Role of Mode Interaction in the Amplification and Suppression of Echoes*

G. F. HERRMANN, D. E. KAPLAN, AND R. M. HILL
Lockheed Palo Alto Research Laboratory, Palo Alto, California 94304
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Phenomena similar to spin echo have recently been observed in a variety of physical systems whose only common feature is that they comprise large collections of oscillators distributed over some narrow frequency range. When these oscillators represent collective modes, any nonlinear interaction associated with echo generation also introduces coupling among the modes, thereby drastically altering the echo process. One consequence is the creation, under suitable conditions, of an unstable regime in which echo amplification of the type recently observed in a ferrimagnet can occur. The coupling must, however, be restricted to modes which are very close to each other in frequency. Specifically, we study in detail a multimode system in which oscillations are assumed to couple only if their frequency separation is within a range σ. If τ is the pulse interval, then echo behavior is characterized by the product $\sigma\tau$. For $\sigma\tau \ll 1$, echo processes are no different than in systems of isolated particles. For $\sigma\tau \sim 1$, in a conservative system, they are enhanced by energy transfer among modes and produce amplified echoes. Conditions for such behavior exist in nonuniform media when oscillation modes are quasilocalized in space. For $\sigma\tau \gg 1$, the echo process is suppressed because of phase mixing among interacting modes. This is typically the case for plane-wave modes in uniform media. If the nonlinearity originates in the dynamics of individual localized particles, all modes interact mutually on an equal basis and no echo is expected.

I. INTRODUCTION

A. Background

IN recent years echo phenomena have been reported in a growing number of physical systems. In addition to spin echo observed first in nuclei[1] and later in electrons,[2] echoes have been reported in relation to optical transitions,[3] in a ferrite,[4] in a plasma,[5] in relation to a rotational molecular transition,[6] and in a superconductor.[7] In view of this proliferation it is interesting to note that, following its discovery, spin echo remained for many years a unique phenomenon of its kind. The initial model for spin echo was tailored specifically to the dynamics of precessing spins and gave no hint of the universality of the underlying process. Strictly speaking, the model applied only to spin systems but it permitted one important generalization. Since the dynamics of any two-level quantum-mechanical system closely parallel those of a spin-$\frac{1}{2}$ particle,[8] echoes associated with spectroscopic transitions between nondegenerate atomic levels could also be predicted. However, the restrictiveness of the model became apparent with the unexpected observation of echoes in a classical system, that is, with the discovery of cyclotron echoes. It has since been recognized that the specific mechanism responsible for spin echo is just one of many possible echo-generating mechanisms and that echo processes constitute a large class of phenomena, similar in certain aspects but of diverse physical origin.

A typical pulse and echo pattern is shown in Fig. 1. A series of short incident pulses is followed at prescribed intervals by reradiated pulses, or echoes, from the medium. The principal characteristic of all these echoes is that the time intervals between echoes and incident pulse, and between the echoes themselves, always correspond to original intervals among incident pulses, or combinations of these. Echo amplitudes in their functional relationship to pulse amplitudes and time intervals, on the other hand, show the widest variation and depend on the particular echo mechanism in question. We shall limit detailed discussion almost exclusively to the simplest of these echoes, namely, the two-pulse echo, in which two incident pulses, at $t=0$ and at $t=\tau$, are followed by an echo at $t=2\tau$.

The common feature of systems exhibiting the phenomenon is that in some sense they can be viewed as very large collections of resonators (or oscillators, as we prefer to say[9]) whose frequencies are distributed over a range comparable to the frequency range included in the Fourier transform of the pulses. Two properties of

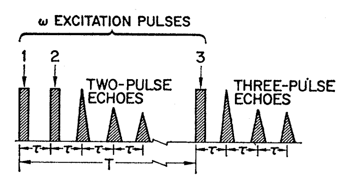

FIG. 1. Characteristic pulse and echo sequences. Pulses are incident at $t=0$, $t=\tau$, and $t=T$. Echoes are reradiated at $t=2\tau$, 3τ, etc., and at $t=T+\tau$, $T+2\tau$, etc.

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[2] J. P. Gordon and K. D. Bowers, Phys. Rev. Letters 1, 369 (1958).
[3] N. A. Kurnit, I. D. Abella, and S. R. Hartmann, Phys. Rev. Letters 13, 567 (1964).
[4] D. E. Kaplan, Phys. Rev. Letters 14, 254 (1965).
[5] R. M. Hill and D. E. Kaplan, Phys. Rev. Letters 14, 1062 (1965).
[6] R. M. Hill, D. E. Kaplan, G. F. Herrmann, and S. K. Ichiki, Phys. Rev. Letters 18, 105 (1967).
[7] I. W. Goldberg, E. Ehrenfreund, and M. Weger, Phys. Rev. Letters 20, 539 (1968).
[8] R. P. Feynman, F. L. Vernon, and R. L. Hellwarth, J. Appl. Phys. 28, 41 (1957).

[9] The terms resonance, oscillator, oscillation mode, or simply mode will be used interchangeably depending on the context. All correspond to a dynamic system which in the linear approximation limit obeys the mathematical equations for a harmonic oscillator.

importance characterize the response of such a system. The first is the so-called "phase mixing" which occurs as a consequence of the spread in frequencies. When the system is excited coherently by a sharp pulse, the phase angles of the oscillators initially excited in unison soon become distributed among all possible values in a quasirandom manner. The macroscopic excitation, which equals the vector sum of all oscillator excitations, therefore tends to decay rapidly to zero, and the reradiation from the medium is accordingly very small. Energy thus remains stored in the oscillations until degraded by intrinsic relaxation processes. The second property relates to phase coincidence. Consider a set of oscillators of different frequencies whose phase angles coincide at $t=0$, and then after spreading apart return again to coincidence at $t=\tau$. After $t=\tau$ this process simply repeats so that all oscillators are again in phase at $t=2\tau$, and then at $t=3\tau$, $t=4\tau$, etc. It is this property which underlies the repetitive time-interval pattern characteristic of resonance echoes.

The actual generation of an echo requires, in addition, some nonlinearity. Each pulse impresses a certain disturbance on the medium. Echoes cannot arise merely as a result of linear superposition of these disturbances but require some nonlinear interaction among them. At high enough pulse levels almost any physical system abounds in nonlinearities, and there is usually no difficulty in satisfying this requirement. It might seem, therefore, that any collection of oscillators with sufficiently long oscillation lifetimes should be capable of exhibiting echoes.

So far we have been vague concerning the precise meaning of the expression "system of oscillators." In principle, the term "oscillator" can be applied to any resonance of the system. In some systems exhibiting echoes, the oscillators correspond to isolated atomic particles; in others, they must be viewed as collective oscillation modes.[4,7,10] In the linear approximation there is no distinction between the two cases, and the question may be raised whether the concepts developed in the interpretations of echoes for isolated particles can be automatically extended to collective modes. Such a generalization would be sweeping, indeed, since it implies that almost any sufficiently complex system which can sustain oscillations will exhibit echoes under suitable conditions. Even the simplest of all oscillating systems, a vibrating string, is then a suitable candidate. Upon further consideration it becomes clear that such a generalization is quite dangerous unless one takes into account the specific manner in which *nonlinear* interactions affect the system. In this respect there is a fundamental difference between the situation obtaining in the case of isolated oscillators and the case of collective modes. In the former, nonlinearity involves each oscillator individually and the complete dynamic behavior can be derived on the basis of single-particle models. In the latter, the nonlinear interactions, as a rule, involve many modes simultaneously and can only be handled within the framework of a many-body problem.

The effect of nonlinear intermode coupling is most strikingly exhibited in the phenomenon of amplified echoes recently observed in a ferrite.[11] Two aspects sharply distinguish these echoes from any of the single-particle systems previously studied. The first is amplification itself. With a small first pulse and a large second pulse, echoes have been observed with energies exceeding those of the first pulse by more than 10^5. The second is the initial exponential-like increase of the echo with pulse separation. Since a growing exponential character is not normally associated with the motion of an isolated oscillator, the behavior suggests that energy transfer takes place among the modes and that the excitation of some modes grows at the expense of others. Nonlinear intermode coupling in this case apparently gives rise to an unstable regime in which the disturbances associated with the echo grow exponentially. Another observation of interest is the fact that in totally homogeneous internal dc magnetic fields not only is there no amplification but in general no echo at all, in spite of the availability of a rich spin-wave spectrum. Depending on conditions, mode interaction can therefore enhance or impair the echo process.

It is the purpose of this paper to analyze certain of these effects in detail, and attempt to draw some general conclusions concerning echoes in a system of collective modes. Since echo amplification is the most interesting consequence of intermode coupling, this phenomenon naturally deserves a major part of our attention, and will be analyzed in considerable detail. We do not presume to describe the general behavior of an arbitrary system of oscillators with arbitrary nonlinear interactions, and must therefore concentrate on systems with clear and simple echo properties. Even then the problem is quite complex. Difficulties inherent in any nonlinear many-body theory are compounded by the fact that the interesting effects occur in an inhomogeneous medium. Our starting point always remains the safe ground of linear approximation from which we take only cautious steps into the nonlinear morass with the aid of traditional techniques of expansion and linearization. Needless to say, we ignore all nonlinear effects which do not bear directly on the echo phenomenon. And, in order to make any headway at all, we must exploit every possible simplifying assumption. Initially we confine detailed mathematical analysis to the most elementary and most restricted system exhibiting the desired properties. Once this is accomplished, the aim of further, less precise, analysis is to indicate that the same qualitative behavior may be expected in more

[10] L. O. Bauer, F. A. Blum, and R. W. Gould, Phys. Rev. Letters **20**, 435 (1968).

[11] D. E. Kaplan, R. M. Hill, and G. F. Herrmann, Phys. Rev. Letters **20**, 1156 (1968).

general systems. In following a route from the particular to the general we shall be in a position to point out the important physical mechanisms in terms of a specific, reasonably well-understood model. Recognizing the futility of aiming at true generalizations or exhaustive solutions for systems of such complexity, we can still arrive at conclusions whose validity extends considerably beyond the specialized model considered.

B. Formulation of the Problem

The specific point of view from which the analysis is approached is motivated by considering some of the difficulties arising in attempting to apply single-particle echo concepts to a system in which all oscillation modes interact with each other on a more or less equal basis. Consider two oscillators at frequencies ω_1 and ω_2. During the interval τ the relative phase of the oscillators changes by $(\omega_1-\omega_2)\tau$, which for most oscillator pairs in the usual echo regime is a very large number. The nonlinear interaction is thus itself subject to phase mixing—any given oscillator interacts with a set of oscillators whose phases are distributed among all angles, and, moreover, are constantly changing. Since the effects of the interaction, and particularly the direction of energy transfer, depend on phase relations among the oscillators, it is hard to see how any systematic process can take place under these circumstances. It thus appears necessary to restrict the *frequency range* over which oscillators interact with each other, and consider systems in which nonlinear coupling is not independent of frequency, but decreases with increased frequency difference.

More specifically, we study systems in which a given oscillator of frequency ω couples nonlinearly primarily to oscillators whose frequency ω' is restricted to $|\omega'-\omega|<\sigma$, where σ is some narrow frequency range. The product $\sigma\tau$ is then a measure of the relative phase drift in a "packet" of interacting oscillators during the interval τ. Depending on whether $\sigma\tau$ is large, small, or intermediate, one can consider three regimes.

(i) $\sigma\tau\ll1$. In this case any group of interacting oscillators can be regarded as a "packet" within which all phases remain equal throughout the experiment. With minor modifications the single-particle echo model for noninteracting modes can be applied to this case.

(ii) $\sigma\tau\ll1$. In this case any systematic nonlinear effect is precluded by phase mixing and the appearance of an echo is unlikely.

(iii) The intermediate region in which $\sigma\tau$ is of order 1. In this region it is shown that unstable growth and amplification may occur.

Many hypothetical systems possessing the required property can be envisioned by constructing suitable modes in some space, e.g., position space, momentum space, velocity space, and, more generally, phase space. For the moment, interest is confined to nonuniform media where frequency is in some sense a function of position, and where coupling arises from the spatial overlap of modes at slightly differing frequencies. In order to provide some degree of practical motivation for the largely abstract arguments of subsequent sections, we permit ourselves a short digression on this subject.

C. Spatial Distribution of Oscillation Modes

It is not our intention to present here an analysis of stationary oscillation modes in nonuniform media, but merely to indicate, perhaps somewhat imprecisely, circumstances under which mode coupling of the type postulated may arise in practice.

Consider a medium which consists of coupled atomic particles localized in space, and can support various oscillation modes. If one assumes that the important nonlinear effects originate solely in the motion of the individual particles, then nonlinear coupling among a set of modes depends directly on their spatial overlap. In this respect there are two extreme possibilities—there is no overlap at all, or there is total overlap of all modes over the entire sample. The first situation will occur, for example, if the particles are isolated oscillators as in the original spin-echo experiments. The second occurs in completely homogeneous media. In large samples the modes in the latter case are in the form of plane waves which overlap over the entire samples and are therefore all mutually coupled through the nonlinearity in the individual particle motion. This condition, which is discussed in detail in Appendix A, is not conducive to echo formation.

The interesting cases lie between these two extremes. As a starting point, we consider an infinite medium, with interactions of purely dipolar nature, and confine our attention to so-called longitudinal oscillations. For such oscillations the group velocity is zero and the field lines are confined to the region in which the disturbance occurs. Oscillations can therefore be confined to any arbitrary region of space. The best-known example is provided by longitudinal plasma oscillations in a cold isotropic plasma at the plasma frequency ω_p, which is a function of the electron density. In an inhomogeneous plasma each region tends to oscillate independently at its appropriate local plasma frequency.

The presence of an external magnetic field greatly complicates this picture. The group velocity, however, remains zero and there exist sets of stationary oscillation modes each occupying a suitable two-dimensional surface. In an inhomogeneous field the mode frequency depends on the local field value as well as the orientation of the surface.

In an actual plasma, or ferrimagnet, these modes are modified by a number of factors. Foremost is the finite sample size which results in surface charges and fringing fields through which various modes interact. Additional coupling is provided by electromagnetic waves and by

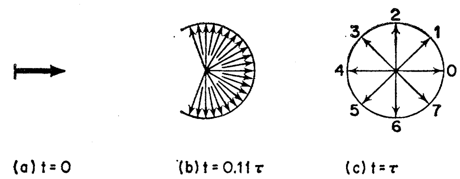

FIG. 2. Evolution of system of oscillators uniformly excited by pulse 1. Each oscillator is represented by vector in complex plane. The circular curve represents locus of vector tips. At $t=\tau$, oscillators at 45° intervals are numbered for future identification. In subsequent figures vectors are omitted and only locus of end points is drawn.

exchange (in ferrimagnets) or thermal effects (in plasmas). The former is important in disturbances which are coarse-grained spatially, and the latter in disturbances which are very fine-grained. The resulting modes, in general, have a very complex spatial distribution. In a highly inhomogeneous field under conditions dominated by volume dipolar interactions, much of the oscillation energy of a given mode remains concentrated about the region corresponding to local resonance conditions and the modes may be regarded as quasilocalized. Overlap among a set of modes is then largest when the frequencies are very close.

Summing up, the desired mode structure may be expected in an inhomogeneous medium, for disturbances in which dipolar interactions predominate. The modes are quasilocalized and interact primarily with neighboring modes at adjacent frequencies.

II. FORMAL PRELIMINARIES

In order to lay the groundwork for the present study we must recapitulate some of the general theory for echo processes. Much of this material can be found elsewhere in expanded form.[12–14]

A. Echo Regime

We impose on the systems under consideration a number of severe restrictions. Although echoes are often observed even when one or more of these restrictions is relaxed, the presentation of the theory is greatly simplified if one adheres to the idealized echo regime defined below.

We confine the discussion to the simplest two-pulse echo sequence which consists of two incident pulses at $t=0$ and $t=\tau$, respectively, and an echo at $t=2\tau$. Other echoes are briefly surveyed in Appendix B.

In the *linear approximation* the system is assumed to comprise a very large number of independent oscillators whose frequencies are spread over a range $\Delta\omega$. The parameter $t_p=1/\Delta\omega$ is a measure of the time required for "phase mixing" to occur, that is, for oscillator phase angles to become distributed among all values following coherent excitation by a sharp pulse.

With t_w denoting pulse widths, we now assume:

(a) $t_w \ll t_p$, i.e., relative phase shifts among oscillators can be ignored for the duration of a pulse.

(b) $\tau \gg t_p$, i.e., complete phase mixing takes place during the interval between pulses.

(c) The distribution $n(\omega)$ of oscillators and coupling to the radiation field is a smooth function of ω. One can therefore divide the system into subensembles, each of frequency range $\delta\omega$ such that $n(\omega)$ and the coupling to the radiation are essentially constant in each subensemble, and at the same time $\tau \gg 1/\delta\omega$.

The nonlinear perturbation can enter in one of two ways—either through interactions with the external field or through interactions involving only the oscillators themselves. Paramagnetic spin echo represents an example of the first case. In the present study we shall be primarily interested in the second. For this case we shall require that the nonlinear perturbation be "small" and assume:

(d) The rate at which momentum or energy is changed by nonlinear effects is very small compared to $\Delta\omega$. Put differently, the characteristic times for nonlinear processes are very long compared to t_p.

A fifth, implicit, assumption can only be vaguely formulated to state that all oscillators are "similar" and exhibit *a priori* identical behavior except in those aspects which depend explicitly on the frequencies.

B. Single-Particle Model

We recapitulate here a simplified echo theory for uncoupled oscillators. We assume a complex convention and refer all motion to the familiar moving frame of reference, which rotates in the complex plane at some mean frequency of the system. The frequency ω will therefore denote the difference between the true oscillator frequency and the frequency of the rotating frame.

Consider first the linear approximation. In the absence of radiation the motion of an oscillator of frequency ω is of the form $ae^{i\omega t}$. In the moving system, a short pulse imparts to all oscillators an equal amplitude[15] increment A. Let pulse 1 at $t=0$ impart the increment A_1, and pulse 2 at $t=\tau$ the increment A_2. Without loss of generality, we can assume that in the rotating frame A_1 and A_2 are real. We can follow the evolution of the system analytically, and also graphically[1,12] by studying the curves corresponding to the loci of the amplitude vector tips for the ensemble (Fig. 2). At

[12] R. W. Gould, Phys. Letters **19**, 477 (1965).
[13] G. F. Herrmann and R. F. Whitmer, Phys. Rev. **143**, 122 (1966).
[14] G. F. Herrmann, R. M. Hill, and D. E. Kaplan, Phys. Rev. **156**, 118 (1967).

[15] Amplitude as used here includes both magnitude and phase in the usual complex convention. In the present discussion it measures a momentum coordinate whose rate of change is proportional to the radiation field. It represents, for example, the electron current—proportional to the linear momentum—in a plasma, and the transverse magnetic moment—proportional to the transverse component of the spin angular momentum—in a ferrimagnet.

$t=0_+$, just after pulse 1, all oscillators have the amplitude A_1 [Fig. 2(a)]. Because of the spread in frequencies the vectors rapidly fan out in a circle [Fig. 2(b)]. At $t=\tau_-$, just prior to pulse 2, the amplitude of a given oscillator is given by $A_1e^{i\omega\tau}$, and since $\Delta\omega\tau\gg 1$, the vector tips lie on a circle which is wound about itself many times [Fig. 2(c)]. On this circle we pick a series of representative points at intervals of 45°, which we shall henceforth follow in tracing the further evolution of the system.

At $t=\tau_+$, immediately following pulse 2, the amplitude is $a(\tau_+)=A_1e^{i\omega\tau}+A_2$ and the ensemble is represented by a circle [Fig. 3(a)], which is displaced from the origin by A_2. Subsequent motion is described by $a(t)=A_1e^{i\omega t}+A_2e^{i\omega(t-\tau)}$. With ω considered as the curve parameter this expression describes the sequence of epicycloids shown in Figs. 3(b)–3(e). The absolute amplitude $|a|$, which is given by

$$|a|^2=A_1^2+A_2^2+2A_1A_2\cos\omega\tau\,, \qquad (2.1)$$

is a periodic function of ω, with period $P(\omega)=2\pi/\tau$. This is reflected also as a periodicity of the epicycloids with respect to the angle φ, the period equaling $P(\varphi)=2\pi(t-\tau)/\tau$.

Interaction with the radiation field depends on the total moment obtained by adding all the individual oscillator amplitudes. This moment can be approximated by an integral of the form

$$\mu=\int n(\omega)(A_1e^{i\omega t}+A_2e^{i\omega(t-\tau)})d\omega\,, \qquad (2.2)$$

which, because of assumptions (c) and (d), is negligible for $t-\tau\gg 1/\Delta\omega$. The vanishing of the moment is also evident in the quasisymmetric manner in which the curves of Fig. 3 surround the origin. An exception occurs at $t=2\tau$ (and also other multiples of τ) when $P(\varphi)=2\pi$ and the epicycloid coalesces to a simple asymmetric closed curve, in which all points which coincided at $t=\tau$ coincide again. It is this asymmetry which causes echoes to occur preferentially at multiples of τ—except that in the linear case, echoes will not occur even then. From the analytic expression

$$a(2\tau)=A_1e^{2i\omega\tau}+A_2e^{i\omega\tau}\,,$$

it is clear that the curve in Fig. 3(e) is merely the superposition of two circular distributions with a vanishing resultant.

In order to produce a finite resultant, some nonlinearity is required. In processes which are responsible for echo generation the nonlinearity usually appears as a function of the oscillator amplitude $|a|$ and does not depend explicitly on the oscillator frequency. Hence, it, too, is a periodic function of ω with the period $P(\omega)=2\pi/\tau$, and the resulting curves, although modified in shape, retain the periodicity in angle $P(\varphi)=2\pi(t-\tau)/\tau$ and the property of quasisymmetry about the origin.

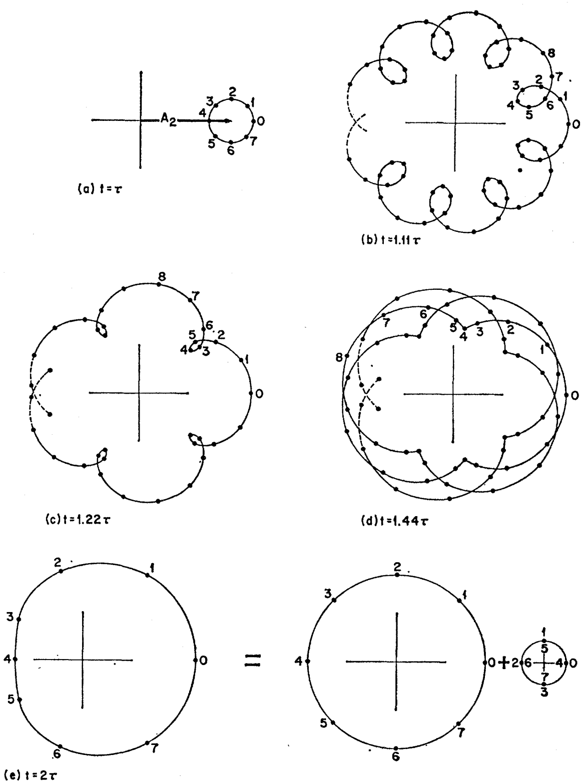

FIG. 3. Evolution of system responding linearly to two pulses. (a) At $t=0$ locus of vector tips is a circle displaced from origin. (b)–(d) Locus is a set of epicycloids symmetrically wound about the origin. (e) Locus is asymmetric figure equaling the superposition of two circles corresponding, respectively, to the disturbances generated by pulses 2 and 1.

As in the linear case, only at multiples of τ will the curve coalesce to an asymmetric simple closed curve. However, in the nonlinear case, a finite resultant moment at these specified times is to be expected.

As an example, consider the simplest anharmonic effect, namely, an amplitude-dependent frequency shift.[16] If the "perturbed" frequency ω' is given by $\omega'=\omega+\alpha a^2$, where α is a small coefficient, then, following the incidence of the two pulses, we have from (2.1)

$$\omega'=\omega+\alpha A_1^2+\alpha A_2^2+2\alpha A_1A_2\cos\omega\tau\,. \qquad (2.3)$$

The second and third terms on the right side are common to all oscillators and can be lumped into the motion of the rotating coordinate frame and henceforth ignored. For the same reason, when considering the period preceding pulse 2, one may ignore the constant term αA_1^2. The oscillator motion can thus be described by

$$a=(A_1e^{i\omega\tau}+A_2)\exp[i(\omega+2\alpha A_1A_2\cos\omega\tau)(t-\tau)].$$

To obtain the resultant at the echo time $t=2\tau$, it suffices to integrate a over a single period $P(\omega)$. Making use of assumption (d) and putting $\theta=\omega\tau\,\mathrm{mod}\,2\pi$, with

[16] W. H. Kegel and R. W. Gould, Phys. Letters 19, 531 (1965).

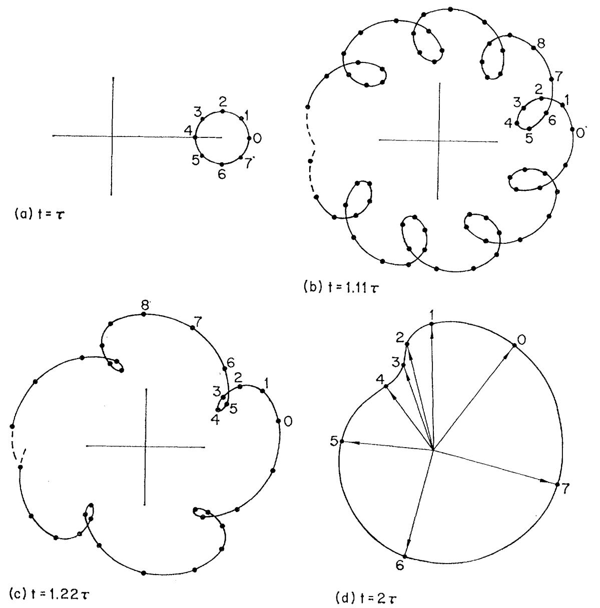

FIG. 4. Echo generation through amplitude-dependent frequency. (b) and (c) Epicycloids are now distorted because of relative forward phase drift of larger amplitude oscillators. (d) At $t=2\tau$, oscillator phases are bunched in upward direction, resulting in finite moment.

$0\leqslant\theta<2\pi$, one has

$$\mu_e=\frac{N}{2\pi}\int_0^{2\pi}(A_1e^{i\theta}+A_2)\exp[i(\theta+2\tau\alpha A_1A_2\cos\theta)]d\theta\,,$$

where N is the number of oscillators. Integration gives

$$\mu_e=N[-A_1J_2(2\tau\alpha A_1A_2)+iA_2J_1(2\tau\alpha A_1A_2)]\,,$$

where J_1 and J_2 are Bessel functions. For small values of $\tau\alpha A_1A_2$,

$$\mu_e=iN\tau\alpha A_1A_2^2\,,$$

that is, the resultant moment, and hence the radiated echo amplitude, increases linearly with the time τ between pulses.

The mechanism is displayed graphically in Fig. 4. The curves, recapitulating part of the evolutionary sequence of Fig. 3, illustrate the distortion of the epicycloids due to the amplitude-dependent phase drift. Figure 4(d) shows that at $t=2\tau$ this drift results in phase bunching towards the upper half of the plane, and a nonvanishing resultant in the upward direction.

The amplitude dependence of the frequency is the only cummulative nonlinear effect in a conservative system of isolated particles. The linear variation of μ with τ (for small τ) simply reflects the uniform rate of the resulting phase drift. In a nonconservative system nonlinear relaxation is also possible. Neither case corresponds to an unstable regime in which the echo varies exponentially with τ. Such behavior requires energy transfer among the oscillators and is therefore incompatible with the single-particle model.

III. ROLE OF NONLINEAR INTERMODE INTERACTION

As indicated in Sec. I B, our aim is to analyze systems of nonlinearly coupled oscillators where the coupling is confined to an effective frequency range σ. We shall proceed to construct the simplest possible mathematical model for such a system and show that it exhibits both echo amplification and suppression, depending on the value of $\sigma\tau$. Next, we shall try to generalize these results to somewhat less restrictive models. Finally, we shall attempt to provide a physical interpretation of the amplification process. The discussion is confined throughout to conservative (i.e., nondissipative) systems, since nonlinear relaxation processes are considerably more difficult to treat. A particularly important dissipative echo mechanism is discussed in Appendix A for the case of a uniform medium.

A. Simple Mathematical Model

Consider a large set of equally spaced oscillation modes at frequencies $\omega_n=\omega_0+n\Omega$, where Ω is a very small frequency increment. Let the modes be all of identical form $\varphi(x)$, but displaced relative to each other so that the nth mode is described by $\varphi_n=\varphi(x-n)$. One may, if one wishes, view x as representing position in a one-dimensional configuration, in which case there will be one oscillation mode per unit of distance. Each mode has a finite effective spread Δx, and therefore overlaps other modes which lie within a frequency range $\sigma=\Omega\Delta x$. We assume that the modes are very densely spaced, and that $\Omega\ll\sigma$ as well as $\Omega\tau\ll1$. This will permit the substitution of integration for summation when sums over all modes are calculated.

In writing the dynamic equations, we shall follow a standard canonical formalism as described for a plasma by Sturrock[17] and for a ferrimagnet by Schlömann.[18] The Hamiltonian $\mathcal{H}$ is given as a function of the variables a_n and their complex conjugates a_n^*. The equations of motion are

$$\dot{a}_n=i\frac{\partial\mathcal{H}}{\partial a_n^*}\quad\text{and}\quad\dot{a}_n^*=-i\frac{\partial\mathcal{H}}{\partial a_n}.\tag{3.1}$$

In the linear approximation the Hamiltonian is a quadratic form. We assume that it is already in the diagonal form

$$\mathcal{H}_2=\sum\omega_na_na_n^*.\tag{3.2}$$

Hence, to first order, $\dot{a}_n=i\omega_na_n$ and $a_n(t)=a_n(0)e^{i\omega_n}$ as stipulated.

[17] P. A. Sturrock, in *Proceedings of the International School of Physics "Enrico Fermi": Course 25*, edited by M. N. Rosenbluth (Academic Press Inc., New York, 1965).

[18] E. Schlömann, Raytheon Company Technical Report No. R-48, 1959 (unpublished). For earlier noncanonical treatment of nonlinearity and instability in a ferrimagnet, see H. Suhl, J. Phys. Chem. Solids 1, 209 (1957).

The nonlinear interaction is given in terms of higher-order polynomials in a_n. Now, to first order, a_n oscillates at ω_n or approximately at ω_0. A product of the form $a_{n1}a_{n2}$ varies approximately as $e^{2i\omega_0 t}$, i.e., at a frequency far removed from that of the oscillators. These terms therefore have little effect. On the other hand, the product $a_{n1}a_{n2}a_{n3}^*$ has a time variation of approximately the frequency ω_0 and therefore affects the motion strongly. We therefore assume a Hamiltonian of the form

$$\mathcal{H}=\sum_n \omega_n a_n a_n^* + \tfrac{1}{2}\sum_{n_1n_2n_3n} C(n_1n_2n_3n) \times a_{n1}a_{n2}a_{n3}^* a_n^*, \quad (3.3)$$

and, accordingly, dynamic equations of the form

$$\dot{a}_n = i\omega_n a_n + \sum_{n_1n_2n_3} C(n_1n_2n_3n) a_{n1}a_{n2}a_{n3}^*. \quad (3.4)$$

In order to determine the form of $C(n_1n_2n_3n)$ we now assume that the anharmonicity is entirely local in character, that is, at any point x it depends only on the amplitude $a(x)$ and not on the amplitude at any other point. The fourth-order Hamiltonian therefore has the form

$$\mathcal{H}_4 = \tfrac{1}{2} q \int |a(x)|^4 dx, \quad (3.5)$$

where q is a small constant. Since

$$a(x) = \sum_n a_n \varphi(x-n),$$

we find by comparison with the fourth-order terms in (3.3) that C is given by the fourfold overlap integral

$$C(n_1n_2n_3n) = q\int \varphi(x-n_1)\varphi(x-n_2)\varphi^*(x-n_3) \times \varphi^*(x-n)dx. \quad (3.6)$$

Note that C is invariant to a simultaneous translation of all n's.

We will now study the response of this system to a pulse sequence of a standard type used in amplified echo experiments,[8] namely, one in which the first of the two incident pulses is extremely weak. Again we assume that a pulse imparts to each mode an equal amplitude increment A. For the first pulse we shall put $A_1=\epsilon$, with $\epsilon \ll A_2$. The disturbance associated with pulse 1 can therefore be regarded as causing a small perturbation of the large-scale motion associated with pulse 2, and we may undertake to linearize the problem with respect to ϵ.

Between pulses 1 and 2 the system may be assumed to be linear because of the smallness of ϵ. At $t=\tau_+$, immediately after pulse 2, the amplitude, just as in the single-particle model, is given by

$$a_n(\tau) = \epsilon e^{i\omega_n\tau} + A_2 = \epsilon e^{i(\omega_0+n\Omega)\tau} + A_2. \quad (3.7)$$

These equations serve as initial conditions for Eqs. (3.4). Equations (3.7) are periodic in n, with a period $P(n)=2\pi/\Omega\tau$. For arbitrary τ, $P(n)$ is not generally an integer. However, since we have assumed a high density of modes, $P(n)$ is a very large number, and can always be converted into an integer by a minor change in the value of τ. For convenience, we will treat $P(n)$ as an integer in the following calculations. Since the basic equations (3.4) and (3.6) are completely symmetric with respect to a translation in n, the solutions corresponding to the initial conditions (3.7) retain the same periodicity.

The echo can thus be obtained by summing a_n at $t=2\tau$ over a single period. We define the echo amplitude per oscillator as

$$a_{\text{echo}} = [P(n)]^{-1} \Big| \sum_{n=1}^{P(n)} a_n(2\tau) \Big|. \quad (3.8)$$

Before linearization we must obtain the zeroth-order solutions to (3.4), i.e., those corresponding to $\epsilon=0$. It is easily checked that these are given by

$$a_n(\epsilon=0) = A_2 \exp\{i[(\omega_0+n\Omega)(t-\tau)+\Theta]\}, \quad (3.9)$$

where

$$\Theta = A_2^2 \int_0^{t-\tau} \sum_{n_1n_2n_3} C(n_1n_2n_3n) \times \exp[i(n_1+n_2-n_3-n)\Omega t']\, dt'$$

is independent of n, because of the symmetry properties of C.

We now introduce a new set of coordinate systems, one for each oscillator, which rotate according to the zeroth-order solution (3.9). The new coordinates $\bar{a}_n$, in what we may call the "interaction representation," are defined by

$$a_n = \bar{a}_n \exp\{i[(\omega_0+n\Omega)(t-\tau)+\Theta]\}, \quad (3.10)$$

which upon substitution into (3.4) yield

$$\frac{d\bar{a}_n}{dt} = i\sum_{n_1n_2n_3} C(n_1n_2n_3n)(\bar{a}_{n1}\bar{a}_{n2}\bar{a}_{n3}^* - A_2^2\bar{a}_n) \times \exp[i(n_1+n_2-n_3-n)\Omega(t-\tau)]. \quad (3.11)$$

These equations can now be linearized by putting

$$\bar{a}_n = A_2 + \epsilon b_n, \quad (3.12)$$

and retaining only linear terms in ϵ. One finds

$$\dot{b}_n = i\sum_{n_1n_2n_3} A_2^2 C(n_1n_2n_3n)(b_{n1}+b_{n2}+b_{n3}^* - b_n) \times \exp[i(n_1+n_2-n_3-n)\Omega(t-\tau)]. \quad (3.13)$$

This represents a system of coupled linear equations between the perturbations associated with pulse 1. The initial conditions at $t=\tau$ are now given by $b_n = e^{i\Omega n\tau}$, and

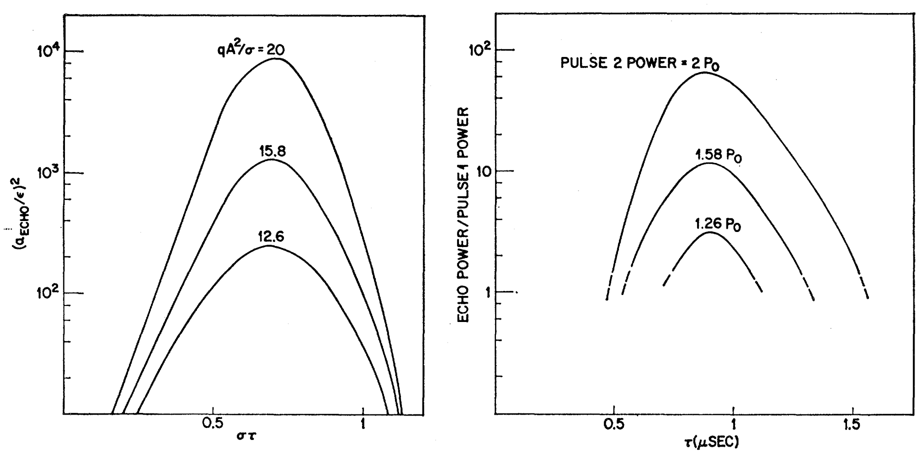

FIG. 5. Echo amplification measured as function of τ for several values of pulse-2 power (right), compared to similar calculated curves. Corresponding curves are matched on the basis of similar dependence on pulse-2 power. Inefficient coupling of the pulses to the sample may be responsible for the large difference in absolute amplification. P_0 equals approximately 1 W and pulse duration is 10 nsec.

the echo amplitude by

$$a_{\text{echo}}=[P(n)]^{-1}\epsilon\left|\sum_{n=1}^{P(n)} b_n(2\tau)e^{i\Omega n\tau}\right|. \tag{3.14}$$

The problem is greatly simplified at this point by making use of the periodicity and introducing the Fourier sum

$$b_n=\sum_m u_m e^{i\Omega\tau nm}, \tag{3.15}$$

where summation is over a period of length $P(n)=2\pi/\Omega\tau$, e.g., from $-\frac{1}{2}P(n)$ to $+\frac{1}{2}P(n)-1$. The inverse transformation is

$$u_m=[P(n)]^{-1}\sum_{n=1}^{P(n)} b_n e^{-i\Omega\tau nm}. \tag{3.16}$$

The initial conditions become $u_1(\tau)=1$, $u_m(\tau)=0$ for $m\neq 1$, and the desired echo amplitude is $a_{\text{echo}}=\epsilon|u_{-1}(2\tau)|$.

One proceeds by substituting (3.15) into the right side of (3.13), multiplying each equation by $e^{-i\Omega\tau nm}$, and forming linear combinations, as in (3.16), to yield $\dot{u}_m$ on the left side. Summation over n_1, n_2, and n_3 is thereupon replaced by integration and the particular form of $C(n_1n_2n_3n)$ is substituted from (3.6). Upon rearranging the order of integration and using the orthogonality relation

$$\sum_{n=1}^{P(n)} e^{i\Omega\tau n(m-m')}=P(n)\delta_{mm'},$$

one obtains a set of equations in which each u_m is coupled only to the corresponding u_{-m}. For $m=1$ one obtains

$$\dot{u}_1=iqA_2{}^2\{|F(t-\tau)|^2[2|F(t)|^2-|F(t-\tau)|^2]u_1 +F(t-\tau)^2F^*(t)F^*(t-2\tau)u_{-1}^*\}, \tag{3.17a}$$

$$\dot{u}_{-1}=iqA_2{}^2\{|F(t-\tau)|^2[2|F(t-2\tau)|^2-|F(t-\tau)|^2]u_{-1} +F(t-\tau)^2F^*(t)F^*(t-2\tau)u_1^*\}, \tag{3.17b}$$

where

$$F(t)=\int\varphi(x)e^{-i\Omega xt}dx. \tag{3.18}$$

Later we shall take a closer look at these equations. For the moment, in the spirit of mathematical simplification, we shall assign to the modes the simplest shape—a Gaussian—by putting

$$\varphi(x)=[\Omega/\sigma(2\pi)^{1/2}]\exp[-(\Omega x)^2/2\sigma^2],$$

where the width σ is in frequency units and represents the interaction range in terms of the frequency separation among modes.

On substituting this expression in Eq. (3.18) we obtain $F(t)=\exp(-\frac{1}{2}\sigma^2t^2)$ and

$$\dot{u}_1=iqA_2{}^2e^{-\sigma^2(t-\tau)^2}[(2e^{-\sigma^2t^2}-e^{-\sigma^2(t-\tau)^2})u_1 +e^{-\sigma^2[(t-\tau)^2+\tau^2]}u_{-1}^*], \tag{3.19a}$$

$$\dot{u}_{-1}=iqA_2{}^2e^{-\sigma^2(t-\tau)^2}[(2e^{-\sigma^2(t-2\tau)^2}-e^{-\sigma^2(t-\tau)^2})u_{-1} +e^{-\sigma^2[(t-\tau)^2+\tau^2]}u_1^*]. \tag{3.19b}$$

Exact solutions can be obtained only numerically. But the character of the solutions can be ascertained from the behavior during the period immediately following pulse 2, that is, for $\sigma(t-\tau)\ll 1$. There are three regimes of interest:

(i) $\sigma\tau\ll 1$. In that case, $u_1\sim 1+iqA_2{}^2(t-\tau)$ and $u_{-1}=iqA_2{}^2(t-\tau)$. The solutions are identical with those obtained in a single-particle model and represent the phase drift associated with an amplitude-dependent frequency. Indeed, a very small σ implies that any oscillator interacts only with oscillators whose motion is identical with its own, and which can therefore be coupled together as a single "oscillator packet."

(ii) $\sigma\tau\ll 1$. In this case u_1 and u_{-1} become, in effect, uncoupled. u_{-1} remains negligible, and so does the echo.

(iii) $\sigma\tau$ is intermediate. Let us choose, for example, a value such that $e^{-\sigma^2\tau^2}=\frac{1}{2}$. Then the solutions are $u_1\sim\cosh\frac{1}{2}qA_2{}^2(t-\tau)$ and $u_{-1}\simeq i\sinh\frac{1}{2}qA_2{}^2(t-\tau)$. These are unstable, exponentially growing solutions. For sufficiently large values of $qA_2{}^2$, u_{-1} may become very large, resulting in greatly amplified echoes.

In the context of an experiment, σ may be assumed fixed while the pulse interval τ is varied. The results indicate that for small τ there is increased amplification with increasing τ, but that for large τ the echo decreases again to zero.

It is instructive at this point to compare the calculation with some experimental results.

We have computed a series of solutions for a_{echo} on the basis of Eqs. (3.19), and a sampling of these is shown in Fig. 5 plotted against $\sigma\tau$. Adjacent to them are displayed some experimental curves obtained in an yttrium-iron-garnet crystal at 10 GHz in echo experiments previously described. The comparison is not intended to be quantitative in any sense, since little is known concerning the experimental mode structure. Moreover, relaxation effects ignored in the theory are probably very important in the experiments.

The qualitative similarity is particularly striking in two respects, both completely unique to this phenomenon. The first is the exponential character of the growth part of the curves, which is symptomatic of instability. The second is the drastic variation of echo power with the power of pulse 2. This behavior is in contrast with single-particle models which predict no more than a quadratic dependence of echo power on either τ or A_2. Experimental and theoretical curves in Fig. 5 are juxtaposed in such a way that an increase in pulse-2 power by 1.26 (1 dB) will produce similar changes in peak echo power. It will be noted that experimental echo amplification is two orders of magnitude lower than the theoretical value. Much of this difference is accounted for by power lost because of poor coupling of the radiation field to the appropriate modes in the sample.

B. Generalizations

The qualitative behavior which we have determined does not depend strongly on the particular shape chosen for φ_n. If φ_n has any well-behaved smooth shape and an effective width σ, then one may assume that the Fourier transform $F(t)$ defined in (3.18) goes to zero for $|t| \gg \sigma^{-1}$. As t varies from zero to ∞, $|F(t)|$ assumes all values between $F(0)$ and zero.

The short-period stability properties of Eq. (3.17) can be investigated by treating the coefficients of u_1 and u_{-1} formally as constants, and calculating the characteristic oscillation frequencies of the system. The character of the solutions is found to depend on the parameter

$$D=[|F(t)|^2-|F(t-\tau)|^2+|F(t-2\tau)|^2]^2 - |F(t)|^2|F(t-2\tau)|^2. \quad (3.20)$$

If $D>0$, then the frequencies are real and distinct and the solutions are periodic, or stable. If $D=0$, then the frequencies are real but degenerate and the solutions are singular, i.e., grow linearly with time. If $D<0$, then the frequencies are complex, and the solutions vary exponentially.

The behavior of the solutions is quite analogous to that found in the case of Gaussian modes.

(i) For $\tau \ll 1/\sigma$ (hence also $t \ll 1/\sigma$ as $\tau < t \leq 2\tau$), $F(t) \sim F(t-\tau) \sim F(t-2\tau) \sim F(0)$; hence $D \sim 0$ and the solutions grow linearly with time.

(ii) For $\tau \gg 1/\sigma$, $F(t) \simeq F(-t) \simeq 0$. Therefore, for $t \sim \tau$, $D>0$ and solutions are stable. In fact, in Eq. (3.17), u_1 and u_{-1} become uncoupled.

(iii) For $t \sim \tau$, the term in square brackets in (3.20) becomes $|F(\tau)|^2-|F(0)|^2+|F(-\tau)|^2$. Since $|F(\tau)|$ and $|F(-\tau)|$ go to zero for large τ, one can always find an intermediate τ for which this term becomes very small; hence $D<0$ and the solution is unstable.

One may ask to what extent the results depend on the special form (3.6) of C. Here again, if $C(n_1n_2n_3n)$ is well behaved, that is, a smooth function which tapers to zero whenever the difference between two of its arguments increases much beyond σ, one can show by lengthy and subtle arguments that, in general, the same three regimes will exist. One should note in this connection that C is a symmetric tensor in any conservative system.

C. Physical Interpretation

The physical process of amplification is compounded of several elements. We shall try to reconstruct it step by step in reference to the mathematical formulation of Sec. III A.

Our starting point is the isolated-particle limit at which the oscillators are noninteracting. In that case all the coefficients C in (3.4) equal zero except for $C_n = C(nnnn)$, and the equation becomes

$$\dot{a}_n = i\omega_n a_n + iC_n a_n a_n a_n^* = i(\omega_n + C_n|a_n|^2)a_n.$$

The nonlinear coefficient $C_n|a_n|^2$ simply constitutes an amplitude-dependent addition to the frequency. Without loss of generality, we will assume here and further on that $q>0$. The oscillator frequency then *increases* with amplitude.

Next we consider energy transfer among modes and its dependence on phase relations. In Eq. (3.4) let us for the sake of brevity denote the sum on the right by F_n, so that (3.4) has the form

$$\dot{a}_n = i\omega_n a_n + iF_n.$$

F_n can be separated into two components in the complex plane: one parallel and the other perpendicular to a_n. The first component changes only the phase of a_n; the second changes its magnitude. If a_n leads F_n slightly, i.e., if the phase of a_n is somewhat larger than that of F_n, then a_n will increase in magnitude with time.

Consider now the zeroth-order solution, corresponding to $\epsilon=0$, as given by Eq. (3.9). In this solution, the amplitudes of all modes remain constant and equal and the phases equally spaced at all times. Let us include the phase angle Θ, which is common to all a_n, in the rotation of the coordinate system. For the remaining phase angle θ_n we have $\theta_n/(t-\tau) = \omega_0 + n\Omega$, and the system can be

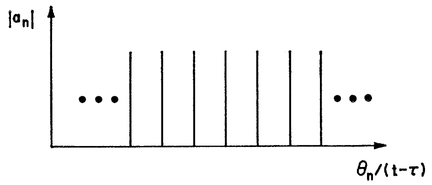

FIG. 6. Amplitudes and phases in system of equally spaced oscillators following a single pulse at $t=\tau$.

displayed in diagram form as a set of equally spaced vertical bars (Fig. 6) whose height represents the mode amplitude, while the phase angle is represented in terms of the abscissa $\theta_n/(t-\tau)$. Since the amplitudes in this solution remain constant, F_n must always remain parallel to a_n (assuming again a positive q; for negative q it would be antiparallel). Suppose that one oscillator—say, the nth—is perturbed by shifting the phase forward relative to its position of symmetry in the ensemble. Then a_n will lead F_n and proceed to grow in amplitude. The energy for this process must come, of course, from the neighboring oscillators with which it interacts.

Suppose, on the other hand, that the perturbation consists of a slight increase in the *amplitude* $|a_n|$. If we assume that the self-interaction term $C_n a_n a_n a_n^*$ predominates in the nonlinear interaction, then this change in amplitude will produce an increase in the frequency and a forward drift of the oscillator phase relative to the ensemble.

We have in this case two reinforcing processes:

(a) An oscillator whose amplitude is larger than the average of the neighboring modes with which it interacts drifts forward in phase relative to the neighborhood.

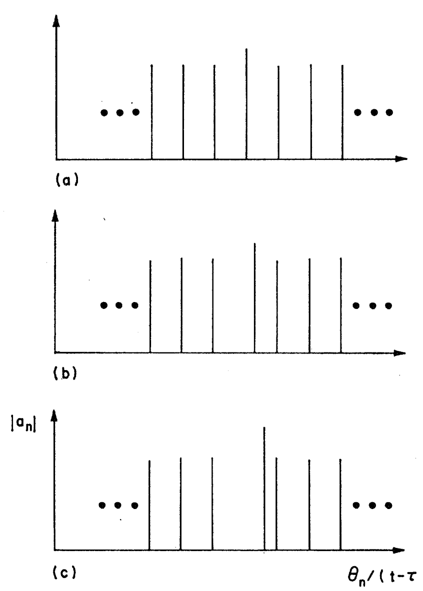

FIG. 7. Unstable growth of perturbation. (a) Single oscillator is perturbed by increasing its amplitude relative to ensemble. (b) Oscillator phase drifts forward relative to ensemble. (c) Thereupon the amplitude increases further.

(b) An oscillator whose phase is in advance of the average for this neighborhood will acquire energy from neighboring modes and its amplitude will grow.

The resulting unstable process, which is a combination of amplitude growth and forward phase drift, is illustrated in Fig. 7. The opposite processes, namely, amplitude decrease and backward phase drift, are similarly linked by positive feedback.

The extension of these considerations to the periodic perturbation of the ensemble associated with pulse 1 is straightforward, but again the behavior is critically dependent on the regime of $\sigma\tau$. In order for process (a) to occur, the predominant interaction of a given oscillator must be with other oscillators whose amplitudes and phases are relatively close to its own. On the other hand, in order for process (b) to occur it must also interact with oscillators whose behavior differs from its own, since otherwise the required phase shift between the oscillator and its interaction neighborhood will not take place. In other words, if $\sigma\tau$ is very large, process (a) will not take place, and if $\sigma\tau$ is very small, process (b) will not take place. This is described graphically in Fig. 8.

The operation of the instability and the manner in which it leads to amplified echoes are illustrated in Fig. 9.

In the above discussion we have assumed $q>0$, i.e., a predominantly positive nonlinear interaction. For $q<0$, the phase drifts occur in the opposite direction, but the outcome is otherwise unaffected.

D. Unstable Behavior Following a Single Pulse

The equally spaced, equally excited system displayed in Fig. 6 is unstable with respect to a variety of perturbations. These need not arise exclusively from the prior excitation by pulse 1. When a single strong pulse is incident on the medium, some deviation from uniformity is present because of thermal noise and because

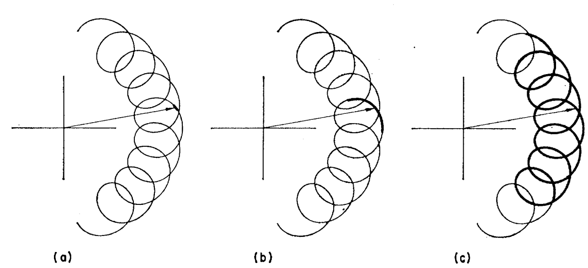

FIG. 8. Regimes of intermode interaction. Epicycloids represent evolution of system shortly after $t=\tau$. Arrow represents a particular oscillator and heavy line its interaction range. (a) Oscillator interacts only with identical oscillators and there is no energy transfer. (b) Oscillator interacts primarily with oscillators whose average phases are smaller than its own and is amplified at their expense. (c) Oscillator interacts with many periods of the disturbance. The interaction becomes phase-mixed, and no echoes arise.

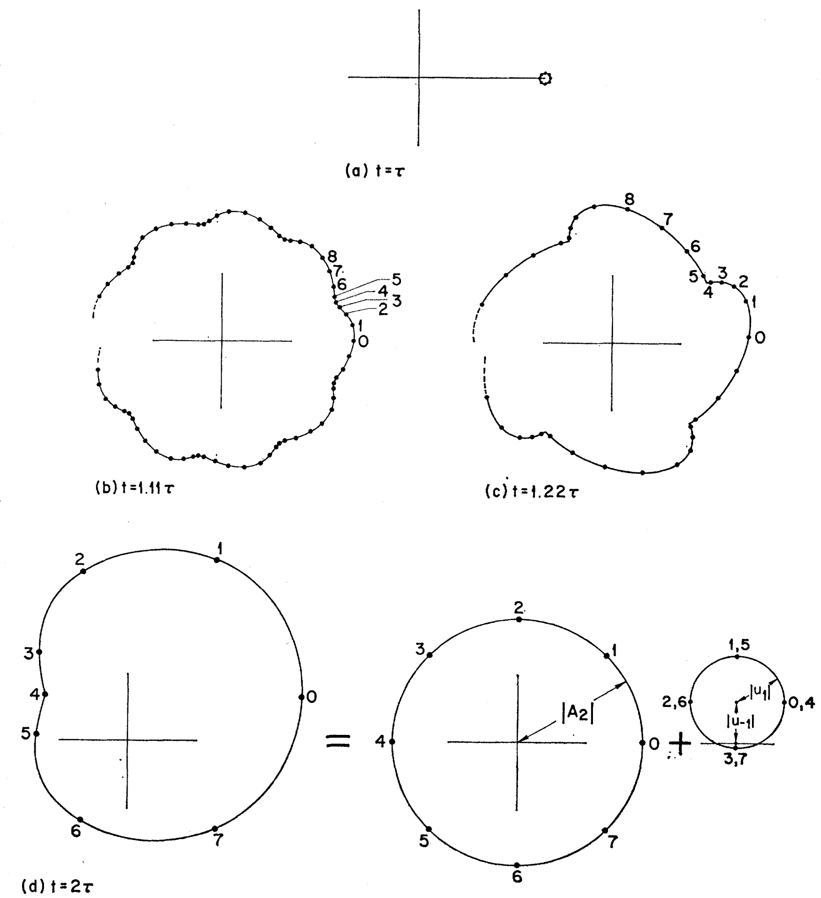

FIG. 9. Evolution of system of coupled oscillators in the amplification regime. (a) Locus given by displaced circle corresponding to very small pulse 1. (b) and (c) Distorted epicycloids show combined effect of phase drift and amplitude change. (d) Final figure, like Fig. 3(e), equals superposition of two circles, but the second circle is here displaced upward from the origin and is much larger than the initial circle in (a).

of the uneven coupling of the radiation to different modes. The perturbation can be Fourier-expanded in terms of the oscillator frequency ω. Fourier components of period $P(\omega)$, for which $P(\omega)\sim\sigma$, are strongly amplified and produce echo spikes at $\tau=1/P(\omega)$. Under conditions of high amplification one may therefore observe a noise pattern consisting of sharp spikes of width $\sim 1/\Delta\omega$ with an envelope corresponding approximately to the gain curve for the amplified two-pulse echo.

IV. CONCLUSION

The collective-mode model developed in the preceding sections accounts well for all the main features of the amplified-echo phenomenon. It also throws light on the more fundamental question posed at the outset, namely, whether echoes may be expected to occur as generally in systems of coupled collective oscillation modes as they do in systems of isolated oscillators. The answer to this question is by and large in the negative. If we consider the set of modes which couple strongly through the nonlinear interaction to a particular mode as a sort of "oscillator packet," then the condition for echo formation appears to be that the phases, and therefore also the amplitudes, in such a packet do not deviate excessively from each other during the interval between pulses. To some extent the packet must retain the characteristic properties of a single oscillator.

The echo process thus remains confined to a regime near the limit corresponding to isolated particles. While small excursions from this limit result in great enhancement of the effect, large excursions result in its disappearance.

APPENDIX A: ECHOES IN A UNIFORM MEDIUM

Echo phenomena have been observed by and large in nonuniform media. Since uniform media support a large number of stationary solutions, in the form of plane waves, these could presumably play the role of oscillators in an echo model. It appears, however, that, in general, echoes will not be generated in such a configuration through a mechanism ordinarily associated with a single stationary particle. This statement may be phrased more precisely as follows.

Consider a uniform medium comprising individual, localized (but coupled) particles whose *a priori* dynamic behavior depends on their position alone (i.e., not on their drift velocity or other possible parameters). Let the nonlinearity arise entirely in the motion of the *individual* particle, i.e., be completely local in character. Then, *in general,* there will be no echo.

The statement can be put in the form of a complementarity principle. The nonlinear interaction is totally localized in ordinary or **r** space. The modes are localized in **k** or "momentum" space. The nonlinearity therefore does not produce effects specific to each particular mode as required for echo formation.

Generalizations with respect to nonlinear systems are often dangerous, since artificial exceptions can almost always be constructed. We therefore claim no more than that under a variety of reasonable conditions the above statement holds. We shall attempt no general proof but will study a number of examples.

One-Dimensional Conservative System

Let $a(x)$ be given by

$$a(x)=\sum_k a_k e^{-ikx}. \tag{A1}$$

The nonlinear interaction part of the Hamiltonian is in the form

$$\mathcal{H}_{\text{nonlinear}}=\int \mathcal{H}(x)dx,$$

where $\mathcal{H}(x)$ is a power series in $a(x)$ and $a^*(x)$.

Following the procedure of Sec. III A, Eq. (A1) is substituted in $\mathcal{H}(x)$ and the equations of motion are obtained from (3.1). Again one need retain only terms of the form $a_{k1}a_{k2}a_{k3}^*$, $a_{m1}a_{m2}a_{m3}a_{m4}^*a_{m5}^*$, etc., whose first-order time dependence falls near the frequency range of the manifold. One then has

$$\dot{a}_k=i\omega_k a_k+i\alpha\sum a_{k1}a_{k2}a_{k3}^* +i\beta\sum a_{m1}a_{m2}a_{m3}a_{m4}^*a_{m5}^*+\cdots, \tag{A2}$$

where α and β are constants and where the summations are subject to the respective constraints $k_1+k_2-k_3=k$ and $m_1+m_2+m_3-m_4-m_4=k$. Note that the nonlinear terms couple all modes on an equal basis irrespective of the differences in their frequencies. Adding all of Eqs. (A2), one obtains for the total moment $\mu=\sum a_k$

$$\dot{\mu}=i\sum\omega_k a_k+\alpha|\mu|^2\mu+\beta|\mu|^4\mu+\cdots. \tag{A3}$$

In an ideal echo regime, μ decays to zero shortly after the incidence of the second pulse. We know that the first term on the right side of (A3), i.e., the linear term, cannot by itself lead to the appearance of an echo. The other terms, however, vanish to a high order with μ. Thus, once μ has become negligible, it can never increase again. Hence, there can be no echo.

Collision Mechanism in a Plasma

It is assumed that the reader is familiar with the cyclotron echo process associated with a velocity-dependent collision frequency.[14] We consider the operation of this process in a uniform plasma.

The current density $j(x)$ can be expanded in terms of plane waves:

$$j(x)=\sum a_k e^{-ikx}. \tag{A4}$$

Conversely,

$$a_k=(1/N)\sum j(x)e^{ikx}dx, \tag{A5}$$

where N is an appropriate normalization factor. $j(x)$ is given by $env_a(x)$, where e is the electron charge, n the electron number density, and $v_a(x)$ the *average* (or drift) electron velocity at x. The velocity v of an individual electron is given by

$$a=v_a(x)+v_R, \tag{A6}$$

where v_R is a random component resulting from collisions.

The collision frequency ν is given as a function of $|v|$ and can be expanded as a power series in v^2. We shall consider only the lowest terms and put $\nu=\nu_0+\alpha v^2$, since higher terms lead to similar results. By summing over all random velocities, one finds that the rate of change of $j(x)$ due to collisions is given by

$$dj/dt=\{\nu_0+\alpha[v_a^2+(5/3)\langle v_R^2\rangle_{\text{av}}]\}j, \tag{A7}$$

where $\langle v_R^2\rangle_{\text{av}}$ is the mean square of v_R. It can now be shown that except for small fluctuation over times of the order $1/\Delta\omega$, $\langle v_R^2\rangle_{\text{av}}$ is independent of x (that is, unless the excitation by the pulses is such as to introduce immediately a strong nonuniformity in $\langle v_R^2\rangle_{\text{av}}$, in which case the medium can no longer be regarded as homogeneous). Putting $v_a=(1/en)j$ in (A7) and using (A5), one obtains

$$\dot{a}_k=[i\omega_k+\nu_0+(5/3)\alpha\langle v_R^2\rangle_{\text{av}}]a_k+(\alpha/ne^2)\sum a_{kl}a_{k2}a_{k3}^*, \tag{A8}$$

where the summation is subject to the condition $k_1+k_2-k_3=k$. Like (A2) above, these equations are added to give $\dot{\mu}$, and identical arguments show that no echo is generated.

The above examples apply to cases where the nonlinearity is strictly localized and arises from the dynamics of individual particles, e.g., individual spins, or individual rotating electrons. Equations very similar to Eq. (A2) apply also to nonlinear dipolar interactions in spin systems. Somewhat more complex nonlinear terms appear in the exchange interaction in a ferrimagnet, but since mode coupling is essentially unrelated to the frequency difference, no echoes in a uniform medium are to be expected from this interaction either. One is led to conclude that field inhomogeneity is required for echoes in a ferrimagnet.

On the other hand, echoes *can* occur in a uniform medium whenever *a priori* properties are not defined by particle position alone. For example, photon echoes can be observed in a uniform gas because of the distribution of Doppler frequencies. In this case, oscillators are defined according to their position in velocity space rather than ordinary space, and all of the above considerations must be modified accordingly.

APPENDIX B: ELEMENTARY DESCRIPTION OF ECHOES

A naïve, totally nonphysical and nonrigorous, and yet convenient description of echoes is based on the manner in which the echoes arise out of various heterodyne or mixing products of the disturbances associated with individual incident pulses. The approach is restricted to systems of isolated particles. Let us start, for the purpose of illustration, with the usual two-pulse sequence and study the response of a single oscillator at frequency ω. Let the pulses produce, respectively, the excitations

$$S_1 = a_1 e^{i\omega t}, \quad S_2 = a_2 e^{i\omega(t-\tau)}.$$

In the linear approximation the combined disturbance is $S_1 + S_2$. In the nonlinear case a driving term must be included which is some nonlinear function of S_1 and S_2. This function is expanded in a power series in S_1, S_2 and their complex conjugates S_1^*, S_2^*. One need consider only products whose time dependence is at the fundamental, e.g., $S_1^2 S_2^*$. The first product of interest is

$$S_1^* S_2^2 = a_1^* a_2^2 e^{i\omega(t-2\tau)}.$$

When integrated over the ensemble, this term vanishes because of phase mixing, except near $t = 2\tau$, corresponding to the principal two-pulse echo. The product $S_1^2 S_2^*$, on the other hand, equals $a_1^2 q_2^* e^{i\omega(t+\tau)}$ and represents a disturbance corresponding to a "virtual echo" at $t = -\tau$, which, of course, is not observable. An echo at $t = 2\tau$ can also be caused by higher-order terms $S_1 S_1^{*2} S_2^2$ $S_1^* S_2^3 S_2^*$, etc. The term $S_1^{*2} S_2^3 = a_1^{*2} a_2^3 e^{i\omega(t-3\tau)}$ yields an echo at $t = 3\tau$, and similarly $S_1^{*n-1} S_2^n$ corresponds to an echo at $t = n\tau$.

A third pulse, at $t = T$, adds a disturbance $S_3 = a_3 e^{i\omega(t-T)}$. The combination $S_1^* S_2 S_3 = a_1^* a_2 a_3 e^{i\omega(t-\tau-T)}$ represents the principal three-pulse echo at $t = T + \tau$. More generally, $S_1^{*n+m-1} S_2^m S_3^n$ corresponds to an echo at $t = nT + m\tau$.

In some cases, e.g., the photon echo, incident pulses are in the form of plane waves. For two incident pulses one has $S_1 = a_1 \exp[i(\omega t - \mathbf{k}_1 \cdot \mathbf{r})]$ and $S_2 = a_2 \times \exp\{i[\omega(t-\tau) - \mathbf{k}_2 \cdot \mathbf{r}]\}$. The echo term $S_1^* S_2^2$ is

$$a_1^* a_2^2 \exp\{i[\omega(t-2\tau) - (2\mathbf{k}_2 - \mathbf{k}_1) \cdot \mathbf{r}]\}.$$

If the pulses are incident along different directions, then $\mathbf{k}_2 \neq \mathbf{k}_1$, and the echo wave vector is given by $\mathbf{k}_e = 2\mathbf{k}_2 - \mathbf{k}_1$. A condition for strong echoes is that $\mathbf{k}_e$ must equal the wave vector of a naturally propagating wave at ω.

One can also describe in this manner more general types of echoes in which the disturbances is impressed not on oscillators but on carriers moving at various velocities, e.g., streaming electrons in a plasma.[19] For example, "spatial echoes" result when two cw signals at ω_1 and ω_2 are impressed at two positions along the stream, at $z = 0$ and $z = d$, respectively. The disturbances transmitted by the set of carriers of velocity v are, respectively,

$$S_1 = a_1 \exp[i\omega_1(t - z/v)]$$

and

$$S_2 = a_2 \exp\{i\omega_2[t - (z-d)/v]\}.$$

The combination $S_1^* S_2$ gives

$$S_1^* S_2 = a_1^* a_2 \exp\{i[(\omega_2 - \omega_1)t + (\omega_1 z - \omega_2 z + \omega_2 d)/v]\}.$$

When integrated over all v, this expression vanishes for all z except for z_e, given by

$$z_e/d = \omega_2/(\omega_2 - \omega_1).$$

At z_e, an echo signal at $\omega_e = \omega_2 - \omega_1$ is obtained.

From the fact that echoes arise in connection with certain products of S_1 and S_2, it is tempting to draw conclusions concerning the dependence of echo amplitudes on incident pulse amplitudes. For example, since the simple two-pulse echo at $t = 2\tau$ arises from $S_1^* S_2^2$, one might conclude that at very low pulse levels the echo varies linearly with pulse 1 and quadratically with pulse 2. However, the products in question represent not the echoes themselves but only driving terms resulting in echoes. The relations among amplitudes can be represented in this manner only in the limit of very small τ.

[19] R. W. Gould, T. M. O'Neil, and J. H. Malmberg, Phys. Rev. Letters **19**, 219 (1967); T. M. O'Neil and J. H. Malberg, Phys. Fluids **11**, 134 (1968).