Critical Region for the Ising Model with a Long-Range Interaction

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A version of the Ising model is developed in which the spin variables can be treated accurately in the continuum approximation. The perturbation series, both above and below the critical temperature T_c, is examined, and it is shown that there is a shift of T_c from its mean-field value proportional to $q^{-2}\ln q$, as well as the well-known shift proportional to q^{-1}; here q is the number of mutually interacting particles. It is shown, using renormalization theory, that there is a perturbation series in $q^{-1}|T-T_c|^{-1/2}$ for which all terms are finite in the limit $q\to\infty$, if the shift of T_c is put in correctly. For the two-dimensional model, the shift is shown to be proportional to $q^{-1}\ln q$. Conditions are derived for a finite system to display critical behavior characteristic of three, two, one, or zero dimensions. It is shown how similar results can be obtained for a model similar to the Heisenberg model and for the standard Ising and Heisenberg models with interactions extending over many neighbors. A comparison is made between previously calculated numerical results for T_c and the asymptotic forms derived here.

1. INTRODUCTION

DEVIATIONS from Landau's[1] "classical" theory of second-order phase transitions have been observed in most systems except superconductors. It has generally been supposed that deviations occur in all systems, but that the width of the temperature range in which such deviations are appreciable depends on the range of the forces responsible for the phase transition. It was argued by Ginzburg[2] on the basis of the Landau-Ginzburg theory of superconductivity[3] and by Thouless[4] using the Bardeen, Cooper, and Schrieffer theory of superconductivity[5] that there should be deviations from the classical theory over a temperature range with a width of order $(a/\xi)^6 T_c$, where a is the mean spacing between electron pairs and ξ is the coherence length—which should be regarded as the effective range of the force between electron pairs. This result is in conflict with an earlier argument due to Pippard[6] which predicts a critical region of width $(a/\xi)^{3/2}T_c$. The deviations from classical behavior in Refs. 2 and 4 are produced by a term proportional to $|T-T_c|^{-1/2}$ in the specific heat. Such a term was also found in studies of magnetic systems with long-range forces by Brout and his collaborators; again the specific heat contained a term proportional to $\kappa(a/l)^3|1-T_c/T|^{-1/2}$, where l is the range of the forces and κ is Boltzmann's constant. A discussion of this work and further references can be found in the book by Brout.[7]

Recent work by Patashinskii and Pokrovskii,[8] Lebowitz *et al.*,[9] Abe,[10] and Vaks, Larkin, and Pikin,[11,12] and a number of other authors has taken the matter further. In particular, Vaks, Larkin, and Pikin[11] have shown that, for the three-dimensional Ising model, there is a shift of the transition temperature from its mean field value proportional to

$$q^{-1}=(a/l)^3\,, \tag{1}$$

and that the specific heat can then be expanded as a series of powers of $q^{-1}|1-T_c/T|^{-1/2}$. This suggests that a resummation of the series should give a function of $q^2(1-T_c/T)$ that might give the critical behavior of the system. The limit $q\to\infty$ with

$$\theta=|1-T_c/T| \tag{2}$$

kept proportional to q^{-2} cannot be taken immediately, however, as is clear from the existence of a term proportional to

$$(q^2\theta)^{-3/2}\ln\theta\,, \tag{3}$$

which diverges logarithmically in this limit. It is shown here that this difficulty can be removed by a further shift of T_c.

In this paper a modified form of the Ising model is studied. It seems to make no essential difference, and leads to a perturbation expansion whose meaning is (to the author, at least) much clearer than the expansion for the standard Ising model. The expression for the free energy looks very much like that of the Landau-

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Ginzburg[3] theory, and this system has recently been the subject of a study by Ferrell.[13]

The perturbation series is a series in a^3/l^3, and gives corrections to the zero-order solution which, above T_c, is just the Gaussian model of Berlin and Kac.[14] Below T_c the main contribution is from the mean field. Well away from T_c, on either side, the series appears to be rapidly convergent, and it is only for θ of the order of q^{-2} that either the terms of the Gaussian model or the perturbation terms make an appreciable contribution to the specific heat.

In this work more emphasis is placed on the specific heat than on the magnetic susceptibility because an infinite specific heat is characteristic of critical behavior, whereas the magnetic susceptibility is infinite at T_c even in the Landau theory, so that corrections to it are inevitably large near T_c.

Once the perturbation series is known on both sides of T_c, it can be checked for consistency. Both the free energy F and the energy E should be continuous through the phase transition. In fact, they extrapolate to different values at the mean-field T_c, but the apparent discontinuities can be removed by a shift of T_c.

This shift of T_c has the effect of systematically cancelling certain terms in the perturbation series. These terms diverge in the limit $q\rightarrow\infty$, with $q^2\theta$ constant, and the shift of T_c is like mass renormalization in field theory. The shift produces counter-terms which approximately cancel the divergent contributions. Even after the shift in T_c proportional to q^{-1} has been allowed for, there remain logarithmically divergent terms, but these can be removed by a further shift of T_c proportional to $q^{-2}\ln q$. Once this has been done, the terms that remain have a convergent limit.

It is important to notice that when the renormalization is carried out no attempt is made to evaluate the terms self-consistently. This is in sharp contrast to the work of Patashinskii and Pokrovskii[8] and of Abe,[10] who get their results by a self-consistent treatment of certain terms in the perturbation series. It is argued here that the corrections produced by self-consistency are no more important than other terms in the series that have been ignored.

It can be checked by the continuity of F and E to the required order in q^{-1} that there is no specific-heat singularity producing appreciable effects over a temperature range greater than $q^{-2}T_c$. It seems very likely that the effects within this range give rise to a singular behavior. In this range all terms in the perturbation series become of order unity, but there is no evidence on the nature of the singularity in this region, or even on the existence of a singularity to this order. Resummation of the perturbation series is necessary.

In Sec. 7 it is shown how the model can be modified when the order parameter is a vector, as it is in the Heisenberg model. The treatment of this problem is not completely satisfactory, since it is not proved that the noncommutation of the operators can be ignored.

In Sec. 8 two- and one-dimensional problems are considered. In two dimensions the critical region is of width q^{-1} and the shift of T_c is proportional to $q^{-1}\ln q$, while in one dimension the critical region is of width $q^{-2/3}$, and there is no shift of T_c larger than this. In one dimension, we know that the critical behavior is analytic everywhere and that there is no singularity in the specific heat. Conditions are derived for a finite system to display critical behavior characteristic of three, two, one, or zero dimensions.

The perturbation series can be compared with the perturbation series for the standard Ising and Heisenberg models. There is found to be a close similarity. The shifts of T_c are calculated in Sec. 9, and it is suggested that the form of the specific heat in the critical region may be the same for this version of the models and the standard version. The results for the standard Ising and Heisenberg models are compared in Sec. 10 with the numerical results of Domb and Dalton,[15] who have calculated critical temperatures for interactions ranging up to third neighbors. It appears that three neighbors is not far enough for the system to display the shift in T_c of order $q^{-2}\ln q$ in three dimensions, but the results for two-dimensional lattices agree well with the calculated proportionality to $q^{-1}\ln q$.

2. FORMULATION OF THE MODEL

We suppose we have a system divided up into N/n cells, each of which contain n Ising spins. Each spin in the cell i interacts with each spin in the cell j, and the energy of interaction is $\mp\frac{1}{2}J_{ij}$, with the minus sign if the spins are parallel and the plus sign if they are antiparallel; the strength of the interaction depends only on the separation between the cells, not on the positions of the spins within a cell. If we write $2S_i$ for the difference between the numbers of up spins and down spins in the cell i, the energy is

$$E=-\sum_i\sum_j J_{ij}S_iS_j+\tfrac{1}{4}n\sum_i J_{ii}. \tag{4}$$

The entropy S is given by

$$\begin{aligned}S/\kappa&=\sum_i \ln\frac{n!}{(\frac{1}{2}n+S_i)!(\frac{1}{2}n-S_i)!}\\&=N\ln 2-\sum_i\left[\tfrac{1}{2}(n+1)\ln\left(1-\frac{4S_i^2}{n^2}\right)+S_i\ln\frac{1+2S_i/n}{1-2S_i/n}\right]\\&\qquad-(N/2n)\ln\tfrac{1}{2}\pi n+O(n^{-1}).\end{aligned} \tag{5}$$

The partition function can be calculated by taking the exponential of $S/\kappa-E/\kappa T$ given by Eqs. (5) and (4), and summing over all possible sets of values of the S_i.

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This sum over the S_i can be replaced by a multiple integral. The error introduced by this step is of the order of $e^{-\alpha n}$, where α is of the order of unity, and so this error can be ignored. We now have a model with some of the simplicity of the Gaussian and spherical models of Kac and Berlin.[14]

For temperatures above the critical temperature, we expand Eq. (5) in powers of S_i^2/n^2 to get the partition function as

$$\mathfrak{z}=\int dS_1\cdots\int dS_{N/n}2^N(\tfrac{1}{2}\pi n)^{-N/2n} \times\exp\left[-\frac{2(n-1)}{n^2}\sum_i S_i^2+\frac{1}{\kappa T}\sum_i\sum_j J_{ij}S_iS_j -\frac{n}{4\kappa T}\sum_i J_{ii}-\left(\frac{4}{3n^3}-\frac{4}{n^4}\right)\sum_i S_i^4+O\left(\frac{S_i^6}{n^5}\right)\right]. \quad (6)$$

Except in the immediate neighborhood of T_c the quadratic terms in the exponent are positive, and the main contribution to the integral comes from the range in which S_i^2/n is of order unity. The quartic terms are of order n^{-1} and can be treated by perturbation theory. The higher-order terms have a negligible effect. This perturbation series, it should be noticed, is not convergent, since the expansion of

$$\int e^{-(1/2)x^2-\alpha x^4}dx$$

in powers of α is not convergent.

Below T_c the integral will be dominated by the values of S_i in the neighborhood of the mean-field value $\bar{S}$, but deviations from the mean-field value will be controlled, except very close to T_c, by the terms in the exponent quadratic in $S_i-\bar{S}$. The expression analogous to Eq. (6) is

$$\mathfrak{z}=\int dS_1\cdots\int dS_{N/n}2^N(\tfrac{1}{2}\pi n)^{-N/2n}\exp\Bigg\{-\tfrac{1}{2}N(1+1/n)\ln\left(1-\frac{4\bar{S}^2}{n^2}\right)-\frac{N\bar{S}}{n}\ln\frac{1+2\bar{S}/n}{1-2\bar{S}/n}+\frac{N\bar{S}^2}{n\kappa T}\sum_j J_{ij}-\frac{N}{4\kappa T}J_{ii}$$
$$+\sum_i(S_i-\bar{S})\left(-\ln\frac{1+2\bar{S}/n}{1-2\bar{S}/n}+\frac{4\bar{S}/n^2}{1-4\bar{S}^2/n^2}+\frac{2\bar{S}}{\kappa T}\sum_j J_{ij}\right)+\tfrac{1}{2}\sum_i(S_i-\bar{S})^2\left(-\frac{4/n}{1-4\bar{S}^2/n^2}+\frac{4/n^2+16\bar{S}^2/n^4}{(1-4\bar{S}^2/n^2)^2}\right)$$
$$+\frac{1}{\kappa T}\sum_i\sum_j J_{ij}(S_i-\bar{S})(S_j-\bar{S})-\sum_i\bar{S}(S_i-\bar{S})^3\left[\frac{16/3n^3}{(1-4\bar{S}^2/n^2)^2}+O\left(\frac{1}{n^4}\right)\right]$$
$$-\sum_i(S_i-\bar{S})^4\left(\frac{4/3n^3+16\bar{S}^2/n^5}{(1-4\bar{S}^2/n^2)^3}+O(n^{-4})\right)+O\left(\frac{(S_i-\bar{S})^5}{n^4}\right)\Bigg\}. \quad (7)$$

This expression has an exponent which consists of the mean-field theory expression for $-F/\kappa T$, a term linear in $S_i-\bar{S}$ which vanishes when $\bar{S}$ is equal to its value in mean-field theory, a negative-definite quadratic term, and cubic and quartic terms which can be treated by perturbation theory.

3. PERTURBATION THEORY

In the expressions (6) and (7) the quadratic terms in the exponent dominate, except very close to the transition temperature. These quadratic terms can be diagonalized by transforming to the variables

$$\sigma_{\mathbf{k}}=(n/N)^{1/2}\sum_i S_ie^{i\mathbf{k}\cdot\mathbf{R}_i}, \quad (8)$$

where $\mathbf{R}_i$ is the center of the cell i. The quadratic part of the exponent of Eq. (6) gives

$$-2\frac{n-1}{n^2}\sum_i S_i^2+\frac{1}{\kappa T}\sum_i\sum_j S_iS_j =\sum_{\mathbf{k}}\left(-\frac{2(n-1)}{n^2}+\frac{G(\mathbf{k})}{\kappa T}\right)\sigma_{-\mathbf{k}}\sigma_{\mathbf{k}}, \quad (9)$$

where

$$G(\mathbf{k})=\sum_j J_{ij}e^{i\mathbf{k}\cdot(\mathbf{R}_i-\mathbf{R}_j)}. \quad (10)$$

The variables $\sigma_{\mathbf{k}}$ and $\sigma_{-\mathbf{k}}$ are complex conjugates of one another, and their real and imaginary parts can be regarded as independent Gaussian variables with mean zero and variance $\frac{1}{8}nh(\mathbf{k})$, where

$$nh(\mathbf{k})=4\langle\sigma_{\mathbf{k}}\sigma_{-\mathbf{k}}\rangle_0=n[1-1/n-nG(\mathbf{k})/2\kappa T]^{-1}. \quad (11)$$

The symbol $\langle\cdots\rangle_0$ denotes an expectation value in the Gaussian distribution. The unperturbed expression for the partition function given by Eq. (6) is

$$\mathfrak{z}_0=2^N\prod_{\mathbf{k}}[h(\mathbf{k})]^{1/2}, \quad (12)$$

giving a free energy

$$F_0=\kappa T[-N\ln 2-\tfrac{1}{2}\sum_{\mathbf{k}}\ln h(\mathbf{k})] \quad (13)$$

and an energy

$$E_0=-\sum_{\mathbf{k}}\tfrac{1}{4}nG(\mathbf{k})h(\mathbf{k}). \quad (14)$$

The first-order term in the perturbation series gives

$$\frac{F_1}{\kappa T}=\left(\frac{4}{3n^3}-\frac{4}{n^4}\right)\sum_i \langle S_i^4\rangle_0$$
$$=\left(\frac{4}{n^3}-\frac{12}{n^4}\right)\sum_i \langle S_i^2\rangle_0\langle S_i^2\rangle_0$$
$$=\frac{1}{4N^2}[1-(3/n)][\sum_{\mathbf{k}} h(\mathbf{k})]^2. \quad (15)$$

The second-order term gives

$$\frac{F_2}{\kappa T}=-\frac{1}{2}\left(\frac{4}{3n^3}-\frac{4}{n^4}\right)^2\sum_i\sum_j (\langle S_i^4S_j^4\rangle_0-\langle S_i^4\rangle_0\langle S_j^4\rangle_0)$$
$$=-\frac{1}{12N^2}[1-(3/n)]^2 \times\sum_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4} h(\mathbf{k}_1)h(\mathbf{k}_2)h(\mathbf{k}_3)h(\mathbf{k}_4)\delta_{\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4}$$
$$-\frac{1}{4N^2}[1-(3/n)]^2\sum_{\mathbf{k}_1}[h(\mathbf{k}_1)]^2[\sum_{\mathbf{k}_2} h(\mathbf{k}_2)]^2, \quad (16)$$

where $\delta_{\mathbf{K}}$ is unity if $\mathbf{K}$ is zero or some other reciprocal lattice vector, and is zero otherwise.

The perturbation series can be represented diagrammatically in a standard way (see, for example, Ref. 7). The diagram consists of undirected lines connecting vertices. Each vertex represents a factor of S_i^4, and each line represents $\langle S_iS_j\rangle_0$. Four lines go through each vertex, and only linked diagrams contribute to the free energy. If we label each line with a wave number $\mathbf{k}$, the line contributes a factor $\frac{1}{4}nh(\mathbf{k})$, while a vertex contributes $(4/3Nn^2)(1-3/n)\delta_{\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4}$. There is also a combinatorial factor for each diagram. Figure 1(a) shows the diagram corresponding to Eq. (15), and Figs. 1(b) and 1(c) show the diagrams for the first and second terms of Eq. (16).

As the limit $n\to\infty$ is taken, we must keep $nG(\mathbf{k})$ constant, so that T_c remains constant. Except in the immediate neighborhood of T_c, $h(\mathbf{k})$ is of the order of unity. There are N/n terms in the sum over $\mathbf{k}$, so that the two terms on the right side of Eq. (13) are of order N and N/n, the right side of Eq. (15) is of order N/n^2, and the right side of Eq. (16) is of order N/n^3. The series is a series in inverse powers of n. The term of order r has $2r$ factors of nh, r factors of $1/Nn^2$, and $r+1$ independent sums over wave number, so that its contribution is of order N/n^{r+1}.

At the mean-field T_c, $h(0)$ is infinite, and, since $G(\mathbf{k})-G(0)$ is quadratic for small k if J_{ij} has a finite range [more precisely, if $\sum_j J_{ij}(\mathbf{R}_i-\mathbf{R}_j)^2$ is finite], we have

$$h(\mathbf{k})\approx(\theta+\alpha k^2)^{-1} \quad (17)$$

for small k, where θ is defined by Eq. (2). It can be seen

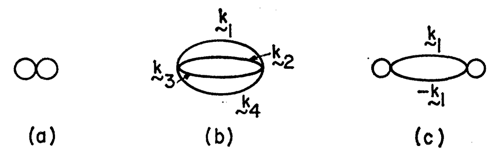

FIG. 1. Lowest-order diagrams for the free energy above T_c.

that F_0 and E_0 given by Eqs. (13) and (14) are finite in the limit $\theta\to 0$ in a three-dimensional system, but the derivative of E_0 diverges, so that the specific heat goes as $1/n\sqrt{\theta}$. F_1 is finite, but its derivative E_1 diverges, and the second term of F_2, containing the factor $\sum[h(k)]^2$, diverges for $\theta=0$. Most of this paper is devoted to a careful examination of the behavior of the series as θ approaches zero.

Below T_c the main contribution to the free energy comes from the constant term in the exponent, and the constant and quadratic terms together give

$$F_0/\kappa T=-N\ln 2+\tfrac{1}{2}N(1+1/n)\ln(1-4\bar{S}^2/n^2)+(N\bar{S}/n)\ln[(1+2\bar{S}/n)/(1-2\bar{S}/n)]$$
$$-\frac{N\bar{S}^2}{n\kappa T}\sum_j J_{ij}+\frac{N}{4\kappa T}J_{ii}-\tfrac{1}{2}\sum_{\mathbf{k}}\ln h(\mathbf{k}), \quad (18)$$

where

$$h(\mathbf{k})=[1/(1-4\bar{S}^2/n^2)-(n^{-1}+4\bar{S}^2/n^3)/(1-4\bar{S}^2/n^2)^2-nG(\mathbf{k})/2\kappa T]^{-1}. \quad (19)$$

The mean magnetization $\bar{S}$ can be chosen to make the coefficient of $S_i-\bar{S}$ vanish, and the third- and fourth-order terms give rise to a perturbation series that can be represented by diagrams with three or four lines at each vertex. The terms of lowest order are shown in Fig. 2, and they give

$$\frac{F_1}{\kappa T}=\frac{1}{4N}\frac{1+12\bar{S}^2/n^2}{(1-4\bar{S}^2/n^2)^3}[\sum_{\mathbf{k}} h(\mathbf{k})]^2$$
$$-\frac{4}{3N}\frac{\bar{S}^2/n^2}{(1-4\bar{S}^2/n^2)^4}\sum_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} h(\mathbf{k}_1)h(\mathbf{k}_2)h(\mathbf{k}_3)\delta_{\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3}$$
$$-\frac{2}{N}\frac{\bar{S}^2/n^2}{(1-4\bar{S}^2/n^2)^4}h(0)[\sum_{\mathbf{k}} h(\mathbf{k})]^2. \quad (20)$$

In general, a diagram with r fourth-order vertices and $2r'$ third-order vertices gives $2r+3r'$ factors of nh, r factors of order $1/Nn^2$, r' factors of order $\bar{S}^2/Nn^5$, and there are $r+r'+1$ independent sums over wave number, so that the total contribution is of order $N(\bar{S}/n)^{2r'}n^{-r-r'-1}$. Since $\bar{S}$ is of order n, this is of order $N/n^{r+r'+1}$, and two third-order vertices contribute in the same order as a single fourth-order vertex.

Close to T_c, but below it, we have

$$\frac{\bar{S}^2}{n^2}\approx-\frac{3}{4}\frac{1-1/n-nG(0)/2\kappa T}{1-3/n}\approx\tfrac{3}{4}\theta\frac{1-1/n}{1-3/n}, \quad (21)$$

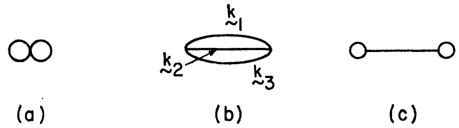

FIG. 2. First-order diagrams for the free energy below T_c.

so that, according to Eq. (19),

$$h(\mathbf{k})\approx(2\theta+\alpha k^2)^{-1} \tag{22}$$

for small k, in contrast with Eq. (17) valid above T_c.

4. CONTINUITY OF F AND E

We have series for the free energy F and the energy E that appear to converge rapidly everywhere except in the vicinity of T_c. Although neither series is useful close to T_c, we can exploit the fact that we have expressions for F and E on both sides of T_c, and we know that F and E are continuous at T_c. For example, a specific-heat anomaly of width ΔT would produce a shift of the energy above T_c relative to the energy below T_c proportional to ΔT, and a shift of F proportional to $(\Delta T)^2$. This might be detected by extrapolation to T_c of the expressions for F and E on either side of T_c. A shift in T_c produces an effect of this sort because there is a discontinuity of $\frac{3}{2}\kappa$ in the specific heat, so that an assumed critical temperature that is higher than the true critical temperature by an amount ΔT will produce an apparent discontinuity in the energy at T_c of $-\frac{3}{2}N\kappa\Delta T$ and a discontinuity in F of $\frac{3}{4}N\kappa(\Delta T)^2/T_c$.

We can look for possible effects over a temperature range of order T_c/n by evaluating F and E up to order n^{-2} and n^{-1}, respectively, at a temperature $T=T_c\pm A/n$, and then letting $A\to 0$. At the temperature $T=T_c+A/n$, we have

$$\sum_{\mathbf{k}}[h(\mathbf{k})]^2\sim\frac{VN}{2\pi^2 n}\int_0\frac{k^2dk}{(A/n+\alpha k^2)^2}\sim\frac{VN}{2\pi\alpha^{3/2}A^{1/2}n^{1/2}}, \tag{23}$$

where V is the volume of a cell. The last term on the right side of Eq. (16) is therefore of order $N/n^{5/2}A^{1/2}$, and does not contribute to F in the order n^{-2}, if the limit $n\to\infty$ is taken before $A\to 0$. In just the same way it can be shown that E_1 calculated from Eq. (15) does not contribute to E in the order n^{-1} under these conditions, and the same applies to all further terms in the perturbation series.

The energy given by mean-field theory tends to zero as $T\to T_c$, but there remains the contribution of the derivative of $\frac{1}{2}\sum\ln h(\mathbf{k})$, which from Eqs. (19) and (21) is

$$E_0=-\sum_{\mathbf{k}}\left[\tfrac{1}{4}n[G(\mathbf{k})]+\tfrac{1}{2}(1-3/n)\kappa T^2\frac{d}{dT}\left(\frac{4\bar{S}^2}{n^2}\right)\right]h(\mathbf{k})$$
$$=-\sum_{\mathbf{k}}[\tfrac{1}{4}nG(\mathbf{k})-(1-1/n)\tfrac{3}{2}\kappa T_c]h(\mathbf{k}). \tag{24}$$

Comparing with Eq. (14) obtained above T_c, we see that, since $h(\mathbf{k})$ is continuous at T_c, the discontinuity in E is

$$E_0(T_c+)-E_0(T_c-)=-(1-n^{-1})\tfrac{3}{2}\kappa T_c\sum_{\mathbf{k}}h_0(\mathbf{k}), \tag{25}$$

where $h_0(\mathbf{k})$ is $h(\mathbf{k})$ at $T=T_c$.

Equations (13) and (18) for F_0 tend to the same limit at the critical temperature. Of the three terms on the right side of Eq. (20), the first tends to the same value as Eq. (15), the second tends to zero (like $\theta\ln\theta$, as will be shown later), and the third tends to a constant because, from Eq. (22), $h(0)$ becomes infinite, like $(2\theta)^{-1}$. The discontinuity in F is therefore

$$\Delta F=\tfrac{3}{4}(\tau T_c/N)[\sum_{\mathbf{k}}h_0(\mathbf{k})]^2, \tag{26}$$

where we have dropped the irrelevant correction of order $1/n$. Equations (25) and (26) are compatible with a shift of T_c equal to $N^{-1}\sum h_0(\mathbf{k})$, with no other deviations from classical behavior. The corrected critical temperature is therefore

$$\kappa T_c\approx\tfrac{1}{2}nG(0)[1+1/n-N^{-1}\sum_{\mathbf{k}}h_0(\mathbf{k})]$$
$$=\tfrac{1}{2}nG(0)\{1-N^{-1}\sum_{\mathbf{k}}[G(\mathbf{k})/G(0)-G(\mathbf{k})]\}. \tag{27}$$

This formula for T_c was first derived by Brout[16] using the random-phase approximation, and has also been derived by Dalton and Domb[17] and by Vaks, Larkin, and Pikin.[11]

5. RENORMALIZATION

The shift of T_c given by Eq. (27) can be included in our perturbation theory by adding a term equal to $-(8/n^3)\langle S_i^2\rangle_0\sum_i S_i^2$ to the quadratic terms of the exponent in Eqs. (6) and (7), and by taking it away again from the perturbing quartic terms. This additional perturbing term is known as a "counter-term" in renormalization theory, and the shift in T_c is analogous to mass renormalization in quantum electrodynamics. If

$$4\langle S_i^2\rangle_0=(n/N)\sum_{\mathbf{k}}h(\mathbf{k}),$$

is evaluated at the critical temperature of the renormalized theory, then T_c is given by

$$T_c=\frac{nG(0)}{2\kappa}\left(1+N^{-1}\sum_{\mathbf{k}}\frac{G(\mathbf{k})}{G(0)-G(\mathbf{k})}\right)^{-1} \tag{28}$$

rather than by Eq. (27). This is the form originally given by Brout,[16] and is identical to T_c given by the spherical model of Berlin and Kac.[14]

[16] R. Brout, Phys. Rev. **118**, 1009 (1960).
[17] N. W. Dalton and C. Domb, Proc. Phys. Soc. (London) **89**, 873 (1966).

We might try to evaluate $\langle S_i^2\rangle_0$ as a function of temperature, and evaluate $h(\mathbf{k})$ self-consistently in this approximation. In diagrammatic perturbation theory this looks very like temperature-dependent Hartree-Fock theory for fermions. There is no justification for doing this, since the corrections are of order n^{-2} and we have neglected other terms of this order. Indeed, the difference between Eqs. (27) and (28) should be regarded as insignificant for the same reason. If $\langle S_i^2\rangle_0$ is evaluated self-consistently as a function of temperature, peculiar results are obtained, and so we take it at a fixed temperature.

Above T_c, the additional diagrams which must be added to the diagrams of Fig. 1 are shown in Fig. 3. The counter-term is represented by a heavy dot. Ignoring the factors of $1-3/n$, we get

$$\frac{F_1}{\kappa T}=\frac{1}{4N}\{\sum_{\mathbf{k}}[h(\mathbf{k})-h_0(\mathbf{k})]\}^2-\frac{1}{4N}[\sum_{\mathbf{k}}h_0(\mathbf{k})]^2, \quad (29)$$

$$\frac{F_2}{\kappa T}=-\frac{1}{12N^2}\sum_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4}h(\mathbf{k}_1)h(\mathbf{k}_2)h(\mathbf{k}_3)h(\mathbf{k}_4)\delta_{\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4}$$
$$-\frac{1}{4N^2}\sum_{\mathbf{k}_1}[h(\mathbf{k}_1)]^2\{\sum_{\mathbf{k}_2}[h(\mathbf{k}_2)-h_0(\mathbf{k}_2)]\}^2. \quad (30)$$

Below T_c, the additional perturbation must be written as

$$(8/n^3)\langle S_1^2\rangle_0\sum_i[\bar{S}^2+2\bar{S}(S_i-\bar{S})+(S_i-\bar{S})^2]. \quad (31)$$

The term in $\bar{S}^2$ must be added to F_0 and taken away from F_1, and the linear term produces effects which are shown diagrammatically in Fig. 4 by a single line ending in a dot. The effect of the diagrams shown in Fig. 4 added to those of Fig. 2 is to alter Eq. (20) to

$$\frac{F_1}{\kappa T}=\frac{1}{4N}\frac{1+12\bar{S}^2/n^2}{(1-4\bar{S}^2/n^2)^3}[\sum_{\mathbf{k}}h(\mathbf{k})]^2-\frac{1}{2N}\sum_{\mathbf{k}}h(\mathbf{k})\sum_{\mathbf{k}_1}h_0(\mathbf{k}_1)$$
$$-\frac{4}{3N}\frac{\bar{S}^2/n^2}{(1-4\bar{S}^2/n^2)^4}\sum_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}h(\mathbf{k}_1)h(\mathbf{k}_2)h(\mathbf{k}_3)\delta_{\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3}$$
$$-\frac{2}{N}\frac{\bar{S}^2/n^2}{(1-4\bar{S}^2/n^2)^4}h(0)[\sum_{\mathbf{k}}h(\mathbf{k})]^2$$
$$+\frac{4}{N}\frac{\bar{S}^2/n^2}{(1-4\bar{S}^2/n^2)^2}h(0)\sum_{\mathbf{k}}h(\mathbf{k})\sum_{\mathbf{k}_2}h_0(\mathbf{k}_1)$$
$$-\frac{2}{N}\frac{\bar{S}^2}{n^2}h(0)[\sum_{\mathbf{k}}h_0(\mathbf{k})]^2-\frac{2}{n}\frac{\bar{S}^2}{n^2}\sum_{\mathbf{k}}h_0(\mathbf{k}). \quad (32)$$

The fourth, fifth, and sixth terms on the right side of Eq. (32), taken together, tend to zero as T tends to T_c, and so the discontinuity of F_1 is gone. The last term

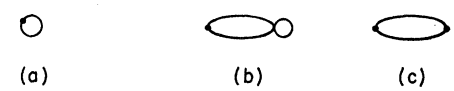

FIG. 3. Diagrams involving counter-terms above T_c.

really belongs with F_0, and it makes a contribution to the energy that just cancels the discontinuity given by Eq. (25). We see that this renormalization removes all specific-heat anomalies with a temperature range of the order of $n^{-1}T_c$.

More delicate methods have to be used to study the behavior over a temperature range within $n^{-2}T_c$ of T_c. It was shown earlier that a diagram with r four-vertices and $2r'$ three-vertices makes a contribution of order $N/n^{r+r'+1}$ to the free energy. Near the transition temperature, however, $h(\mathbf{k})$ can be very large for small k. If we restrict k to be less than K, where αK^2 is of the order of θ, then h is of the order θ^{-1}, as can be seen from Eqs. (17) and (22). If we restrict all the lines in a diagram to have wave number less than K, we get a total contribution to the free energy of the order of

$$(n\theta^{-1})^{2r+3r'}(Nn^{-2})^r(\bar{S}^2/Nn^5)^{r'}[(N/n)K^3V]^{r+r'+1}, \quad (33)$$

where V is the volume of a cell. This follows from the analysis of the contributions of diagrams immediately after Eq. (20). Since $\bar{S}^2/n^2$ is proportional to θ, and K is proportional to $\theta^{1/2}$, the expression (33) is proportional to $N/n^{r+r'+1}\theta^{(r+r'-3)/2}$, so that every term in the series is proportional to $N\theta^{3/2}n^{-1}$ times a power of $(n^2\theta)^{-1}$, and all terms are of the same order when θ is of the order of n^{-2}.

This argument suggests that a quantity like the specific heat (obtained from F by differentiating twice with respect to θ) might have a well-defined value in the limit $n\rightarrow\infty$ with $n^2\theta$ constant. This is not so, however, because some of the terms in the perturbation series diverge in this limit. We can recognize such divergent terms by going to the limit, and then looking for divergent integrals. This corresponds very closely to the methods used for renormalization of quantum field theory; the main difference is that in field theory a cutoff is introduced artificially, while in this problem there is a real cutoff for any finite n, and it is only in the limit $n\rightarrow\infty$ that any terms are really divergent.

The method of determining the degree of divergence of a diagram is described in detail in books on the quantum theory of fields.[18] We consider diagrams with

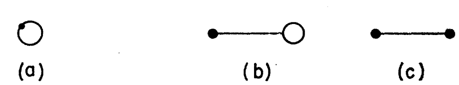

FIG. 4. Additional diagrams involving counter-terms below T_c.

[18] S. S. Schweber, H. A. Bethe, and F. de Hoffmann, *Mesons and Fields* (Row Peterson and Co., Evanston, Ill., 1955), Vol. I, pp. 316–327; S. S. Schweber, *Relativistic Quantum Field Theory* (Row Peterson and Co., New York, 1961), pp. 584–607; J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill Book Co., New York, 1965), pp. 317–344.

FIG. 5. Divergent diagrams cancelled by counter-terms.

r four-vertices, $2r'$ three-vertices, I internal lines and E external lines; we allow external lines because such a diagram may be a component of a larger diagram. We have

$$E+2I=4r+6r',$$

and $2r'$ is odd if, and only if, E is odd. The denominator has I powers of k^2, and there are $I-r-2r'+1$ three-dimensional integrations, so the degree of divergence is

$$D=3(I-r-2r'+1)-2I=3-\tfrac{1}{2}E-r-3r'. \quad (34)$$

For $D=2$ or 1 the integral depends quadratically or linearly on the cutoff at large wave numbers; for $D=0$ the integral may be logarithmically divergent; and for D negative the integral is convergent.

There are 18 diagrams with positive or zero D, but two of these have no internal lines and represent constants, while 11 of them contain one or the other of the components shown in Fig. 5. These two are the diagrams with $E=1$, $r=0$, $r'=\frac{1}{2}$, and $E=2$, $r=1$, $r'=0$, both linearly divergent. These are approximately cancelled by the counter-terms given in Eq. (31). Figure 5(a) contributes

$$\frac{4}{(Nn^3)^{1/2}}\frac{\bar{S}/n}{(1-4\bar{S}^2/n^2)^2}\sigma_0\sum_{\mathbf{k}} h(\mathbf{k}),$$

while the counter-term contributes

$$-[4/(Nn^3)^{1/2}](\bar{S}/n)\sigma_0\sum_{\mathbf{k}} h_0(\mathbf{k}).$$

Since $\bar{S}/n$ is of order θ, and negligible in the limit, the sum of these is

$$\frac{4}{(Nn^3)^{1/2}}\frac{\bar{S}}{n}\sigma_0\sum_{\mathbf{k}}[h(\mathbf{k})-h_0(\mathbf{k})] = -\frac{4}{(Nn^3)^{1/2}}\frac{\bar{S}}{n}\sigma_0\sum_{\mathbf{k}}\frac{2\theta}{(2\theta+\alpha k^2)\alpha k^2}. \quad (35)$$

There are two extra powers of k in the denominator, and the degree of divergence is reduced by two, becoming negative in all cases. Similarly, Fig. 5(b) added to its counter-term contributes something proportional to $\sum[h(\mathbf{k})-h_0(\mathbf{k})]$, and the degree of divergence is again reduced by two, to become negative.

There remain the five diagrams shown in Fig. 6. The first has $E=0$, $r=2$, $r'=0$, and is linearly divergent, while the remainder are logarithmically divergent. We concentrate our attention on Fig. 6(b), which has $E=2$, $r=2$, $r'=0$. The insertion of such an element into a line with wave number $\mathbf{k}$ gives a factor

$$\frac{2}{3N^2}h(\mathbf{k})\sum_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} h(\mathbf{k}_1)h(\mathbf{k}_2)h(\mathbf{k}_3)\delta_{\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3}$$

$$\sim\frac{2V^2}{3n^2(2\pi)^6}h(\mathbf{k})\int d^3k_1\int d^3k_2 \times[(\theta+\alpha k_1^2)(\theta+\alpha k_2^2)(\theta+\alpha|\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}|^2)]^{-1}, \quad (36)$$

where the integrals must be understood to be cut off at some value $K_{\max}$. This asymptotic equation gives only the logarithmic dependence on $K_{\max}$ correctly, and does not keep constant terms correct. Evaluation of the integral gives (see the Appendix)

$$[V^2/24\pi^2\alpha^3][h(\mathbf{k})/n^2]\ln[K_{\max}/(\theta/\alpha)^{1/2}] \quad (37)$$

for small k; the argument of the logarithm is replaced by $K_{\max}/k$ if αk^2 is much larger than θ.

We cannot evaluate this at the critical temperature to get a suitable term to add to the quadratic terms and subtract from the perturbation, because it is infinite at $\theta=0$, but we can choose a fixed value θ_0, proportional to n^{-2}, and evaluate the left side of Eq. (36) for $\mathbf{k}=0$, $\theta=\theta_0$, to get a counter-term

$$-(4/3N^2n)\sum_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}\bar{h}_0(\mathbf{k}_1)\bar{h}_0(\mathbf{k}_2)\bar{h}_0(\mathbf{k}_3)\delta_{\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3}\sum_i S_i^2, \quad (38)$$

where $\bar{h}_0$ is h evaluated at θ_0. When the contribution of this counter-term is added to the expression (36), we get a term of the order of

$$(V^2/24\pi^2\alpha^3)[h(\mathbf{k})/n^2]\ln(\theta_0/\theta)^{1/2} \quad (39)$$

for small k, and of the order of

$$(V^2/24\pi^2\alpha^3)[h(\mathbf{k})/n^2]\ln[(\theta_0/\alpha)^{1/2}/k] \quad (40)$$

for large k, and such a term does no harm to the convergence of the series in the limit $n\rightarrow\infty$, with $n^2\theta$ and $n^2\theta_0$ fixed. Below T_c, that part of the counter-term linear in $S_i-\bar{S}$ cancels the logarithmic divergence given by the diagram of Fig. 6(c).

There remain divergent contributions from Figs. 6(a), 6(d), and 6(e), together with some low-order effects from the counter-terms. Figure 6(a) combined with the

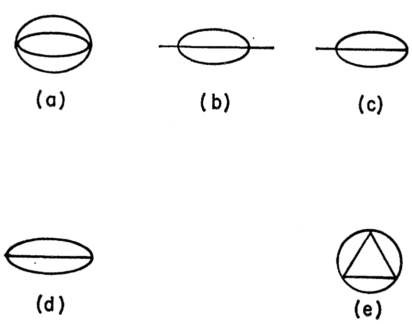

FIG. 6. Divergent diagrams.

first-order effect of the counter-term contributes to $F_2/\kappa T$ an amount

$$-\frac{1}{12N^2}\sum_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4} h(\mathbf{k}_1)h(\mathbf{k}_2)h(\mathbf{k}_3)h(\mathbf{k}_4)\delta_{\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4}+\frac{1}{3N^2}\sum_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4} h(\mathbf{k}_1)\bar{h}_0(\mathbf{k}_2)\bar{h}_0(\mathbf{k}_3)\bar{h}_0(\mathbf{k}_4)\delta_{\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}}$$

$$=-\frac{1}{12N^2}\sum_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4}\{-3\bar{h}_0(\mathbf{k}_1)\bar{h}_0(\mathbf{k}_2)\bar{h}_0(\mathbf{k}_3)\bar{h}_0(\mathbf{k}_4)+4h(\mathbf{k}_1)\bar{h}_0(\mathbf{k}_2)\bar{h}_0(\mathbf{k}_3)\bar{h}_0(\mathbf{k}_4)$$

$$+6[h(\mathbf{k}_1)-\bar{h}_0(\mathbf{k}_1)][h(\mathbf{k}_2)-\bar{h}_0(\mathbf{k}_2)]\bar{h}_0(\mathbf{k}_3)\bar{h}_0(\mathbf{k}_4)+4[h(\mathbf{k}_1)-\bar{h}_0(\mathbf{k}_1)][h(\mathbf{k}_2)-\bar{h}_0(\mathbf{k}_2)][h(\mathbf{k}_3)-\bar{h}_0(\mathbf{k}_3)]\bar{h}_0(\mathbf{k}_4)$$

$$+[h(\mathbf{k}_1)-\bar{h}_0(\mathbf{k}_1)][h(\mathbf{k}_2)-\bar{h}_0(\mathbf{k}_2)][h(\mathbf{k}_3)-\bar{h}_0(\mathbf{k}_3)][h(\mathbf{k}_4)-\bar{h}_0(\mathbf{k}_4)]\}\delta_{\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4}$$

$$+\frac{1}{3N^2}\sum_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4} h(\mathbf{k}_1)\bar{h}_0(\mathbf{k}_2)\bar{h}_0(\mathbf{k}_3)\bar{h}_0(\mathbf{k}_4)\delta_{\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4}. \quad (41)$$

The first term in the curly brackets gives a contribution that depends linearly on the cutoff, but it is independent of temperature, and so gives just a constant term proportional to N/n^3 in the entropy. The second term depends logarithmically on the cutoff, but that dependence is cancelled by the counter-term, leaving something of the order of

$$(V^2/48\pi^2\alpha^3 n^2)\sum_{k>(\theta_0/\alpha)^{1/2}} h(\mathbf{k})\ln[k/(\theta_0/\alpha)^{1/2}] \quad (42)$$

according to Eq. (37). This is another term which is linearly dependent on the cutoff, but whose derivative is independent of the cutoff. The remaining terms are independent of the cutoff and converge for $n\to\infty$. None of these terms give any discontinuity at $T=T_c$.

The contribution of Fig. 6(d) is given in Eq. (20). It is approximately

$$-(4/3N)(\bar{S}^2/n^2)\sum_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} h(\mathbf{k}_1)h(\mathbf{k}_2)h(\mathbf{k}_3)\delta_{\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3}. \quad (43)$$

This tends to zero at T_c, but its derivative does not. However, the counter-term (38), when written in terms of $S_i-\bar{S}$, has a constant term which contributes an amount

$$(4/3N)(\bar{S}^2/n^2)\sum_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} \bar{h}_0(\mathbf{k}_1)\bar{h}_0(\mathbf{k}_2)\bar{h}_0(\mathbf{k}_3)\delta_{\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3}, \quad (44)$$

and this cancels the possible discontinuity in E_1.

The final divergent term is shown in Fig. 6(e). The contribution of this is

$$(6N)^{-3}\sum_{\mathbf{k}}[\sum_{\mathbf{k}_1} h(\mathbf{k}_1)h(\mathbf{k}-\mathbf{k}_1)]^3, \quad (45)$$

together with some unimportant terms in which the wave number changes by a nonzero reciprocal lattice vector at a vertex. This gives a logarithmic term in the free energy, but it is continuous at $T=T_c$, and gives no divergence in any other quantity.

Since the logarithmic terms give no discontinuities in F and E, the shift in T_c implied by the introduction of the counter-term (38) is sufficient, and there are no more anomalous effects to this order.

6. EXISTENCE OF THE LIMIT

The work of the Sec. 5 shows that if the critical temperature is assumed to be

$$T_c=\frac{nG(0)}{2\kappa}\left[1+N^{-1}\sum_{\mathbf{k}}\frac{G(\mathbf{k})}{G(0)-G(\mathbf{k})}-\frac{2}{3N^2}\sum_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}\bar{h}_0(\mathbf{k}_1)\bar{h}_0(\mathbf{k}_2)\bar{h}_0(\mathbf{k}_3)\delta_{\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3}\right]^{-1}, \quad (46)$$

then there is a perturbation series for the specific heat for which every term has a finite value in the limit $n\to\infty$ with $n^2\theta$ kept constant. This must be contrasted with the series of Vaks, Larkin, and Pikin,[11] which has terms that diverge logarithmically in this limit. In this limit only terms up to fourth order of S_i in the exponent of Eq. (6) are significant, and only the zero and second moments of J_{ij} matter. Despite this, it is not possible to write the limiting theory in terms of a function of a continuous variable to replace S_i, as has been proposed by Ferrell,[13] because then the renormalization effects are infinite.

In the papers of Patashinskii and Pokrovskii[8] and Abe,[10] an attempt was made to evaluate the diagram of Fig. 6(b) self-consistently. This does not seem to be desirable since, once the renormalization described here has been carried out, there seems no reason to prefer one class of diagrams to another. A resummation of the complete perturbation series seems to be necessary, and this work does not indicate how it is to be done. If such a resummation were done, it is likely that some sort of critical behavior would be found in this limit. It is certain, at least, that the $|T-T_c|^{-1/2}$ behavior of the specific heat in the Gaussian model, which was also obtained in the work on superconductivity,[2,4] does not represent a real singularity, but is just the first term in an infinite series of increasing powers of $|T-T_c|^{-1/2}$.

7. VECTOR ORDER PARAMETERS

The model formulated in Sec. 2 can be extended to cover the case of an order parameter with more than one real degree of freedom, such as the gap parameter for superconductivity, or the magnetization in the Heisenberg model of a magnetic system. We consider in detail the Heisenberg model, for particles of spin $\frac{1}{2}$, with an interaction

$$\hat{H}=-\sum_i\sum_j\sum_{\alpha\in i}\sum_{\beta\in j}J_{ij}\hat{\mathbf{s}}_\alpha\cdot\hat{\mathbf{s}}_\beta+\tfrac{3}{4}n\sum_i J_{ii}. \quad (47)$$

Here it has been assumed that the interaction between the spins α and β depends only on which cells i and j they lie in; the second term on the right side cancels the interaction of a spin with itself. Both the energy and the entropy can be written in terms of the spin operators $\hat{\mathbf{S}}_i$ for the cells, in complete analogy with Eqs. (4) and (5). The energy is

$$H=-\sum_i\sum_j J_{ij}\hat{\mathbf{S}}_i\cdot\hat{\mathbf{S}}_j+\tfrac{3}{4}n\sum_i J_{ii}, \quad (48)$$

while the number of states for which the eigenvalue of $\hat{\mathbf{S}}_i^2$ is $S_i(S_i+1)$, is equal to

$$\frac{n!(2S_i+1)^2}{(\frac{1}{2}n+S_i+1)!(\frac{1}{2}n-S_i)!}$$
$$=(2S_i+1)^2\exp\Big(n\ln n-(\tfrac{1}{2}n+S_i+\tfrac{1}{2})\ln(\tfrac{1}{2}n+S_i+\tfrac{1}{2})$$
$$-(\tfrac{1}{2}n-S_i-\tfrac{1}{2})\ln(\tfrac{1}{2}n-S_i-\tfrac{1}{2})$$
$$+\ln\frac{\sqrt{n}}{(2\pi)^{1/2}(\frac{1}{2}n+S_i+1)(\frac{1}{2}n-S_i)}+O(n^{-1})\Big)$$
$$\approx(2S_i+1)^2\exp(n\ln 2-(2/n)(1-2/n)(S_i+\tfrac{1}{2})^2$$
$$-(4/3n^3)(S_i+\tfrac{1}{2})^4+\tfrac{1}{2}\ln(8/\pi n^3)). \quad (49)$$

In the case of the Ising model it was possible to replace the sum over S_i by an integral, making errors that can be shown to be exponentially small for large n. It is plausible that a similar replacement can be made in this case since the important eigenvalues of $\hat{\mathbf{S}}_i^2$ are large, so that it should be possible to replace the vector operators by ordinary vectors. If this replacement is made, we get

$$\mathfrak{z}\approx\int d^3S_1\cdots\int d^3S_{N/n}2^N\left(\frac{8}{\pi^3n^3}\right)^{N/2n}$$
$$\times\exp\Big[-\frac{2}{n}\Big(1-\frac{2}{n}\Big)\sum_i(S_i^{(x)2}+S_i^{(y)2}+S_i^{(z)2})$$
$$+\sum_i\sum_j\frac{J_{ij}}{\kappa T}(S_i^{(x)}S_j^{(x)}+S_i^{(y)}S_j^{(y)}+S_i^{(z)}S_j^{(z)})$$
$$-\frac{4}{3n^3}\sum_i(S_i^{(x)2}+S_i^{(y)2}+S_i^{(z)2})^2-\frac{3n}{4\kappa T}\sum_i J_{ii}\Big], \quad (50)$$

which is very similar to Eq. (6).

It has not been proved that expression (50) is accurate to the same order as Eq. (6), and there is at least one contribution to the shift of the critical temperature which has been dropped in the transition from Eqs. (48) and (49) to Eq. (50). This term arises because a combination of operators such as

$$\hat{S}_i^{(z)}J_{ij}\hat{S}_i^{(x)}\hat{S}_j^{(x)}J_{ij}\hat{S}_i^{(y)}\hat{S}_j^{(y)}\hat{S}_j^{(z)}$$

has nonzero expectation value for $i\neq j$; the effect is to decrease the critical temperature by an amount

$$\frac{n}{12\kappa^2T}\sum_{j\neq i}J_{ij}^2$$
$$=\frac{1}{12\kappa^2T}\left[\frac{n^2}{N}\sum_{\mathbf{k}}G(\mathbf{k})G(-\mathbf{k})-\frac{n^3}{N^2}(\sum_{\mathbf{k}}G(\mathbf{k}))^2\right], \quad (51)$$

which is of order n^{-1}.

Equation (50) can be treated in just the same way as Eq. (6) has been treated in the bulk of this paper. The only difference is that each diagram has a different combinational factor, because every line in the diagram can represent the correlation of any of the three components of the spin vector. For example, the contributions of the diagrams shown in Figs. 5(b) and 6(b) are both increased by a factor of $\frac{5}{3}$. The expression for the critical temperature analogous to Eq. (46) is

$$T_c=\frac{nG(0)}{2\kappa}\Big(1+\frac{2}{n}+\frac{5}{3N}\sum_{\mathbf{k}}\frac{G(0)}{G(0)-G(\mathbf{k})}+\frac{1}{6\kappa TG(0)}\sum_{j\neq i}J_{ij}^2$$
$$-\frac{10}{9N^2}\sum_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}\bar{h}_0(\mathbf{k}_1)\bar{h}_0(\mathbf{k}_2)\bar{h}_0(\mathbf{k}_3)\delta_{\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3}\Big)^{-1}$$
$$=\frac{nG(0)}{2\kappa}\Big[1+\frac{5}{3N}\sum_{\mathbf{k}}\frac{G(\mathbf{k})}{G(0)-G(\mathbf{k})}$$
$$+\frac{1}{3NG(0)^2}\sum_{\mathbf{k}}G(\mathbf{k})G(-\mathbf{k})-\frac{1}{3n}$$
$$-\frac{n}{3N^2G(0)^2}(\sum_{\mathbf{k}}G(\mathbf{k}))^2$$
$$-\frac{10}{9N^2}\sum_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}\bar{h}_0(\mathbf{k}_1)\bar{h}_0(\mathbf{k}_2)\bar{h}_0(\mathbf{k}_3)\delta_{\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3}\Big]^{-1}. \quad (52)$$

These combinatorial factors do not affect the arguments about renormalizability, but there is no reason why they should not have a crucial effect on the behavior in the critical region, where the perturbation series must be summed completely. They must have such an effect if Eq. (50) gives an adequate description of the critical region, as there are good reasons to believe that the critical behavior of the Ising and Heisenberg models are different. On the other hand, we should not expect the critical behavior of the Heisenberg model to depend on the spin of the individual particles, since expressions similar to Eqs. (49) and (50) can be derived for particles with spin greater than one half. The coefficients of $(S_i+\frac{1}{2})^2$ and $(S_i+\frac{1}{2})^4$ will depend on the spin, but this can affect only the position, width, and strength of the critical region, and cannot affect the functional form of the specific heat in this region.

8. TWO-DIMENSIONAL AND ONE-DIMENSIONAL SYSTEMS

In two dimensions the arguments about renormalization are altered. A diagram with E external lines, I internal lines, r four-vertices, and $2r'$ three-vertices, with $E+2I=4r+6r'$, has I powers of k^2 in the denominator and $I-r-2r'+1$ two-dimensional integrations, so that the degree of divergence is

$$D=2-2r-4r', \tag{53}$$

and the only divergent diagrams are those with a single vertex, such as those shown in Fig. 5, and they are logarithmically divergent. These diagrams must be approximately cancelled by a suitable counter-term, and the remaining terms in the series will be functions of $(n\theta)^{-1}$; not only simple powers, but also terms like $(n\theta)^{-1}\ln(n\theta)$.

As in the three-dimensional case, the counter term is taken to be $\sum h(\mathbf{k})$ evaluated at some suitable temperature. The temperature cannot be taken to be the critical temperature $\theta=0$, since this gives a divergent result, but it can be taken to be some value θ_0 of the order of n^{-1}, which is on the edge of the critical region. This gives

$$T_c=[nG(0)/2\kappa][1-n^{-1}+N^{-1}\sum_{\mathbf{k}}\bar{h}_0(\mathbf{k})]^{-1}$$

$$=\frac{nG(0)}{2\kappa}\left(1+N^{-1}\sum_{\mathbf{k}}\frac{G(\mathbf{k})}{(1+\theta_0)G(0)-G(\mathbf{k})}\right)^{-1}. \tag{54}$$

The order of magnitude of this shift can be found by substituting Eq. (17) for $\bar{h}_0(\mathbf{k})$; the result is

$$N^{-1}\sum_{\mathbf{k}}\bar{h}_0(\mathbf{k})\approx\frac{A}{n}(2\pi)^{-2}\int\frac{d^2k}{\theta_0+\alpha k^2}\approx\frac{A}{2\pi n\alpha}\ln\frac{K_{\max}}{(\theta_0/\alpha)^{1/2}}$$

$$=(A/4\pi n\alpha)\ln n+O(n^{-1}). \tag{55}$$

Here A is the area of a cell, the cutoff wave number $K_{\max}$ is of the order of $A^{-1/2}$, and $\alpha K_{\max}{}^2$ is of the order of unity, so that the shift in T_c from its mean-field value is proportional to $n^{-1}\ln n$.

For the Heisenberg model this shift is multiplied by a factor of $\frac{5}{3}$. The nature of the critical behavior must be quite different, as it is known from the work of Mermin and Wagner[19] that there can be no spontaneous magnetization in the two-dimensional Heisenberg model.

For a one-dimensional system no diagrams are divergent and no renormalization is necessary. The terms we have considered, which give a series in powers of $n\theta^{-1/2}$ for three dimensions and in powers of $n\theta^{-1}$ for two dimensions, give a series in powers of $n\theta^{-3/2}$ for one dimension, and so the width of the critical region is proportional to $n^{-2/3}$. Since there are no phase transitions in a one-dimensional system with forces of finite range,[20] the critical region must join smoothly the two regions in which mean-field theory is good. The problem can be formulated immediately in terms of a simple transfer matrix.

A system need not be truly two- or one-dimensional to exhibit behavior typical of a two- or one-dimensional system; that is, it need not consist of a single layer or a single line of cells. The decisive question is whether or not it is valid to replace the sums over wave number which occur in the terms of the perturbation series by integrals. Since effects in the critical region come from the low wave number region of the integrals, quite stringent conditions may have to be met before such a replacement is made.

We consider a slab large in two of its dimensions but only C cells across, where C is a small number. We take the z axis to be perpendicular to the slab. It is first necessary to diagonalize the quadratic terms in the exponent of Eq. (6), and the running waves given in Eq. (8) will no longer be adequate, unless periodic boundary conditions are imposed. For simplicity we assume that the cells are arranged in a simple cubic lattice, and that the interaction has strength J within a cell or between neighboring cells, and is zero otherwise; the cells on the boundary interact with one less cell than those in the interior. If we take the z coordinates of the cells to be $l, 2l, \cdots, cl$, where l is the length of the side of a cell, suitable variables are

$$\sigma_{k_x,k_y,\nu}=(2n/N)^{1/2}\sum_i S_i\exp(ik_xX_i+ik_yY_i)\times\sin[\nu\pi Z_i/(C+1)l]. \tag{56}$$

The equation corresponding to Eq. (11) is

$$h(k_x,k_y,\nu)=\left[1-n^{-1}-\frac{nJ}{2\kappa T}\left(1+2\cos k_xl+2\cos k_yl+2\cos\frac{\nu\pi}{C+1}\right)\right]^{-1}$$

$$\approx\left[\theta+\frac{1}{7}k_x{}^2l^2+\frac{1}{7}k_y{}^2l^2+\frac{1}{7}(\nu^2-1)\frac{\pi^2}{(C+1)^2}\right]^{-1}, \tag{57}$$

[19] N. D. Mermin and H. Wagner, Phys. Rev. Letters 17, 1133 (1966).

[20] See Ref. 1, p. 482.

where the critical temperature is shifted down by an amount

$$\Delta T_c/T_c = -(1/7)[\pi^2/(C+1)^2], \tag{58}$$

because the boundary condition depresses the maximum possible value of $\cos[\nu\pi/(C+1)]$ below unity.

The properties of the critical region are determined by values of k of the order of $(\theta/\alpha)^{1/2}$, and the critical range of θ in a three-dimensional system is of order n^{-2}. It is clear from the form of Eq. (57) that the sum over ν can only be replaced by an integral if

$$C \gg n. \tag{59}$$

If C is much less than n, the critical behavior is determined entirely by the terms with $\nu=1$ and all higher values of ν give a negligible contribution, so that the behavior is characteristic of a two-dimensional system. Since, in the critical region, spins are correlated right across the slab, the effective number of particles in a cell is nC rather than n. This can be seen in detail by considering the perturbation series. As we go from one term to the next of one higher order, we get one factor of N^{-1}, one extra sum over wave number, and two factors of h. We can write

$$N^{-1}\sum_{k_xk_y} = \frac{l^2}{(2\pi)^2 nC}\int dk_x \int dk_y, \tag{60}$$

and so we get a critical region of width proportional to $(nC)^{-1}$.

To evaluate the shift of T_c we need to know $\sum h(\mathbf{k})$. As we have seen, this makes a contribution to the temperature shift of order n^{-1} in the three-dimensional case. Since the main contribution comes from large values of k, it is not much affected by the finite value of C. To get the difference between the shift for a three-dimensional system and for this two-dimensional system, it is sufficient to evaluate $\sum h(\mathbf{k})$ using the quadratic approximation for h given in Eq. (57) and using a crude cutoff at large k. We have

$$N^{-1}\sum_{k_xk_y\nu} h(k_x,k_y,\nu)$$

$$\approx \frac{l^2}{2\pi nC}\int k dk \sum_\nu \left(\theta+\frac{1}{7}k^2l^2+\frac{1}{7}(\nu^2-1)\frac{\pi^2}{(C+1)^2}\right)^{-1}$$

$$= \frac{7}{4\pi nC}\sum_\nu \ln\frac{\theta+(1/7)K_{\max}{}^2 l^2}{\theta+(1/7)(\nu^2-1)[\pi^2/(C+1)^2]}. \tag{61}$$

The numerator of the logarithm gives a term which can be evaluated by replacing the sum by an integral, but it is the denominator which is important. By using a contour integral method, this can be shown to make a contribution

$$-\frac{7}{4\pi nC}\ln\left[\sin\pi\left(1-7\theta\frac{(C+1)^2}{\pi^2}\right)^{1/2}\right] + \frac{7}{4\pi nC}\ln\left(\frac{\pi^2}{7(C+1)^2}-\theta\right)^{1/2},$$

in addition to what would be obtained if the sum were replaced by an integral. Since θ has to be taken of the order $(nC)^{-1}$, this gives the shift

$$\Delta T_c/T_c = -(7/4\pi nC)\ln n, \tag{62}$$

in addition to the shifts proportional to n^{-1} and C^{-2}, respectively.

It may be observed that the sum over ν for those terms which are not particularly large for small ν does give a result larger by a factor of $(C+1)/C$ than would be an infinite system. This, however, only results in a shift of T_c of order $(nC)^{-1}$; this is inside the critical region for the two-dimensional system, and should be ignored.

After this discussion of a slab C cells across, it is easy to see what happens for a system cuboidal in shape, A cells long, B cells across, and C cells deep, where

$$A > B > C. \tag{63}$$

For a system to have a two-dimensional critical region, it is necessary that inequality (59) be violated, and that the sum over k_y can be replaced by an integral when θ is of the order of $(nC)^{-1}$. From Eq. (57) it can be seen that this implies

$$B^2/C \gg n \gg C. \tag{64}$$

If this inequality is violated, the system may have a one-dimensional critical region. Since there are effectively nBC particles in each cell when the system is behaving one-dimensionally, the width of the one-dimensional critical region is proportional to $(nBC)^{-2/3}$. For the sum over k_x to be replaceable by an integral in this region, it is necessary that

$$A^3/BC \gg n \gg B^2/C. \tag{65}$$

Finally, in the case

$$n \gg A^3/BC, \tag{66}$$

the system behaves as if it had zero dimensions in the critical region; that is, the spins are correlated across the whole system. The critical behavior is entirely determined by the lowest Fourier component, which, with the boundary conditions we have chosen, is

$$\sigma_{1,1,1} = \left(\frac{8}{ABC}\right)^{1/2}\sum_i S_i \times \sin\frac{\pi X_i}{(A+1)l}\sin\frac{\pi Y_i}{(B+1)l}\sin\frac{\pi Z_i}{(C+1)l}. \tag{67}$$

Substituting in Eq. (6) we have for the critical region

$$\mathfrak{Z}\approx 2^N\int_{-\infty}^{\infty} d\sigma_{1,1,1}$$

$$\times\exp\left\{-2\sigma_{1,1,1}^2\left[1-n^{-1}-\frac{J}{2\kappa T}\left(1+2\cos\frac{\pi}{A+1}+2\cos\frac{\pi}{B+1}+2\cos\frac{\pi}{C+1}\right)\right]-\frac{32}{3n^3ABC}\sigma_{1,1,1}^4\right\}. \quad (68)$$

The width of this critical region is of order $(nABC)^{-1/2} = N^{-1/2}$.

We have derived conditions for critical regions of the various types to exist, which are expressed by the inequalities (59), (64), (65), and (66). It is not suggested that the critical behavior is everywhere characteristic of a particularly dimensionality when one of these inequalities is satisfied. We know, for example, that sufficiently close to the critical temperature the specific-heat curve must be rounded for any finite system. We cannot discuss this in detail without knowing the nature of the critical behavior.

9. STANDARD ISING AND HEISENBERG MODELS

So far, a special case of the Ising and Heisenberg models has been considered, in which the interaction between two spins depends only on the cells in which the spins are located. The results can readily be generalized to cover the standard form of the models by using the close analogy between the diagrams which have been used here to represent terms in the perturbation series and the diagrams used to represent the perturbation series for the standard Ising and Heisenberg models (see Ref. 7). Each line in one of our diagrams corresponds to the sum over all lines connecting two points with only two vertices along the line; that is, a line joining the sites i and j gives a factor R_{ij}, where

$$R_{ij}=\frac{J_{ij}}{2\kappa T}+\sum_k\frac{J_{ik}J_{kj}}{(2\kappa T)^2}+\sum_k\sum_l\frac{J_{ik}J_{kl}J_{lj}}{(2\kappa T)^3}+\cdots,$$

$$\sum_k R_{ik}\left(\delta_{kj}-\frac{J_{kj}}{2\kappa T}\right)=\frac{J_{ij}}{2\kappa T}. \quad (69)$$

The sums over intermediate sites in this expression are not restricted in any way. The diagrams which are important in the case of a long-range interaction are those in which not more than four lines go through any site. The leading terms in the series for the Ising model spin-spin correlation function are

$$4\langle s_is_j\rangle=\delta_{ij}+R_{ij}-\sum_k(\delta_{ik}+R_{ik})R_{kk}(\delta_{kj}+R_{kj})$$
$$+\tfrac{2}{3}\sum_k\sum_l(\delta_{ik}+R_{ik})(R_{kl})^3(\delta_{lj}+R_{lj})+\cdots. \quad (70)$$

We can express the inverse of the Fourier transform of this function in terms of $G(\mathbf{k})$, defined by Eq. (10), to get

$$1-\frac{G(\mathbf{k})}{2\kappa T}+\frac{1}{N}\sum_{\mathbf{k}'}\frac{G(\mathbf{k}')/2\kappa T}{1-G(\mathbf{k}')/2\kappa T}$$
$$-\frac{2}{3N^2}\sum_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}\frac{G(\mathbf{k}_1)G(\mathbf{k}_2)G(\mathbf{k}_3)}{[2\kappa T-G(\mathbf{k}_1)][2\kappa T-G(\mathbf{k}_2)][2\kappa T-G(\mathbf{k}_3)]}$$
$$\times\delta_{\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3-\mathbf{k}}+\cdots. \quad (71)$$

The third and fourth terms of this expression correspond to the diagrams of Figs. 5(b) and 6(b). The critical temperature is given by

$$T_c=\frac{G(0)}{2\kappa}\left(1+N^{-1}\sum_{\mathbf{k}}\frac{G(\mathbf{k})}{G(0)-G(\mathbf{k})}\right.$$
$$\left.-\frac{2}{3N^2}\sum_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}\frac{G(\mathbf{k}_1)G(\mathbf{k}_2)G(\mathbf{k}_3)}{[(1+\theta_0)G(0)-G(\mathbf{k}_1)][(1+\theta_0)G(0)-G(\mathbf{k}_2)][(1+\theta_0)G(0)-G(\mathbf{k}_3)]}\delta_{\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3}\right)^{-1}. \quad (72)$$

The first correction is identical with the first one given in Eq. (46), and the second correction is almost exactly the same as the second in Eq. (46) except that the cutoff for large values of k_1,k_2,k_3 is provided by the factors of G in the numerator, rather than by the maximum wave number allowed for a system made up of cells.

For the Heisenberg model the series is rather more complicated. The series analogous to Eq. (70) is

$$\tfrac{4}{3}\langle\hat{\mathbf{S}}_i\cdot\hat{\mathbf{S}}_j\rangle=\delta_{ij}+R_{ij}-\tfrac{5}{3}\sum_k(\delta_{ik}+R_{ik})R_{kk}(\delta_{kj}+R_{kj})+\tfrac{2}{3}R_{ii}\delta_{ij}$$
$$-\tfrac{1}{3}\sum_k\sum_l(\delta_{ik}+R_{ik})(J_{kl}/2\kappa T)^2(\delta_{lj}+R_{lj})-\tfrac{2}{3}R_{ij}J_{ij}/2\kappa T+\frac{10}{9}\sum_k\sum_l(\delta_{ik}+R_{ik})(R_{kl})^3(\delta_{lj}+R_{lj})+\cdots. \quad (73)$$

The inverse of the Fourier transform of this is

$$1-\frac{G(\mathbf{k})}{2\kappa T}+\frac{5}{3N}\sum_{\mathbf{k}'}\frac{G(\mathbf{k}')}{2\kappa T-G(\mathbf{k}')}-\frac{2}{3N}\left[1-\frac{G(\mathbf{k})}{2\kappa T}\right]^2\sum_{\mathbf{k}'}\frac{G(\mathbf{k}')}{2\kappa T-G(\mathbf{k}')}$$

$$+\frac{1}{3N}\sum_{\mathbf{k}'}\frac{G(\mathbf{k}')G(\mathbf{k}-\mathbf{k}')}{(2\kappa T)^2}+\frac{2}{3N}\left[1-\frac{G(\mathbf{k})}{2\kappa T}\right]^2\sum_{\mathbf{k}'}\frac{G(\mathbf{k}')G(\mathbf{k}-\mathbf{k}')}{2\kappa T[2\kappa T-G(\mathbf{k}')]}$$

$$-\frac{10}{9N^2}\sum_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}\frac{G(\mathbf{k}_1)G(\mathbf{k}_2)G(\mathbf{k}_3)}{[2\kappa T-G(\mathbf{k}_1)][2\kappa T-G(\mathbf{k}_2)][2\kappa T-G(\mathbf{k}_3)]}\delta_{\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3-\mathbf{k}}+\cdots. \quad (74)$$

The fifth and sixth terms in these expressions arise because the expectation value of $\mathcal{S}^{(x)}\mathcal{S}^{(y)}\mathcal{S}^{(z)}$ at a site does not vanish. The fourth and sixth terms have a negligible effect on the transition temperature, as they are multiplied by a factor which is particularly small there, and so the result for the critical temperature is

$$T_c=\frac{G(0)}{2\kappa T}\left[1+\frac{5}{3N}\sum_{\mathbf{k}}\frac{G(\mathbf{k})}{G(0)-G(\mathbf{k})}+\frac{1}{3NG(0)^2}\sum_{\mathbf{k}}G(\mathbf{k})G(-\mathbf{k})\right.$$

$$\left.-\frac{10}{9N^2}\sum_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}\frac{G(\mathbf{k}_1)G(\mathbf{k}_2)G(\mathbf{k}_3)}{[(1+\theta_0)G(0)-G(\mathbf{k}_1)][(1+\theta_0)G(0)-G(\mathbf{k}_2)][(1+\theta_0)G(0)-G(\mathbf{k}_3)]}\delta_{\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3}\right]^{-1}. \quad (75)$$

The correspondence between this and Eq. (52) is not quite as close as the correspondence between Eqs. (72) and (46), but it must be remembered that $G(\mathbf{k})$ does not have quite the same meaning in the two equations; in particular, $\sum G(\mathbf{k})$ vanishes in the standard Ising or Heisenberg model.

These results for the critical temperature are consistent with the results of Vaks, Larkin, and Pikin for the Ising[11] and Heisenberg[12] models. They include the terms of order n^{-1} but omit the terms of order $n^{-2}\ln n$ However, their series for the specific heats are not purely functions of $n\theta^{-1/2}$, but include terms such as $n^{-3}\theta^{-3/2}\ln\theta$. If the shift of T_c given here is included, such terms become $n^{-3}\theta^{-3/2}\ln(\theta/\theta_0)$, which is of the required form when θ_0 is of order n^{-2}. The same happens to other functions calculated by Vaks, Larkin, and Pikin that appear to increase logarithmically with the range of the interaction.

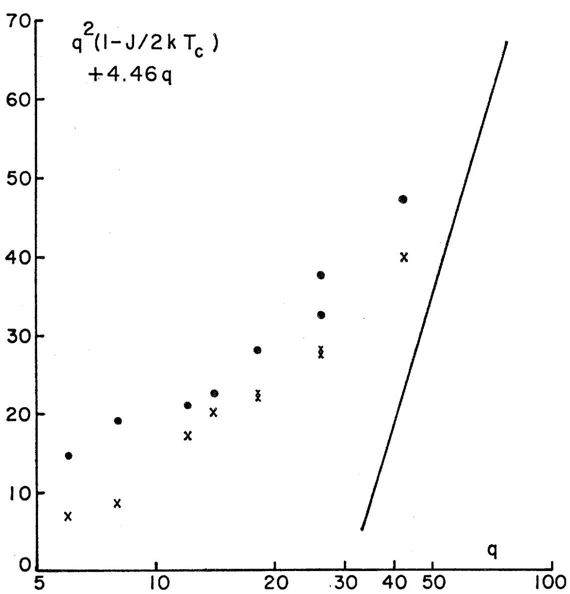

FIG. 7. The dots show the calculated values of $q^2(1-J/2\kappa T_c)+4.46q$ as a function of q (on a logarithmic scale) for the three-dimensional Ising model. The crosses show $q^2\{(1-J/2\kappa T_c)+N^{-1}\sum G(\mathbf{k})[G(0)-G(\mathbf{k})]^{-1}\}$ as a function of q. The straight line is a line of the slope predicted for large q.

In the critical region the diagrams of the models formulated in Secs. 2 and 7 contribute the same as the diagrams of the standard Ising and Heisenberg models, except possibly for a combinatorial factor, because the factor $G(\mathbf{k})/2\kappa T$ which comes in the numerator of the Fourier transform of R_{ij} is essentially unity in this region. It is not known if the combinatorial factors are identical, but no evidence has been found that they are different.

10. COMPARISON WITH NUMERICAL RESULTS

Domb and Dalton[15] have calculated the critical temperature for the Ising and Heisenberg models for a number of different lattices, with an interaction that extends over the first one, two, or three shells of nearest neighbors. In this section we examine these results and compare them with the asymptotic formulas given in Eqs. (72) and (75).

We assume that the interaction is of strength $2q^{-1}J$, and extends over a range R, where

$$(q+1)V=\tfrac{4}{3}\pi R^3. \quad (76)$$

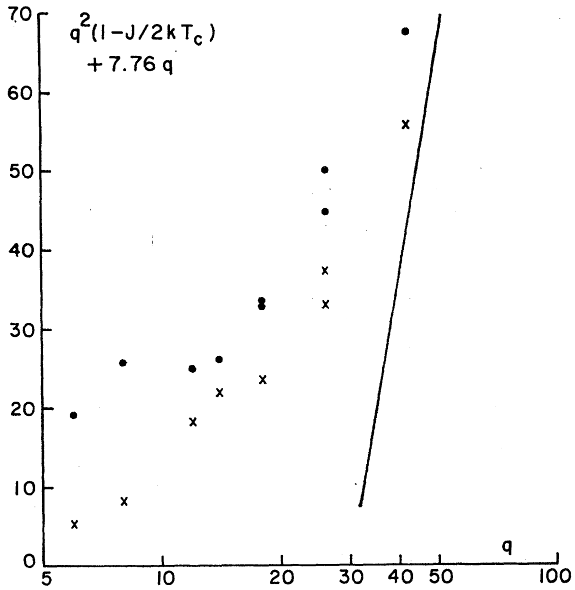

FIG. 8. The dots show the calculated values of $q^2(1-J/2\kappa T_c)+7.76q$ as a function of q (on a logarithmic scale) for the three-dimensional Heisenberg model. The crosses show $q^2\{(1-J/2\kappa T_c)+(5/3N)\sum G(k)[G(0)-G(k)]^{-1}+(3q)^{-1}\}$ as a function of q. The straight line is a line of the slope predicted for large q.

In the limit of large q, we have

$$G(\mathbf{k})\approx(J/qV)\int_{r<R}e^{i\mathbf{k}\cdot\mathbf{r}}d^3r$$
$$=3J[(\sin kR/k^2R^2)-(\cos kR/k^2R^2)]. \quad (77)$$

With this form of G, we get the first term in the relative shift of T_c for the Ising model equal to $4.46q^{-1}$, which is given by Dalton and Domb.[17] To evaluate the second term in Eq. (72), we can use the results obtained in the Appendix. The cutoff $K_{\max}$ is of order R^{-1}, and α is minus the coefficient of k^2 in $G(\mathbf{k})/G(0)$, which is $\frac{1}{10}R^2$. The resulting value is $(2000/27q^2)\ln q$, so that we get

$$\kappa T_c\sim\tfrac{1}{2}J(1+4.46q^{-1}-74.1q^{-2}\ln q)^{-1}. \quad (78)$$

For the Heisenberg model we get, from Eq. (75),

$$\kappa T_c\sim\tfrac{1}{2}J(1+7.76q^{-1}-123.5q^{-2}\ln q)^{-1}. \quad (79)$$

In Figs. 7 and 8 the calculated values of

$$q^2(1-J/2\kappa T_c)+4.46q \quad (80)$$

for the Ising model and the corresponding expression for the Heisenberg model are plotted as functions of $\ln q$. The straight lines to which the values should be asymptotic are shown. The fit is not good.

Since Dalton and Domb[17] have also calculated the quantity

$$\sum_{\mathbf{k}}G(\mathbf{k})/[G(0)-G(\mathbf{k})]$$

for the lattices in question, we have plotted on the same figures

$$q^2\left(1-\frac{J}{2\kappa T_c}+N^{-1}\sum_{\mathbf{k}}\frac{G(\mathbf{k})}{G(0)-G(\mathbf{k})}\right) \quad (81)$$

and the corresponding expression for the Heisenberg model. This does not appreciably improve the fit to the asymptotic form.

The results for the two-dimensional Ising model can be compared with the results given in Eqs. (54) and (55), or rather with the modified formulae valid for the standard Ising model. With

$$(q+1)A=\pi R^2 \quad (82)$$

and

$$G(\mathbf{k})\approx\frac{J}{qA}\int_{r<R}e^{i\mathbf{k}\cdot\mathbf{r}}d^2r$$
$$=(J/qA)\pi R^2(1-\tfrac{1}{4}k^2R^2+\cdots), \quad (83)$$

we have $\alpha=\frac{1}{4}R^2$, and so

$$\kappa T_c\sim\tfrac{1}{2}J(1+q^{-1}\ln q)^{-1}. \quad (84)$$

In Fig. 9 a plot of $q(J/\kappa T_c-1)$ against $\ln q$ is given for the numerical results for the two-dimensional Ising model, and here the points lie close to a line of the expected slope.

11. CONCLUSIONS

We have shown that a version of the Ising model with long-range interactions can be formulated for which the behavior in the neighborhood of the critical point can be calculated in the following intuitively appealing manner. An expression similar to the Ginzburg-Landau expression is used for the free energy F, and the partition function is found by summing $\exp(-F/\kappa T)$ over all states of the system. It is, however, necessary

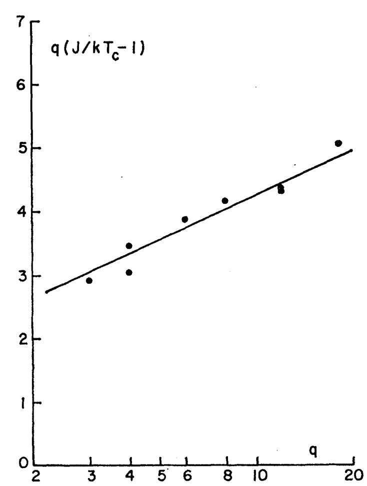

FIG. 9. $q(J/2\kappa T_c-1)$ as a function of q (plotted logarithmically) for the two-dimensional Ising model. The straight line is a line of the slope predicted for large q.

to renormalize the theory and allow for the shifts of critical temperature proportional to q^{-1} and $q^{-2}\ln q$, where q is the number of particles interacting with a given particle. Once this is done, it is found that there is a critical range of temperature, of width proportional to q^{-2} where the perturbation series in powers of q^{-1} is no longer useful. In this range, the specific heat is a function of $q^2\theta$, where θ is the temperature measured from the renormalized T_c. This implies, for example, that all specific-heat curves pass through the same value at T_c, even if T_c has not been chosen to coincide with the real critical temperature.

In two dimensions the shift is proportional to $q^{-1}\ln q$, and the specific heat in the critical region is a function of $q\theta$.

It is suggested here, but not proved, that the critical properties of the ordinary Ising and Heisenberg models can be found in the same way from a Ginzburg-Landau free energy. The shifts of T_c and the widths of the critical region which are calculated are independent of this assumption. If the assumption is correct, the behavior in the critical region for a system with long-range forces should depend on the nature of the order parameter (how many degrees of freedom it has, for example), and on the dimensionality of the system, but not on such details as the nature of the lattice, and the magnitude of the spin of the particles.

The results derived here can be applied to other systems where the Ginzburg-Landau free energy can be used, such as superconductors. For a superconductor similar results are obtained if we equate the volume of a cell with ξ^3 and the number of pairs in a cell with $\xi^3 k_F^2/\xi_0$, where ξ is the coherence length of the superconductor at zero temperature, k_F is the Fermi wave number of the electrons, and ξ_0 is the zero-temperature coherence length of the superconductor in its pure state. For a clean superconductor this is equal to ξ, while for a dirty superconductor ξ^2 is equal to ξ_0 times the electron mean free path. Something of this sort has been done by Ginzburg[2] for the three-dimensional case, by Langer and Amgebaokar[21] for the one-dimensional case, and by Abrahams and Woo[22] and Ferrell[13] for the two-dimensional case. In particular, Ferrell has derived the equivalent of Eq. (62) for the superconducting film.[23]

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APPENDIX

We need the asymptotic value of the integral

$$\int d^3k_1 \int d^3k_2 \times[(\theta+\alpha k_1^2)(\theta+\alpha k_2^2)(\theta+\alpha|\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}|^2)]^{-1} \quad \text{(A1)}$$

for small k and θ, where the region of integration is such that k_1, k_2 and $|\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}|$ are all less than $K_{\max}$. We have

$$\int d^3k_1[(\theta+\alpha k_1^2)(\theta+\alpha|\mathbf{k}_1-\mathbf{k}'|^2)]^{-1} = \frac{\pi}{\alpha k'}\int_0^{K_{\max}} \frac{k_1 dk_1}{\theta+\alpha k_1^2} \ln\frac{\theta+\alpha(k_1+k')^2}{\theta+\alpha(k_1-k')^2} + O\left(\frac{k'}{\alpha^2 K_{\max}^2}\right). \quad \text{(A2)}$$

For small k' this integral can be written as a contour integral and gives

$$(2\pi^2/\alpha^2 k')\tan^{-1}[k'/2(\theta/\alpha)^{1/2}].$$

Substitution of this back into Eq. (A1) gives the result

$$\frac{4\pi^4}{\alpha^3}\ln\frac{K_{\max}}{k}$$

or

$$\frac{4\pi^4}{\alpha^3}\ln\frac{K_{\max}}{(\theta/\alpha)^{1/2}}, \quad \text{(A3)}$$

according to whether k is greater than or less than $(\theta/\alpha)^{1/2}$.

[21] J. S. Langer and V. Ambegaokar, Phys. Rev. **164**, 498 (1967).

[22] E. Abrahams and J. W. F. Woo, Phys. Letters **27A**, 117 1968).

[23] R. A. Ferrell (private communication through M. Cyrot).