma\sqrt{\pi} = 10$; (2) $k_0/\sigma\sqrt{\pi} = 100$; (3) asymptotic expansion; it is equal to 1 for $-2 \leq x \leq 2$ and equal to curve (2) for the other values of x. Note that the asymptotic expansion approximates curve (2) already reasonably well though the condition given in Eq. (A.9) requires $k_0/\sigma\sqrt{\pi} \geq 150$. After differentiation of the asymptotic expansion with respect to k_0/σ only the edges contribute and this leads immediately to Eqs. (A.8) and (A.11).

Furthermore when in the line wing only the most outside lying components contribute, for all other hfs components

$$x_{F,F'} + (\ln R_{F_1F'} k_0/\sigma\sqrt{\pi})^{1/2} \ll x_{F_0,F_0'} + (\ln R_{F_0,F_0'} k_0/\sigma\sqrt{\pi})^{1/2} = \Delta. \quad (A.10)$$

Similarly for Δ'. Equations (A.9) and (A.10) give the precise requirements for which the first-order asymptotic approximation in Eq. (A.8) holds. Sometimes, when the splitting of, for instance, the upper level is very small, Eq. (A.10) breaks down. If the terms in Eq. (A.10), originating in the upper level, are of the same order, this means that these terms practically coincide and this splitting can be neglected. It is then often possible to fulfill Eq. (A.10) only taking into account the splitting of the lower level.

Let us now proceed further with the integrals in Eq. (A.6). It is easily verified that with Eq. (A.8) the integrals give $\frac{1}{2}\pi(\Delta+\Delta')$, and we arrive along the same lines as for the simple line at the expression for the Fourier transform of the integral kernel for a Doppler profile with hfs. Equation (10) is to be replaced by

$$\sim 1 - \tfrac{1}{8}\pi[\sigma/k_0(\ln R_{F_0,F_0'} k_0/\sigma\sqrt{\pi})^{1/2} + \sigma/k_0(\ln R_{F_1,F_1'} k_0/\sigma\sqrt{\pi})^{1/2}]. \quad (A.11)$$

This is a striking contrast with hyperfine structure of a Lorentz-broadened line, where no difference is found with the formula for the simple line. There all components contribute equally, since the line shape falls off slowly.

For the same situation in a similar optical problem, see Ref. 20. When the separation is so great that the lines can be considered as independent, or when for two successive components

$$\left(\ln\frac{Rk_0}{\sigma\sqrt{\pi}}\right)^{\frac{1}{2}} + \left(\ln\frac{R'k_0}{\sigma\sqrt{\pi}}\right)^{\frac{1}{2}} \ll (\delta x - \delta x'), \quad (A.12)$$

we find for the Fourier transform instead of Eq. (10)

$$\sim 1 - \frac{\pi}{4} \sum_{F,F'} \frac{\sigma}{k_0(\ln R_{F,F'} k_0/\sigma\sqrt{\pi})^{1/2}}. \quad (A.13)$$

Note that Eq. (A.12) is the reverse of Eq. (A.9). This case has been treated by Holstein.[8] In general, however, the situation described by Eq. (A.11) prevails, and, in any case, only this formula is consistent with the asymptotic expansion for $k_0 L \to \infty$. But, for instance, for a metal-like Cs of which, for not too high temperatures, the hfs separation (in Doppler breadths) is large, Eq. (A.13) may sometimes apply.

We finally mention that as well the Fourier transform can be found if $\mathfrak{L}(x)$ is the line shape for statistical broadening.[22] Because this case seems not to be of very great practical importance, we only mention the result

$$1 - \frac{C}{2}\frac{k_0}{\sigma}\int_\Delta^\infty \mathfrak{L}^2(y)dy \sim \tfrac{1}{4}\Gamma(\tfrac{1}{6})\Gamma(\tfrac{5}{6})B(\sigma/k_0)^{1/3}. \quad (A.14)$$

$$B = (\tfrac{2}{3}\pi\beta^{1/2}N)^{2/3}$$

with β and N as defined by Margenau.[22] In the formula $C/2$ appears since the line is not symmetric.

APPENDIX B

In Sec. II we have stated that, in order to be able to determine the eigenvalues and eigenfunctions of Eq. (15), it is necessary to solve an additional integral equation. The integral equation depends on the particular value of the exponent α in the Fourier transform Eq. (17). It has been given by Widom.[17] In this Appendix we shall be concerned with the solution of the integral equation for $\alpha=1$, given in Eq. (B.1). This enables us to determine the eigenfunctions of Eq. (15) when the enclosure is a slab and the spectral line shape a Doppler profile with or without hfs. The integral equation is

$$\lambda f(\xi) = \frac{1}{2\pi}\int_{-1}^{+1} d\xi' \ln\left[\frac{1-\xi\xi'+[(1-\xi^2)(1-\xi'^2)]^{1/2}}{1-\xi\xi'-[(1-\xi^2)(1-\xi'^2)]^{1/2}}\right] f(\xi'). \quad (B.1)$$

We write this equation symbolically as

$$\lambda f(\xi) = \int_{-1}^{+1} d\xi' M(\xi,\xi') f(\xi').$$

Since

$$M(-\xi,-\xi') = M(\xi,\xi')$$

the eigenfunctions of Eq. (B.1) must be even or odd.[23] This is already obvious from the original Eq. (15). We introduce the new variables $\xi=\sin\varphi$, $\xi'=\sin\psi$, $-\pi/2 \le \varphi,\psi \le \frac{1}{2}\pi$, and obtain after some algebra

$$\lambda f(\sin\varphi) = \pi^{-1}\int_{-\pi/2}^{+\pi/2} d\psi \cos\psi \, \ln|\cos\tfrac{1}{2}(\varphi+\psi)/\sin\tfrac{1}{2}(\varphi-\psi)|\, f(\sin\psi). \tag{B.2}$$

We split up this equation into one for the even eigenfunctions f_+ and one for the odd ones f_-.

$$\lambda f_+(\sin\varphi) = \pi^{-1}\int_{-\pi/2}^{+\pi/2} d\psi \cos\psi \ln|\cot\tfrac{1}{2}(\varphi-\psi)| f_+(\sin\psi), \tag{B.3}$$

$$\lambda f_-(\sin\varphi) = -\pi^{-1}\int_{-\pi/2}^{+\pi/2} d\psi \cos\psi \ln|2\sin(\varphi-\psi)| f_-(\sin\psi).$$

We introduce now the Fourier expansion of $\ln|\cot\frac{1}{2}\vartheta|$ and $-\ln 2|\sin\vartheta|$,[24]

$$\tfrac{1}{2}\ln|\cot\tfrac{1}{2}\vartheta| = \sum_{n=0}^{\infty}\frac{\cos(2n+1)\vartheta}{2n+1}, \quad -\ln 2|\sin\vartheta| = \sum_{n=1}^{\infty}\frac{\cos 2n\vartheta}{n}, \quad -\pi<\vartheta<\pi. \tag{B.4}$$

At this stage it is convenient to shift the integration interval from $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$ to $[0,\pi]$. We expand the functions f_+ and f_- in a sine series on $[0,\pi]$. Because of their even and odd character, the functions contain only even and odd terms.

$$f_+(\cos\varphi) = \sum_{m=0}^{\infty} a_{2m}\sin(2m+1)\varphi, \quad f_-(\cos\varphi) = \sum_{m=0}^{\infty} a_{2m+1}\sin(2m+2)\varphi. \tag{B.5}$$

We substitute Eqs. (B.4) and (B.5) into Eq. (B.3). The integrations are readily performed and we get

$$\lambda \sum_{m=0}^{\infty} a_{2m}\sin(2m+1)\varphi = \frac{2}{\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{2m}\sin(2n+1)\varphi\left[\frac{1}{(2n+1)^2-(2m)^2} - \frac{1}{(2n+1)^2-(2m+2)^2}\right]. \tag{B.6}$$

Similarly for f_-. Now multiply both the left- and right-hand sides of Eq. (B.6) with $\sin(2k+1)\varphi$ and integrate over $[0,\pi]$. Because of the orthogonality of the sine functions, Eq. (B.6) becomes the following matrix equation:

$$\frac{\pi\lambda}{2}a_{2k} = \sum_{m=0}^{\infty}\alpha_{2k,2m}a_{2m}, \quad \frac{\pi\lambda}{2}a_{2k+1} = \sum_{m=0}^{\infty}\alpha_{2k+1,2m+1}a_{2m+1}, \quad k=0,1,\ldots.$$

With according to Eq. (B.6)

$$\alpha_{k,m} = [(k+1)^2 - m^2]^{-1} - [(k+1)^2-(m+2)^2]^{-1}.$$

The solution of the integral equation, Eq. (B.1), is now equivalent with the determination of the eigenvalues and eigenvectors of Eq. (B.7). Since hardly anything is known about the solution of such problems, except about bounds on the eigenvalues, which do not interest us, we have treated the systems in Eq. (B.7) numerically. The matrix is truncated and the eigenvalues and eigenvectors are determined. This has been done for a 5×5 up to a 10×10 matrix, in order to see if the eigenvectors and eigenvalues converge as a function of the truncation order. Excellent convergence is found. It appears that the first seven eigenvalues of each matrix equation in Eq. (B.7) are equal within a few percent to the diagonal elements of the matrix.[25]

Moreover since the integral kernel in Eq. (B.1) is Hermitian, the eigenfunctions must be mutually orthogonal. If we designate by $a_{m,j}$, the mth element of the jth eigenvector, the elements must obey the following relationship (for the even eigenfunctions):

$$\sum_{m,n=0}^{\infty} a_{2m,j}a_{2n,i}\int_0^{\pi}\sin(2m+1)\varphi\sin(2n+1)\varphi\sin\varphi d\varphi = A_j\delta_{i,j}. \tag{B.8}$$

Similarly for the odd eigenfunctions.

This feature can be used as a check on the calculations. It appears that the orthogonality relations are fulfilled within 10^{-10} (absolute error). In the tables the results of the calculations have been given: in Table III the first ten expansion coefficients of the first three even eigenfunctions and the corresponding eigenvalues, and in Table IV the same quantities for the first two odd ones. The eigenfunctions have been normalized to unity (i.e., $\int_{-1}^{+1} f_j^2(\xi)d\xi = 1$). Note that since the coefficients in the tables have been rounded off, the orthogonality relations are now somewhat less better fulfilled (~ 10^{-6}). Transforming back from the interval $[0, \pi]$ to the original interval $[-1, 1]$, we have from the representation Eq. (B.5) [since $U_m(\cos\vartheta) = \sin(m+1)\vartheta/\sin\vartheta$]

$$f_{j,+}(\xi) = (1-\xi^2)^{1/2} \sum_{m=0}^{\infty} a_{2m,j} U_{2m}(\xi), \quad f_{j,-}(\xi) = (1-\xi^2)^{1/2} \sum_{m=0}^{\infty} a_{2m+1,j} U_{2m+1}(\xi). \tag{B.9}$$

The $U_m(\xi)$ are the Tschebyscheff polynomials of the second kind.[26] We remind the reader that these eigenfunctions are so important because, as has been stated in Sec. II, the eigenfunctions ψ_j of Eq. (15) converge to the f_j for $k_0 L \to \infty$ or $\psi_j(\xi = 2x/L) \sim f_j(\xi)$.

Therefore Tables III and IV give the expansion coefficients of the eigenfunctions of Eq. (15) when the line shape is a Doppler profile with or without hfs, Eq. (24).

The convergence of the series in Eq. (B.9) can be enhanced a little if we transform from the Tschebyscheff polynomials to the Jacobi polynomials $P_n^{(1,1)}(\xi)$. This is to be expected since these Jacobi polynomials constitute an orthogonal set with the weight function[27] $(1-\xi^2)$, see Eq. (B.9). The transformation matrix can immediately be calculated from a formula given by Szegö[28]

$$(\sin\vartheta)^2 P_n^{(1,1)}(\cos\vartheta) = \frac{1}{2\Gamma(\frac{3}{2})} \frac{\Gamma(n+3)}{\Gamma(n+\frac{5}{2})} \sum_{\nu=0}^{\infty} f_\nu^{(\frac{3}{2})} \sin(n+2\nu+1)\vartheta \tag{B.10}$$

with $f_0^{(\frac{3}{2})} = 1, \quad f_{\nu,n}^{(\frac{3}{2})} = \dfrac{(1-\frac{3}{2})(2-\frac{3}{2})\cdots(\nu-\frac{3}{2})}{\nu!} \dfrac{(n+1)(n+2)\cdots(n+\nu)}{(n+\frac{5}{2})(n+\frac{7}{2})\cdots(n+\nu+\frac{3}{2})}$.

TABLE III. Eigenvalues λ_j and expansion coefficients $a_{2m,j}$ of corresponding even eigenfunctions.

$\frac{1}{2}\lambda\pi$	1.356 74	0.363 880	0.210 558
$m=0$	+0.848 83	+0.214 95	+0.122 95
1	−0.074 25	+0.829 87	+0.460 83
2	−0.001 47	−0.352 81	+0.563 20
3	−0.000 71	+0.043 57	−0.566 02
4	−0.000 26	−0.005 09	+0.169 38
5	−0.000 12	−0.000 61	−0.030 41
6	−0.000 06	−0.000 41	+0.002 24
7	−0.000 03	−0.000 23	−0.000 74
8	−0.000 02	−0.000 14	−0.000 30
9	−0.000 01	−0.000 10	−0.000 22

TABLE IV. Eigenvalues λ_j and expansion coefficients $a_{2m+1,j}$ of corresponding odd eigenfunctions.

$\frac{1}{2}\lambda\pi$	0.570 213	0.266 591
$m=0$	+0.885 07	+0.366 89
1	−0.215 83	+0.715 84
2	+0.010 55	−0.474 88
3	−0.001 89	+0.098 23
4	−0.000 58	−0.013 62
5	−0.000 29	+0.000 06
6	−0.000 16	−0.000 52
7	−0.000 10	−0.000 28
8	−0.000 06	−0.000 18
9	−0.000 04	−0.000 13

However, for further applications the expansion in Eq. (B.9) is much more suitable because Tschebyscheff polynomials are related to trigonometric functions. The main feature of the expansion in Jacobi polynomials lies in the fact that these polynomials fit into the picture of the solution of the integral equations given by Widom for general α. We shall discuss this briefly in Appendix C.

APPENDIX C

In this Appendix we shall solve the integral equation given by Widom for $\alpha = \frac{1}{2}$. As has been mentioned in Sec. II, we shall then be able to determine the eigenvalues and eigenfunctions of Eq. (15) for a slab when the spectral line shape is given by the Lorentz profile or (what amounts to the same) the Voigt profile with $a \neq 0$. The integral equation is[29]

$$\lambda f(\xi) = \frac{\cos\frac{1}{4}\pi}{\Gamma(\frac{1}{2})} \int_{-1}^{\xi} \frac{d\xi' f(\xi')}{(\xi-\xi')^{1/2}} - \frac{\sin\frac{1}{4}\pi}{\pi\Gamma(\frac{1}{2})} \int_{-1}^{\xi} \frac{d\xi'}{(\xi-\xi')^{1/2}(1-\xi^{1/2})^{1/4}} \int_{-1}^{+1} \frac{d\zeta(1-\zeta^2)^{1/4} f(\zeta)}{\xi'-\zeta} . \tag{C.1}$$

As is customary $⨍$ designates the Cauchy principal value. In the second term of the right-hand side of Eq. (C.1) the integrations have been interchanged. This is allowed in this case.[30] Let us write for the time being

$$g(\xi')=\pi^{-1}⨍_{-1}^{+1} d\xi(1-\xi^2)^{1/4}f(\xi)/(\xi'-\xi), \tag{C.2}$$

and let us expand both functions $f(\xi')$ and $g(\xi')$ as follows

$$f(\xi')=\sum_0^\infty b_k P_k(\xi')\equiv(1-\xi'^2)^{1/4}\sum_{k=0}^\infty c_k U_k(\xi'),\quad g(\xi')=(1-\xi'^2)^{1/4}\sum_0^\infty d_k P_k(\xi'). \tag{C.3}$$

The functions P_k and U_k are the Legendre and Tschebyscheff polynomials of the second kind, respectively. We shall consider the coefficients c_k as the ones to be determined from the integral equation. The other coefficients appearing in the equivalent expansions are connected to the c_k by linear transformations. Equation (C.1) becomes, with the aid of Eqs. (C.2) and (C.3) and after introduction of the new variables $\xi=-\cos\vartheta$, $\xi'=-\cos\varphi$,

$$\lambda(\sin\vartheta)^{-\frac{1}{2}}\sum_k c_k(-1)^k\sin(k+1)\vartheta=\frac{1}{(2\pi)^{1/2}}\sum_n(-1)^n(b_n-d_n)\int_0^\vartheta d\varphi\,\frac{\sin\varphi P_n(\cos\varphi)}{(\cos\varphi-\cos\vartheta)^{1/2}}\ . \tag{C.4}$$

The integrals are easily calculated[31] and Eq. (C.4) becomes

$$\lambda\sum_0^\infty c_k(-1)^k\sin(k+1)\vartheta=2^{-\frac{1}{2}}\sin\vartheta\sum_0^\infty(-1)^n(b_n-d_n)P_n^{-\frac{1}{2}}(\cos\vartheta). \tag{C.5}$$

We multiply both the left- and right-hand sides of Eq. (C.5) with $\sin(m+1)\vartheta$ and integrate over ϑ. Because of the orthogonality of the functions $\sin(m+1)\vartheta$, we have for $m=0,1,\ldots$

$$\tfrac{1}{2}\lambda\pi(-1)^m c_m=2^{-\frac{1}{2}}\sum_n(-1)^n(b_n-d_n)\int_0^\pi\sin\vartheta\sin(m+1)\vartheta P_n^{-\frac{1}{2}}(\cos\vartheta)d\vartheta .$$

This can be put in matrix form

$$\tfrac{1}{2}\lambda\pi\vec{c}=A\vec{b}-A\vec{d}\,, \tag{C.6}$$

where the vectors $\vec{c}$, $\vec{b}$, and $\vec{d}$ are given, respectively, by $(c_0,c_1,c_2,\ldots)$, $(b_0,b_1,b_2,\ldots)$, $(d_0,d_1,d_2,\ldots)$, and the matrix elements of A by

$$\alpha_{m,n}=(-1)^{m+n}2^{-\frac{1}{2}}\int_0^\pi\sin\vartheta\sin(m+1)\vartheta P_n^{-\frac{1}{2}}(\cos\vartheta)d\vartheta . \tag{C.7}$$

For the Legendre functions appearing in this expression the following relation exists[32]:

$$P_n^{-1/2}(\cos\vartheta)=[2/(2n+1)](2/\pi)^{1/2}(\sin\vartheta)^{-1/2}\sin(n+\tfrac{1}{2})\vartheta .$$

The matrix elements are now calculated and become for $m>n$:

$$\alpha_{m,n}=\frac{(-1)^{m+n}}{4(2n+1)}\left(\frac{(-1)^{m+n}-1}{2}\,\frac{\Gamma(\frac{1}{2}(m-n))}{\Gamma(\frac{3}{2}+\frac{1}{2}(m-n))}+\frac{(-1)^{m+n}+1}{2}\,\frac{\Gamma(\frac{1}{2}+\frac{1}{2}(m+n))}{\Gamma(2+\frac{1}{2}(m+n))}\right)\ ;$$

and for $n\geq m, n-m\neq 1$:

$$\alpha_{m,n}=\frac{(-1)^{m+n}}{4(2n+1)}\left(\frac{(-1)^{m+n}+1}{2}\right)\left(\frac{\Gamma(\frac{1}{2}+\frac{1}{2}(m+n))}{\Gamma(2+\frac{1}{2}(m+n))}-\frac{\Gamma(-\frac{1}{2}+\frac{1}{2}(n-m))}{\Gamma(1+\frac{1}{2}(n-m))}\right)\ ,\quad \alpha_{n,n+1}=-\tfrac{1}{4}\sqrt{\pi}\ . \tag{C.8}$$

We proceed now to the calculation of the transformations that connect the vectors $\vec{c}$, $\vec{b}$, and $\vec{d}$. See Eqs. (C.2) and (C.3). According to Eq. (C.2) the following relation exists between $g(\xi')$ and $f(\xi)$

$$g(\xi')=\pi^{-1}⨍_{-1}^{+1}(1-\xi^2)^{1/4}f(\xi)/(\xi'-\xi).$$

If we expand $f(\xi)$ in this formula in the Tschebyscheff polynomials of the second kind, Eq. (C.3), we find by a well-known formula[33]

$$(1-\xi'^2)^{1/4}\sum_0^\infty d_k P_k(\xi') = \sum_0^\infty c_j T_{j+1}(\xi'), \tag{C.9}$$

where the functions $T_j(\xi')$ are the Tschebyscheff polynomials of the first kind. If we now write in matrix notation $\vec{d} = W\vec{c}$, then we have for the matrix elements of W, upon multiplication of both the left- and right-hand sides of Eq. (C.9) with $(1-\xi'^2)^{-1/4} P_n(\xi')$ and integration over $[-1,+1]$,

$$W_{n,j} = (n+\tfrac{1}{2})\int_0^\pi d\psi\,(\sin\psi)^{1/2} P_n(\cos\psi)\cos(j+1)\psi. \tag{C.10a}$$

For the evaluation of this integral we use the well-known representation of the Legendre polynomials[34]

$$P_n(\cos\psi) = \sum_{k=0}^n g_k g_{n-k}\cos(n-2k)\psi = \sum_{k=0}^n g_k g_{n-k} e^{i(n-2k)\psi}.$$

The integral can be written

$$W_{n,j} = (n+\tfrac{1}{2})\,\mathrm{Re}\int_0^\pi d\psi \sum_{k=0}^n g_k g_{n-k} e^{i(n-2k+j+1)\psi}(\sin\psi)^{\frac{1}{2}}. \tag{C.10b}$$

Now integrate the following integral over a semicircle $|z|=1$ in the upper half of the complex plane and the line $[-1,1]$

$$\frac{1}{(2i)^{1/2}}\sum_{k=0}^n g_k g_{n-k}\oint z^{n-2k+j-\frac{1}{2}}(1-z^2)^{\frac{1}{2}}\,dz.$$

For $j-n>0$ the integrand has no singularities. For $j-n=0$ we indent the contour at $z=0$. Following standard methods we arrive at

$$\int_0^\pi d\varphi\,P_n(\cos\varphi)(\sin\varphi)^{\frac{1}{2}}\cos(j+1)\varphi = -\mathrm{Re}\int_{-1}^{+1} dx\sum g_k g_{n-k} x^{n-2k+j-\frac{1}{2}}(1-x^2)^{\frac{1}{2}}/(2i)^{\frac{1}{2}}.$$

This integral is readily evaluated in terms of β functions. After substitution of the expression for g_k we have

$$\frac{\pi}{\Gamma(\frac{3}{2})}W_{n,j} = -(n+\tfrac{1}{2})\tfrac{1}{4}[1+(-1)^{n+j+1}]\sum_{k=0}^n \frac{\Gamma(k+\frac{1}{2})}{\Gamma(k+1)}\,\frac{\Gamma(n-k+\frac{1}{2})}{\Gamma(n-k+1)}\,\frac{\Gamma(\frac{1}{2}(n+j)-k+\frac{1}{4})}{\Gamma(\frac{1}{2}(n+j)-k+\frac{7}{4})}. \tag{C.11}$$

This formula is as well valid for $j-n<0$ by analytic continuation. The transformation V of the vector $\vec{b}$ into $\vec{c}$ is of the same type. The defining relation is [see Eq. (C.3)]

$$\sum_0^\infty b_k P_k(\xi') = (1-\xi'^2)^{1/4}\sum_0^\infty c_j U_j(\xi'),$$

which gives for $n=0,1,2,\ldots$

$$b_n = (n+\tfrac{1}{2})\sum_j c_j\int_0^\pi d\psi\,(\sin\psi)^{1/2}P_n(\cos\psi)\sin(j+1)\psi.$$

Following the analysis given above it is easily shown that, if we write $\vec{b} = V\vec{c}$, the matrix V has the matrix elements

$$\frac{\pi}{\Gamma(\frac{3}{2})}u_{n,j} = (n+\tfrac{1}{2})\tfrac{1}{4}[1+(-1)^{n+j}]\sum_{k=0}^n \frac{\Gamma(k+\frac{1}{2})}{\Gamma(k+1)}\,\frac{\Gamma(n-k+\frac{1}{2})}{\Gamma(n-k+1)}\,\frac{\Gamma(\frac{1}{2}(n+j)-k+\frac{1}{4})}{\Gamma(\frac{1}{2}(n+j)-k+\frac{7}{4})}. \tag{C.12}$$

Compare this with Eq. (C.11). The system given in Eq. (C.6) becomes now in matrix notation

$$\tfrac{1}{2}\lambda\pi\vec{c} = A(V-W)\vec{c}. \tag{C.13}$$

The matrix $A(V-W)$ is symmetric, since the integral operator given in Eq. (C.1) is Hermitian (this is not obvious but has been shown by Widom), and the functions $(1-\xi^2)^{1/4}\,U_m(\xi)$ constitute an (complete and) orthogonal system on $[-1,+1]$.

Moreover the matrix elements that connect even and odd functions must be zero, since, as again has been shown by Widom, the integral kernel has a definite parity. Of course, this must be the case because Eq. (15) has a definite parity.

The calculation of the matrices is straightforward. The matrix multiplication is performed numerically. The above-mentioned features of the resulting matrix $A(V-W)$ may serve as checks on the calculations. The matrix is split up into a matrix for the even eigenfunctions and one for the odd eigenfunctions. These two are truncated, and eigenvectors and eigenvalues are calculated numerically as a function of the truncation order. Excellent convergence was found. Again, as in the case treated in Appendix B, the eigenvalues are equal within a few percent to diagonal elements of the matrices. The results for the first three eigenfunctions are given in Table V and for the first two odd ones in Table VI. The eigenfunctions have been normalized [i.e., $\int_{-1}^{+1} f_j{}^2(\xi)d\xi=1$].

The eigenfunctions constitute again an orthogonal system. As has been mentioned in Sec. II, the eigenfunctions ψ_j of Eq. (15) for a Voigt ($a\neq 0$) or Lorentz profile converge for $k_0L\to\infty$ to the eigenfunctions $f_j(\xi)$ found in this Appendix, or

$$\psi_j(\xi=2x/L)\sim f_j(\xi).$$

Therefore, the Tables V and VI give as well the coefficients in Eq. (28).

The solutions for the integral equations for $\alpha=1$ and $\alpha=\frac{1}{2}$ obtained in the Appendices B and C, respectively, suggest the form of the solution for general α. The generalization is that a suitable set of functions for the expansion of the eigenfunctions is furnished by $(1-\xi^2)^{\alpha/2}\,P_n{}^{(\alpha,\alpha)}(\xi)$, where the $P_n{}^{(\alpha,\alpha)}(\xi)$ are the Jacobi polynomials. Note that

$$U_m(\xi)\propto P_n^{(\frac{1}{2},\frac{1}{2})}(\xi).$$

The expansion of the eigenfunctions of Eq. (15) for the statistical line profile [see Appendix A, Eq. (A.14)] would be, therefore,

$$(1-\xi^2)^{\frac{1}{6}}\sum_{n=0}^{\infty} c_n P_n^{(\frac{1}{3},\frac{1}{3})}(\xi).$$

For the calculation of the expansion coefficients and the eigenvalues λ_j, one may proceed along the lines given in this Appendix. Because interest for this case seems to be lacking and the calculations are much more difficult, we have considered it not worth doing.

TABLE V. Eigenvalues λ_j and expansion coefficients $b_{2m,j}$ of corresponding eigenfunctions.

$\frac{1}{2}\lambda\pi$	1.61910	0.774 241	0.582 911
$m=0$	+0.793 06	+0.077 02	+0.031 88
1	−0.087 48	+0.715 52	+0.272 16
2	−0.004 63	−0.341 93	+0.509 48
3	−0.001 92	+0.041 99	−0.524 29
4	−0.000 87	−0.006 28	+0.161 66
5	−0.000 46	−0.001 32	−0.030 74
6	−0.000 28	−0.000 89	+0.001 74
7	−0.000 18	−0.000 57	−0.001 13
8	−0.000 12	−0.000 39	−0.000 61
9	−0.000 09	−0.000 30	−0.000 50

TABLE VI. Eigenvalues λ_j and expansion coefficients $b_{2m+1,j}$ of corresponding odd eigenfunctions.

$\frac{1}{2}\lambda\pi$	0.980 805	0.658 020
$m=0$	+0.765 53	+0.189 12
1	−0.224 70	+0.625 70
2	+0.008 70	−0.447 27
3	−0.003 27	+0.095 34
4	−0.001 36	−0.014 65
5	−0.000 78	−0.000 53
6	−0.000 48	−0.000 95
7	−0.000 32	−0.000 60
8	−0.000 22	−0.000 42
9	−0.000 17	−0.000 34

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