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Validity of the Semiclassical Approximation in Maser Theory

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A simple model of a maser or laser consisting of a single-mode field coupled to N identical two-level atoms is considered. The semiclassical approximation is equivalent to neglecting the statistical correlations between the atoms and the field. The accuracy of this approximation is investigated by writing equations of motion for the correlations. To obtain a complete, self-consistent set of equations, it is necessary to include correlations of the field with itself (related to the coherence of the field) and correlations between different atoms. Relaxations and an energy source are introduced phenomenologically into the equations of motion. They are then solved for the case of steady-state oscillation. It is found that the correlations are smaller than the terms kept in the semiclassical theory by the order of 1 over the number of photons present in the field, larger if thermal photons abound, but smaller if the field relaxation is dominant. Expressions are also found for the amplitude and phase fluctuations of the field. The latter yield a linewidth for the maser oscillator in agreement with earlier calculations, but obtained by a different method.

I. INTRODUCTION

MOST successful theories[1–8] of masers and lasers to date have been semiclassical; that is, they treat the electromagnetic field classically while retaining a quantum-mechanical description of the radiating atoms. Some theories[9–12] have started with a quantized radiation field and then made the approximation of neglecting correlations[13] between the atoms and the field. This brings them back to a semiclassical theory. The purpose of the present work is to discover how big the atom-field correlations are and thereby determine the validity of semiclassical theories.

One's first guess might be that quantum-mechanical corrections to the semiclassical theory would be smaller by a factor of 1 over the number of photons present, and this is at least partly right. However, the two existing papers[11,14] which attack this problem have obtained different results. Both assume the density matrix to consist of a term which factors into a product of a density matrix for the field and one for the atoms (this term gives the semiclassical theory), plus a term containing atom-field correlations. Willis[14] assumes an expansion in powers of $\tilde{\gamma}N\tilde{\gamma}$, where $\tilde{\gamma}$ is essentially the atom-field coupling parameter divided by the frequency and N is the number of active atoms present. He claims that the correlations are smaller by a factor $(\tilde{\gamma}N\tilde{\gamma})^2$, but his choice of expansion parameter is not adequately justified, nor does he include relaxations. Weidlich and Haake[11] include relaxations, but assume the field relaxation is dominant to find the correlations smaller by a factor of 1 over the number of atoms present. Both papers are, at best, incomplete. Other authors who have used a quantized field have sought only statistical properties of the field and have not looked at the correlation question.

Since the calculations in the remainder of the paper are somewhat involved, we shall describe here, in words, what we have done. Our model of a maser is as simple as possible, but still contains the features of interest. It consists of a single-mode radiation field confined to a resonant cavity. This field interacts with a large number N of identical two-level atoms. The atoms are so confined that they all see essentially the same amplitude of the field, so that spatial coordinates are not needed. We do not consider atomic motion, inhomogeneous broadening, multimode operation, or other such phenomena often required for a complete description of a real laser.[8]

To minimize working with operators which may not commute, we use expectation values of the operators as our basic variables. We sum the expectation values over all the atoms present to obtain macroscopic variables which describe the average behavior of the atoms without involving us in complexities arising from the high degree of degeneracy[14] in the system. The equations of motion are obtained by taking the expectation value of the Heisenberg equations for the operators. The nonlinearity of the problem brings second moments or second-order correlations into the equations for the first moments. Neglecting the correlations gives the semiclassical theory. To evaluate the second-order

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[1] K. Shimoda, T. C. Wang, and C. H. Townes, Phys. Rev. **102**, 1308 (1956).
[2] N. G. Basov and A. M. Prokhorov, Zh. Eksperim. i Teor. Fiz. **30**, 560 (1956) [English transl.: Soviet Phys.—JETP **3**, 426 (1956)].
[3] A. N. Oraevskii, Radiotekhnika i Elektronica **4**, 718 (1959) [English transl.: Radio Engineering and Electronics **4**, 228 (1959)].
[4] H. Statz and G. deMars, in *Quantum Electronics*, edited by C. H. Townes (Columbia University Press, New York, 1960), p. 530.
[5] D. Kleppner, H. M. Goldenberg, and N. F. Ramsey, Phys. Rev. **126**, 603 (1962).
[6] L. W. Davis, Proc. IEEE **51**, 76 (1963).
[7] C. L. Tang, J. Appl. Phys. **34**, 2935 (1963).
[8] W. E. Lamb, Phys. Rev. **134**, A1429 (1964).
[9] E. T. Jaynes and F. W. Cummings, Proc. IEEE **51**, 89 (1963).
[10] E. R. Buley and F. W. Cummings, Phys. Rev. **134**, A1454 (1964).
[11] W. Weidlich and F. Haake, Z. Physik **186**, 203 (1965).
[12] J. H. Shirley, Am. J. Phys. **36**, 949 (1968).
[13] We use the word "correlation" in the statistical sense: The correlation between two fluctuating variables x and y is $\langle xy\rangle - \langle x\rangle\langle y\rangle$, where the angular brackets denote a statistical (ensemble) average.
[14] C. R. Willis, J. Math. Phys. **6**, 1984 (1965).

correlations we write equations of motion for them, again using the Heisenberg equations. Of course, these equations then contain third-order correlations. Since we expect the second-order correlations to be small corrections in the equations for the first moments, we also expect that the third-order correlations will be small corrections in the equations for the second-order correlations. Hence we truncate our system of equations by neglecting the third-order correlations.

We now have a system of nonlinear differential equations for the first and second moments. To make the model realistic, we introduce relaxation and pumping into these equations phenomenologically. We then look for a solution describing steady-state oscillation. If we find the correlations small in this steady state, it is likely that they are also small in a dynamic situation approaching the steady state and our objective is achieved.

When the procedure outlined above was carried out, several interesting results were obtained. The equations of motion for the atom-field correlations involve, also, field-field correlations and correlations between different atoms. To obtain a complete set, equations must be written for these other types of correlations as well. The field-field correlations, which are related to the coherence of the field, are driven only by the atom-field correlations. Hence semiclassical theories neglecting the latter will not contain incoherence generated, for example, by spontaneous emission from the atoms. The necessity of including atom-atom correlations has not been recognized by previous authors. Both Willis[14] and Weidlich and Haake[11] omit them by assumption.

Once a complete set of equations was found and solved, the correlations turned out to be smaller than the product of the first-order moments by a factor of the order of the square of the coupling parameter divided by the product of two relaxation rates. If atomic relaxation is dominant, as it is in most solid-state devices, the factor is of the order of 1 over the number of photons present. However, if the field relaxation is dominant, as it is in the hydrogen maser, the factor is of the order of 1 over the number of atoms present. In either case the semiclassical approximation is extremely good except when very close to threshold.

Among the correlation variables are those correlating a variable with itself. These can be interpreted as a measure of the fluctuations of that variable. Hence, as a byproduct of our work, we have obtained expressions for the fluctuations in the phase and amplitude of the field. The phase fluctuations in turn give us information about the coherence of the field generated by the atoms.

II. DEVELOPMENT OF EQUATIONS OF MOTION

The Hamiltonian for our maser model is

$$\mathcal{H}=\hbar[\omega a^\dagger a+\sum_j \tfrac{1}{2}\omega\sigma_j{}^z+\sum_j b(\sigma_j{}^+a+\sigma_j{}^-a^\dagger)]. \qquad (1)$$

The first term represents the Hamiltonian for the radiation field, which we treat as a single quantum-mechanical harmonic oscillator of angular frequency ω. The creation and annihilation operators, $a^\dagger$ and a, obey $[a,a^\dagger]=1$. The second term represents the Hamiltonian of N identical atoms indexed by j. For each j the σ matrices can be represented by

$$\sigma^z=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\quad \sigma^+=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix},\quad \sigma^-=\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}.$$

The remainder of the Hamiltonian represents the interaction between the atoms and the field. We have omitted nonresonant terms like $\sigma^+a^\dagger$ (rotating-field approximation). The coupling parameter b is assumed real. We have also assumed that the cavity mode for the field is exactly tuned to the resonance frequency of the atoms. This assumption considerably simplifies the algebra without, we believe, losing any features of physical interest for the present investigation.

From the Hamiltonian (1) we find the following Heisenberg equations of motion for the operators:

$$\begin{aligned} i\dot{a}&=\omega a+b\sum_j \sigma_j{}^-,\\ i\dot{\sigma}_j{}^-&=\omega\sigma_j{}^- -b\sigma_j{}^z a, \qquad (2)\\ i\dot{\sigma}_j{}^z&=2b(\sigma_j{}^+a-\sigma_j{}^-a^\dagger).\end{aligned}$$

We now introduce the expectation values of these operators, summed over all the atoms:

$$\begin{aligned} A&=\langle a\rangle e^{i\omega t},\\ M&=\sum_j\langle\sigma_j{}^-\rangle e^{i\omega t}, \qquad (3)\\ W&=\sum_j\langle\sigma_j{}^z\rangle.\end{aligned}$$

We have also factored out the gross time dependence. Physically, A can then be interpreted as the complex amplitude of the field, so scaled that A^*A is the mean photon number, M is proportional to the complex amplitude of the magnetization or electric polarization acquired by the atoms, and W is the population difference between the two levels. We refer to A, M, and W as macroscopic variables. Their equations of motion are obtained from the expectation values of the Heisenberg equations (2). The difficulty is that the resulting expectation values of operator products like $\langle\sigma^z a\rangle$ cannot in general be factored into a product of macroscopic variables.

It is convenient at this stage to define what we shall call residual operators:

$$\begin{aligned}\delta a &= a e^{i\omega t} - A\,,\\ \delta m &= \sum_j \sigma_j^- e^{i\omega t} - M\,,\\ \delta w &= \sum_j \sigma_j^z - W\,.\end{aligned} \tag{4}$$

These operators have zero expectation value, but carry the commutation properties of the original operators, for example,

$$[\delta m,\delta w] = 2\sum_j \sigma_j^- e^{i\omega t} = 2(M+\delta m)\,. \tag{5}$$

We can now express the expectation value of operator products by the product of the expectation values plus a correlation or the expectation value of the product of two of the residual operators. The equations of motion for the macroscopic variables then become

$$\begin{aligned}\dot{A} &= -ibM\,,\\ \dot{M} &= ib(WA+\langle\delta w\delta a\rangle)\,,\\ \dot{W} &= -2ib(M^*A+\langle\delta m^\dagger\delta a\rangle)+\text{c.c.}\,,\end{aligned} \tag{6}$$

where c.c. stands for complex conjugate. Subtracting these equations from (2), we obtain the equations of motion for the residual operators:

$$\begin{aligned}\delta\dot{a} &= -ib\delta m\,,\\ \delta\dot{m} &= ib(W\delta a+A\delta w+\delta w\delta a-\langle\delta w\delta a\rangle)\,,\\ \delta\dot{w} &= -2ib(M^*\delta a+A\delta m^\dagger+\delta m^\dagger\delta a-\langle\delta m^\dagger\delta a\rangle)+\text{H.c.}\,,\end{aligned} \tag{7}$$

where H.c. stands for Hermitian conjugate.

Were it not for the correlation terms, Eqs. (6) would be a complete set of equations for the macroscopic variables. In semiclassical theories the correlation terms are dropped.[12] Our task is to evaluate the correlation terms by finding and solving equations for them.

For notational convenience we introduce italic capital letters for our correlation variables. In a complete, self-consistent set of equations, correlations corresponding to all possible pairs of residual operators occur. For the atom-field correlations we define

$$D = \langle\delta w\delta a\rangle\,, \tag{8a}$$

$$E = \langle\delta m\delta a^\dagger\rangle\,, \tag{8b}$$

$$F = \langle\delta m\delta a\rangle\,. \tag{8c}$$

For the field-field correlations we define

$$G = \langle\delta a^\dagger\delta a\rangle\,, \tag{8d}$$

$$R = \langle\delta a\delta a\rangle\,. \tag{8e}$$

The variable G gives us information about the coherence of the radiation field. For any state of the field $|\psi\rangle$, G is the normalization of the state $\delta a|\psi\rangle$, and hence is positive or zero. G can be zero if and only if $\delta a|\psi\rangle$ is zero, that is, if $|\psi\rangle$ is an eigenstate of the annihilation operator or a pure coherent state in the sense of Glauber.[15] A nonzero value of G gives an indication of how far the state of the field departs from pure coherence. The total number of photons present is $n\equiv\langle a^\dagger a\rangle = A^*A+G$. We can think of A^*A as representing the coherent part of the field, and G the incoherent part.

For the atom-atom correlations we define

$$C = \langle\delta m\delta m^\dagger\rangle - \tfrac{1}{2}(N-W)+J\,, \tag{8f}$$

$$B = \langle\delta w\delta m\rangle + M + K\,, \tag{8g}$$

$$H = \langle\delta m\delta m\rangle + L\,, \tag{8h}$$

$$V = \langle\delta w\delta w\rangle - N + U\,, \tag{8i}$$

where the auxiliary variables J, K, L, and U are defined by

$$\begin{aligned}J &= \sum_j |\langle\sigma_j^-\rangle|^2\,,\\ K &= \sum_j \langle\sigma_j^z\rangle\langle\sigma_j^-\rangle e^{i\omega t}\,,\\ L &= \sum_j \langle\sigma_j^-\rangle^2 e^{2i\omega t}\,,\\ U &= \sum_j \langle\sigma_j^z\rangle^2\,.\end{aligned} \tag{9}$$

In the double sum over the atoms represented by $\langle\delta m\delta m^\dagger\rangle$ there are terms representing not only the correlation between different atoms, but also the correlation of an atom with itself. We know that an atom will always be correlated with itself, so it seemed appropriate to exclude such terms from our basic variables. The peculiar looking definitions (8f)–(8i) were chosen to represent only correlations between different atoms. Thus

$$C = \sum_{j\neq k} \langle(\sigma_j^- - \langle\sigma_j^-\rangle)(\sigma_k^+ - \langle\sigma_k^+\rangle)\rangle\,,$$

where the sum is over all pairs of different atoms. Similar expressions hold for B, H, and V. These definitions then lead to the appearance of the auxiliary variables. One might think that the auxiliary variables could be expressed in terms of the macroscopic variables, e.g., $J = M^*M/N$. Sometimes this is true, but there are cases where it is not. We have chosen to retain a separate notation for the auxiliary variables. They are sort of a nuisance since they are required for quantitative completeness but have no effect on the qualitative results to be obtained.

Now that we have a sufficient arsenal of notation, we are at last ready to derive equations of motion for the correlation variables. We shall not work out all nine, but only one of each type to illustrate the features encountered and the approximations made. The basic

[15] R. J. Glauber, Phys. Rev. **131**, 2766 (1963).

technique is to expand the derivative of a correlation with the aid of Eqs. (7) for the residual operators, and then reexpress the resulting correlations in terms of the definitions (8). For G we thus have, without approximation,

$$\begin{aligned}\dot{G}&=\langle\delta\dot{a}^\dagger\delta a\rangle+\langle\delta a^\dagger\delta\dot{a}\rangle\\&=\langle(ib\delta m^\dagger)\delta a\rangle+\langle\delta a^\dagger(-ib\delta m)\rangle\\&=ib(E^*-E).\end{aligned}$$

Note that changes in G are derived solely from the imaginary part of E. Thus if the field is initially coherent ($G=0$), incoherence will develop only if there are atom-field correlations present. This result depends only on the linear form of the Heisenberg equation for the field, as shown by Paul.[16]

For the atom-field correlation E we find

$$\begin{aligned}\dot{E}&=\langle\delta m(ib\delta m^\dagger)\rangle+\langle ib(W\delta a+A\delta w+\delta w\delta a-D)\delta a^\dagger\rangle\\&=ib[C+\tfrac{1}{2}(N-W)-J]+ibW(G+1)+ibAD^*\\&\qquad+ib\langle\delta w\delta a\delta a^\dagger\rangle,\end{aligned}$$

where we have used the commutator of δa and $\delta a^\dagger$. Here we see the appearance of the atom-atom and field-field correlation variables on the right, explaining why we must fuss with them. We also see a third-order correlation term. If we wrote equations for third-order correlations, we would find fourth-order correlations, and so on. We must truncate the equations somewhere. We choose to neglect the third-order correlation[17] relative to the second-order correlation times a macroscopic variable. This is equivalent to writing

$$\langle xyz\rangle\approx\langle x\rangle\langle y\rangle\langle z\rangle+\langle x\rangle\langle\delta y\delta z\rangle+\langle y\rangle\langle\delta x\delta z\rangle+\langle z\rangle\langle\delta x\delta y\rangle$$

or

$$\langle xyz\rangle\approx\langle x\rangle\langle yz\rangle+\langle y\rangle\langle xz\rangle+\langle z\rangle\langle xy\rangle-2\langle x\rangle\langle y\rangle\langle z\rangle,$$

where x, y, and z stand for any of our operators. Neither Willis nor Weidlich and Haake used all the terms on the right when they met products of three operators.

After this basic approximation the E equation becomes

$$\dot{E}=ib(C+AD^*+WG+\tfrac{1}{2}N+\tfrac{1}{2}W-J).$$

Note that if there are initially no correlations, polarization, or fields present, we still have on the right the term $\frac{1}{2}(N+W)$ which equals the number of atoms in the upper state. This term, which arises from the commutation of the atomic operators, is the source of spontaneous emission in our theory. It drives E, which in turn drives G, thus creating incoherent radiation. The E equation is the only one containing this source.

For the atom-atom correlations, let us look at the equation for C:

$$\dot{C}=\langle\delta\dot{m}\delta m^\dagger\rangle+\langle\delta m\delta\dot{m}^\dagger\rangle-\tfrac{1}{2}\dot{N}+\tfrac{1}{2}\dot{W}+\dot{J}.$$

In addition to the derivative of δm given in (7), we need the time derivatives of N, W, and J. In our simple model we can assume N is constant. We get an expression for $\dot{W}$ from (6). We will neglect the correlation term in (6), since it is expected to be small, and we do not need an accurate value of W to find a lowest-order value for C. The reader who keeps the correlation term from (6) will find that we have in effect assumed $W\gg1$. Similarly we can neglect correlations in the equation of motion for J, finding from (2) and (9) that

$$\dot{J}=ib(AK^*-A^*K).$$

So far we then have

$$\begin{aligned}\dot{C}=ib(W\langle\delta a\delta m^\dagger\rangle&+A\langle\delta w\delta m^\dagger\rangle\\&+\langle\delta w\delta a\delta m^\dagger\rangle-M^*A+K^*A)+\text{c.c.}\end{aligned}$$

To reduce $\langle\delta w\delta m^\dagger\rangle$ to the definition (8g) we must use the commutation relation (5):

$$\begin{aligned}\langle\delta w\delta m^\dagger\rangle&=\langle\delta m\delta w\rangle^*=\langle\delta w\delta m+[\delta m,\delta w]\rangle^*\\&=(B-M-K)^*+2\langle M+\delta m\rangle^*=(B+M-K)^*.\end{aligned}$$

The M and K terms from the definition of B then cancel those from $\dot{W}$ and $\dot{J}$. Finally we neglect the third-order correlations to wind up with

$$\dot{C}=ib(WE^*+AB^*-WE-A^*B).$$

The next step is the introduction of relaxation terms. Several authors have studied relaxation from a quantum-mechanical viewpoint.[11,18–22] All we shall do here is adopt their conclusions, that is, add damping and equilibrium terms to our equations of motion. We ignore the stochastic driving terms, which simulate the fluctuations associated with dissipation mechanisms, since they average to zero and our equations are only for average values. For the field we introduce the damping constant $\beta\equiv\omega/2Q$, where Q is the (loaded) quality factor of the resonant cavity. For the atoms we introduce the damping constants γ_1 and γ_2 for W and M, respectively. These gammas correspond to the reciprocals of the T_1 and T_2 relaxation times in magnetic resonance theory.[23] We also add to the W equation a pumping term I to supply the energy lost to relaxation mechanisms. In the absence of interaction ($b=0$), A and M will relax to zero, but W will in general relax to a nonzero value which we call I/γ_1.

[16] H. Paul, Ann. Physik **16**, 403 (1965).

[17] This is the same as neglecting the third-order cumulants, which represent that portion of an average not expressible in terms of lower-order cumulants. Assuming a Gaussian distribution for our variables would give similar results since a Gaussian has no cumulants higher than second order. Cumulants in physics have been discussed by R. Kubo, J. Phys. Soc. Japan **17**, 1100 (1962).

[18] W. Weidlich and G. Haake, Z. Physik **185**, 30 (1965).

[19] W. H. Louisell, *Radiation and Noise in Quantum Electronics* (McGraw-Hill Book Co., New York, 1964).

[20] R. K. Wangsness and F. Bloch, Phys. Rev. **89**, 728 (1953).

[21] I. R. Senitzky, Phys. Rev. **119**, 670 (1960); **131**, 2827 (1963).

[22] M. Lax, Phys. Rev. **145**, 110 (1966).

[23] A. Abragam, *The Principles of Nuclear Magnetism* (Clarendon Press, Oxford, England, 1961).

For the relaxation of the correlations we assume that the field and atomic relaxation mechanisms are independent, and further that the relaxation mechanisms for each atom are independent. We can then simply sum the relaxation rates for the two variables to find the relaxation rates for their correlations. Note that if we had left self-correlations in the definitions of the atom-atom correlations, they would relax differently. In the absence of interaction, all the correlations should relax to zero, since we assume that the relaxations do not generate any correlations. The one exception is the autocorrelation G, which will relax to the value $\bar{n} = [\exp(\hbar\omega/kT)-1]^{-1}$, where T is the temperature of the cavity, since the cavity losses generate incoherent thermal photons.

For the auxiliary variables we distinguish between strong and weak atomic-relaxation mechanisms. Weak relaxation is the end result of an extended series of small nudges to an atom by the relaxation mechanism, as, for example, collisions which only slightly disturb the phase of the radiating atom. For weak relaxation $\langle\sigma_j\rangle^2$ will relax twice as fast as $\langle\sigma_j\rangle$. Strong relaxation is the average result of catastrophic events to individual atoms; for example, a collision so hard that the atom completely forgets what state it was in, or in a hydrogen maser,[5] the atom escapes from the storage bulb. If the catastrophic events are randomly distributed in time, we can represent strong relaxation by replacing a sum over a large number of atoms by an integral over the times at which the atoms appear and are exposed to the strong relaxation mechanism:

$$\sum_j \langle\sigma_j\rangle \to \int_{-\infty}^{t} \langle\sigma(t-t_j)\rangle I_0 e^{-\gamma_0(t-t_j)} dt_j .$$

Here I_0 represents the rate of appearance of new atoms, and the exponential is the probability that the atom survives that long without a catastrophe. When differentiated it gives a relaxation term with the relaxation rate γ_0. The problem with the auxiliary variables is that they contain a product of $\langle\sigma\rangle$'s but only one sum over the atoms. In the relaxation of the auxiliary variables the weak relaxation rate is doubled, but the strong relaxation is not. Let γ_0 be the strong relaxation rate, presumably the same for σ^z and $\sigma^\pm$; and let γ_1', γ_2' be the weak relaxation rates. Then, for example, M relaxes at the rate $\gamma_2=\gamma_0+\gamma_2'$, whereas J relaxes at the rate $\gamma_0+2\gamma_2'$. The source terms for σ^z must also be distinguished. We let $\sigma_0=\sigma^z(0)$ represent the average state of the atoms supplied by the pump and σ_1 the average state after the weak relaxation γ_1'. Then the source term I in the W equation is made up of $I_0\sigma_0$ from the strong relaxation and $N\gamma_1'\sigma_1$ from the weak relaxation. Similarly, U will have the source $I_0\sigma_0^2+2W\gamma_1'\sigma_1$.

This completes the development of our equations of motion. As a reference point we now list all of these equations. For the macroscopic variables including the correlation corrections we have

$$\dot{A}=-\beta A-ibM, \tag{10a}$$

$$\dot{M}=-\gamma_2 M+ibWA+ibD, \tag{10b}$$

$$\dot{W}=I-\gamma_1 W-2ib(M^*A-MA^*)+2ib(E-E^*). \tag{10c}$$

For the correlation variables we have

$$\dot{G}=2\beta\bar{n}-2\beta G-ib(E-E^*), \tag{11a}$$

$$\dot{R}=-2\beta R-2ibF, \tag{11b}$$

$$\dot{D}=-(\beta+\gamma_1)D+ib(-2AE^*+2A^*F -2M^*R+2MG-B+M+K), \tag{11c}$$

$$\dot{E}=-(\beta+\gamma_2)E+ib(C+AD^*+WG +\tfrac{1}{2}N+\tfrac{1}{2}W-J), \tag{11d}$$

$$\dot{F}=-(\beta+\gamma_2)F+ib(-H+AD+WR+L), \tag{11e}$$

$$\dot{B}=-(\gamma_1+\gamma_2)B+ib(2ME-2M^*F +2A^*H-2AC+WD+AV), \tag{11f}$$

$$\dot{C}=-2\gamma_2 C+ib(WE^*+AB^*-WE-A^*B), \tag{11g}$$

$$\dot{H}=-2\gamma_2 H+2ib(WF+AB), \tag{11h}$$

$$\dot{V}=-2\gamma_1 V+4ib(MD^*+A^*B-M^*D-AB^*). \tag{11i}$$

For the auxiliary variables without correlation corrections we have

$$\dot{J}=-(\gamma_0+2\gamma_2')J+ib(AK^*-A^*K), \tag{12a}$$

$$\dot{K}=\gamma_1'\sigma_1 M-(\gamma_0+\gamma_1'+\gamma_2')K +ib(-2AJ+2A^*L+AU), \tag{12b}$$

$$\dot{L}=-(\gamma_0+2\gamma_2')L+2ibAK, \tag{12c}$$

$$\dot{U}=I_0\sigma_0^2+2W\gamma_1'\sigma_1-(\gamma_0+2\gamma_1')U -4ib(AK^*-A^*K). \tag{12d}$$

Gordon[24] has recently attacked the laser problem by expanding the density matrix as some operator elements times a weight function P, which depends on c-number variables. He then finds an equation of motion for the weight function in a Fokker-Planck form. Moments of the weight function correspond to expectation values of the operators. By computing moments from Gordon's Eq. (3.9), one can reproduce Eqs. (10) and (11) above, except that in Gordon's model the auxiliary variables are neglected. It was particularly gratifying to find that the relaxation model used by Gordon (attributed to Lax[22]) for the density matrix gives just the source and relaxation terms which we introduced in Eqs. (10) and (11).

III. SOLUTIONS OF EQUATIONS

The solution of the macroscopic equations (10) without the correlations has been discussed in our previous publication[12] and will not be reiterated here. We merely

[24] J. P. Gordon, Phys. Rev. **161**, 367 (1967).

note the solution for steady-state oscillation:

$$M=i(\beta/b)A\,, \tag{13a}$$

$$W=\beta\gamma_2/b^2, \tag{13b}$$

$$A^*A=I/4\beta-\gamma_1\gamma_2/4b^2. \tag{13c}$$

The phase of A is undetermined. If we let $r=|A|$ and define the relative pumping parameter $z=I/I_{\text{th}}$, where $I_{\text{th}}=\beta\gamma_1\gamma_2/b^2$, then (13c) becomes

$$r^2=(\gamma_1\gamma_2/4b^2)(z-1)\,, \tag{13d}$$

with the threshold for oscillation at $z=1$.

The steady-state solution of Eqs. (12) for the auxiliary variables is somewhat messy for the general relaxation rates given. We here contrast two extreme cases to illustrate the difference. If the relaxation is entirely weak, we find a steady-state value

$$J=\frac{\beta\gamma_1\sigma_1}{4b^2}\frac{(z-1)}{z}=\frac{|M|^2}{N}.$$

But if the relaxation is entirely strong, we get

$$J=\frac{\beta\gamma_0}{2b^2}\frac{z(z-1)}{(4z-3)}=\frac{2z^2}{(4z-3)}\frac{|M|^2}{N},$$

which has a quite different dependence on the pumping, but the same order of magnitude otherwise. In the following we shall carry the auxiliary variables as J, K, etc., except that we will use the result $\mathrm{Re}L=-J$, which is independent of the relaxations.

We now face the equations for the correlations. If the correlations are negligible in Eqs. (10), then we can use the steady-state solutions (13) to evaluate A, M, and W in Eqs. (11). We then have nine coupled linear equations for the correlations. Of the nine correlation variables, G, C, and V are real and the rest complex. In terms of real variables there are 15 coupled equations. But by taking real and imaginary parts in (11) and forming some simple linear combinations, we can separate the 15 equations into two sets of six and one set of three.

$$(d/dt)\,\mathrm{Im}R=-2\beta\,\mathrm{Im}R-2b\,\mathrm{Re}F\,, \tag{14a}$$

$$(d/dt)\,\mathrm{Im}D=-(\beta+\gamma_1)\,\mathrm{Im}D-2br\,\mathrm{Re}E+2br\,\mathrm{Re}F-2\beta r\,\mathrm{Im}R-b\,\mathrm{Re}B, \tag{14b}$$

$$(d/dt)\,\mathrm{Re}E=-(\beta+\gamma_2)\,\mathrm{Re}E+br\,\mathrm{Im}D\,, \tag{14c}$$

$$(d/dt)\,\mathrm{Re}F=-(\beta+\gamma_2)\,\mathrm{Re}F-br\,\mathrm{Im}D-(\beta\gamma_2/b)\,\mathrm{Im}R+b\,\mathrm{Im}H\,, \tag{14d}$$

$$(d/dt)\,\mathrm{Re}B=-(\gamma_1+\gamma_2)\,\mathrm{Re}B-2\beta r\,\mathrm{Re}E-2\beta r\,\mathrm{Re}F-2br\,\mathrm{Im}H-(\beta\gamma_2/b)\,\mathrm{Im}D\,, \tag{14e}$$

$$(d/dt)\,\mathrm{Im}H=-2\gamma_2\,\mathrm{Im}H+2(\beta\gamma_2/b)\,\mathrm{Re}F+2br\,\mathrm{Re}B\,; \tag{14f}$$

$$(d/dt)(G+\mathrm{Re}R)=2\beta\bar{n}-2\beta(G+\mathrm{Re}R)+2b\,\mathrm{Im}(E+F)\,, \tag{14g}$$

$$(d/dt)\,\mathrm{Re}D=-(\beta+\gamma_1)\,\mathrm{Re}D-2br\,\mathrm{Im}(E+F)-2\beta r(G+\mathrm{Re}R)+b\,\mathrm{Im}(B-M-K)\,, \tag{14h}$$

$$(d/dt)\,\mathrm{Im}(E+F)=-(\beta+\gamma_2)\,\mathrm{Im}(E+F)+(\beta\gamma_2/b)(G+\mathrm{Re}R)+2br\,\mathrm{Re}D+b(C-\mathrm{Re}H)+b(\tfrac{1}{2}N+\tfrac{1}{2}W-J+\mathrm{Re}L)\,, \tag{14i}$$

$$(d/dt)\,\mathrm{Im}B=-(\gamma_1+\gamma_2)\,\mathrm{Im}B-2\beta r\,\mathrm{Im}(E+F)-2br(C-\mathrm{Re}H)+(\beta\gamma_2/b)\,\mathrm{Re}D+brV\,, \tag{14j}$$

$$(d/dt)(C-\mathrm{Re}H)=-2\gamma_2(C-\mathrm{Re}H)+2(\beta\gamma_2/b)\times\mathrm{Im}(E+F)+4br\,\mathrm{Im}B\,, \tag{14k}$$

$$(d/dt)V=-2\gamma_1V-8\beta r\,\mathrm{Re}D-8br\,\mathrm{Im}B\,; \tag{14l}$$

$$(d/dt)(G-\mathrm{Re}R)=2\beta\bar{n}-2\beta(G-\mathrm{Re}R)+2b\,\mathrm{Im}(E-F)\,, \tag{14m}$$

$$(d/dt)\,\mathrm{Im}(E-F)=-(\beta+\gamma_2)\,\mathrm{Im}(E-F)+(\beta\gamma_2/b)\times(G-\mathrm{Re}R)+b(C+\mathrm{Re}H)+b(\tfrac{1}{2}N+\tfrac{1}{2}W-J-\mathrm{Re}L)\,, \tag{14n}$$

$$(d/dt)(C+\mathrm{Re}H)=-2\gamma_2(C+\mathrm{Re}H)+2(\beta\gamma_2/b)\,\mathrm{Im}(E-F)\,. \tag{14o}$$

We have chosen the arbitrary phase of A to vanish so that $A=r$.

Equations (14a)–(14f) form a set completely decoupled from the other nine. Since they contain no inhomogeneous terms, we can assume that they are initially zero and then they will remain zero. This trivial disposition of six of the variables does not occur if the cavity is off tune.

Equations (14g)–(14l) also form an independent set. Since they do contain inhomogeneous terms, they will have a nontrivial steady-state solution. This can be found by setting the derivatives to zero and systematically eliminating variables among the six equations. The algebra is simplified by working with normalized correlation variables, or what statisticians call correlation coefficients of the form $\langle\delta x\delta y\rangle/\langle x\rangle\langle y\rangle$.

We define

$$g=(G+\mathrm{Re}R)/r^2, \tag{15a}$$

$$d=(\mathrm{Re}D)/Wr\,, \tag{15b}$$

$$f=\mathrm{Im}(E+F)/\mathrm{Im}Mr\,, \tag{15c}$$

$$k=\mathrm{Im}B/\mathrm{Im}MW\,, \tag{15d}$$

$$h=(C-\mathrm{Re}H)/|M|^2, \tag{15e}$$

$$v=V/W^2. \tag{15f}$$

Using (13a) and (13b) for M and W, we substitute these definitions into (14g)–(14l), set the derivatives to zero, and find the steady-state equations:

$$g-f=u, \tag{16a}$$

$$(\beta+\gamma_2)f-\gamma_2 g-2\gamma_2 d-\beta h=\gamma_2 p, \tag{16b}$$

$$(\beta+\gamma_1)d+\tfrac{1}{2}\gamma_1(z-1)(f+g)-\beta k=-\beta q, \tag{16c}$$

$$h-f-2k=0, \tag{16d}$$

$$(\gamma_1+\gamma_2)k+\tfrac{1}{2}\gamma_1(z-1)(f+h)-\gamma_2 d-\gamma_2 v=0, \tag{16e}$$

$$v+(z-1)(d+k)=0, \tag{16f}$$

where the inhomogeneous terms are defined by

$$p=[\tfrac{1}{2}(\eta z+1)-2(J/W)]/r^2, \tag{17a}$$

$$q=[1+b(\mathrm{Im}K)/\beta r]/W, \tag{17b}$$

$$u=\bar{n}/r^2. \tag{17c}$$

We have used $N=(\beta\gamma_2/b^2)\eta z$, where the factor

$$\eta=(\gamma_0+\gamma_1')/(\gamma_0\sigma_0+\gamma_1'\sigma_1)$$

is of the order of unity.

Considering the relaxation rates to be all of the same order of magnitude and the pumping parameter z to be of the order of unity, we can see immediately from Eqs. (16) that the normalized correlations are all of the same order of magnitude as the inhomogeneous terms p, q, and u. The ratio J/W is less than or of the order of unity, so that p is of the order of $1/r^2$, or, from (13d), $b^2/\gamma_1\gamma_2$. Similarly the ImK term in q is of the order of unity so that q is of the order of $1/W$ or $b^2/\beta\gamma_2$. Of course, u is just the ratio of thermal energy to coherent energy in the cavity. As a generality we can therefore say that the correlation coefficients are of the order of the square of the coupling parameter divided by the product of two relaxation rates. For typical systems this ratio may be 10^{-8} or smaller. Small values of the correlation coefficients mean we can neglect the correlations in the macroscopic equations (10b) and (10c), which is what we set out to demonstrate.

To actually solve Eqs. (16) we use (16a), (16d), and (16f) to remove g, h, and v from the other three equations, leaving

$$-2\gamma_2 d-2\beta k=\gamma_2 p+\gamma_2 u, \tag{18a}$$

$$(\beta+\gamma_1)d+\gamma_1(z-1)f-\beta k=-\beta q-\tfrac{1}{2}\gamma_1(z-1)u, \tag{18b}$$

$$(\gamma_1+\gamma_2)zk+\gamma_1(z-1)f+\gamma_2(z-2)d=0. \tag{18c}$$

These have been solved in a straightforward manner, but the solutions are rather lengthy and no more enlightening than the qualitative remarks just made. For example, if we insert the solution for f into (16a), we find

$$g=(\Gamma_1 p+\Gamma_2 q+\Gamma_3 u)/\Gamma_4(z-1), \tag{19}$$

where

$$\Gamma_1=\gamma_2[\beta\gamma_1 z+2\beta\gamma_2(z-1)+\gamma_1(\gamma_1+\gamma_2)z],$$
$$\Gamma_2=2\beta\gamma_2[\beta(z-2)-(\gamma_1+\gamma_2)z],$$
$$\Gamma_3=2\beta^2\gamma_1(z-1)+2\beta\gamma_1^2(z-1)-\beta\gamma_1\gamma_2(z^2-6z+4)+2\beta\gamma_2^2(z-1)+\gamma_1\gamma_2(\gamma_1+\gamma_2)z^2,$$
$$\Gamma_4=2\gamma_1[\beta^2+\beta\gamma_1-\beta\gamma_2(z-3)+\gamma_2(\gamma_1+\gamma_2)z].$$

However, in most maser systems the relaxation rates are widely different. We have previously shown[12] how this fact could be used to simplify the dynamic macroscopic equations (10) for two cases. We now consider these same two cases for the solution of Eqs. (18).

In solid-state lasers we usually have $\gamma_2\gg\beta$, γ_1; that is, the polarization relaxes much faster than the field or population difference. If the correlation coefficients are all the same order of magnitude, we can neglect the β term in (18a) and obtain immediately

$$d=-\tfrac{1}{2}(p+u).$$

We specialize even further by noting that, for the laser case, J/W is the order of M^2/W^2 or γ_1/γ_2, i.e., small compared to the first term in p. Also, at optical frequencies $\bar{n}\ll 1$, so that u is small compared to p. Hence the dominant contributor to driving the correlations is the first term of p, which is traceable back to the terms in (11d), which we earlier identified as the source of spontaneous emission. For the laser case we thus have

$$d=-\tfrac{1}{4}(\eta z+1)/r^2,$$

and similarly, from (18c),

$$k=\tfrac{1}{4}(\eta z+1)(z-2)/zr^2.$$

The other correlation coefficients are also of the order of 1 over the number of photons present, which for lasers well above threshold may run from 10^8 to as high as 10^{15}.

Now that we know the answer, we can try to argue why it should be expected. For our laser case, the polarization relaxes faster than the field, or, in other words, the field persists after the polarization M or correlation E driving it has decayed. The polarization and correlation present is only that which is immediately driven by the field present. Hence the correlations should bear the same relation to the polarization as do the amounts of incoherent and coherent field driving them. But the ratio of spontaneous (incoherent) to stimulated (coherent in our case) emission is well known to be 1 over the number of photons present.[25] At best this is an order-of-magnitude argument which cannot reveal the complex dependence on the pumping parameter z.

In the case of a hydrogen maser[5] we have $\beta\gg\gamma_1$, γ_2; that is, the field relaxes much faster than the atomic

[25] L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Co., New York, 1955), p. 399.

variables. If we neglect the $\gamma_2 d$ terms in (18a), we immediately obtain

$$k=-\tfrac{1}{2}(\gamma_2/\beta)(p+u),$$

and, from (18b),

$$d=-q-\tfrac{1}{2}(\gamma_1/\beta)(z-1)u-\tfrac{1}{2}(\gamma_2/\beta)(p+u).$$

Since p is of the order $1/r^2$ or $b^2/\gamma_1\gamma_2$, then $(\gamma_2/\beta)p$ is of the order of $b^2/\beta\gamma_1$ or $1/W$. Hence the correlation coefficients are of the order of 1 over the number of active atoms present, a qualitatively different result from the laser case, but agreeing with Weidlich and Haake,[11] who also assume this case.

We can argue for this result too by noting that it is now the atomic polarization which persists, while the field rapidly damps to that value immediately generated by the polarization present. Hence the correlation coefficients should be determined from the atomic side. Now atom-atom correlations occur only through the mediation of the field, since we assume the atoms are not directly coupled. An atom is correlated with that part of the field that it itself has generated. A second atom will become correlated with the first atom by interacting with the field generated by the first atom. But only $1/N$ of the field seen by the second atom has been generated by the first, where N is the number of radiating atoms. Hence the correlation coefficients should be of the order of $1/N$.

Actually, for the hydrogen maser, the thermal photon number $\bar{n}$ is about 4500, so that the thermal noise term u dominates the correlations generated by the radiation process or by stimulated emission. For typical operating conditions the correlation coefficients are about 10^{-8}.

The remaining three correlation equations (14m)–(14o) were put aside since they contain a new feature. If we set the derivatives to zero and try to solve for their steady state, we soon come to a contradiction: Equations (14m)–(14o) do not have a steady-state solution. To find what kind of solution they do have, we can form the following linear combinations:

$$P=G-\mathrm{Re}R-(b/\beta)\,\mathrm{Im}(E-F), \quad (20a)$$

$$Q=G-\mathrm{Re}R-2(b/\beta)\,\mathrm{Im}(E-F)+(b/\beta)^2(C+\mathrm{Re}H). \quad (20b)$$

The physical significance of these variables will be elucidated in Sec. IV. In terms of P and Q, Eqs. (14m)–(14o) become

$$(d/dt)(G-\mathrm{Re}R)=-2\beta P+2\beta\bar{n}, \quad (21a)$$

$$(d/dt)P=-(\beta+\gamma_2)P-\beta Q+2\beta\bar{n}-\tfrac{1}{2}\gamma_2(\eta z+1), \quad (21b)$$

$$(d/dt)Q=-2(\beta+\gamma_2)Q+2\beta\bar{n}-\gamma_2(\eta z+1). \quad (21c)$$

We can now find steady-state solutions of (21b) and (21c).

$$P=[2\beta(\beta+2\gamma_2)\bar{n}-\gamma_2^2(\eta z+1)]/2(\beta+\gamma_2)^2,$$
$$Q=[2\beta\bar{n}-\gamma_2(\eta z+1)]/2(\beta+\gamma_2).$$

Normalizing P and Q by dividing by r^2, we obtain the same kind of results that we found for the other correlation coefficients. However, when we put the steady-state value of P into (21a), we find

$$G-\mathrm{Re}R=\beta\gamma_2^2(\beta+\gamma_2)^{-2}(2\bar{n}+\eta z+1)t. \quad (22)$$

Although we have managed to confine this linear increase with time to a single variable, it actually permeates the entire system, making all our solutions pseudo-steady states. From (20a), ImE will contain this increase and hence will eventually become significant in Eq. (10c), upsetting our solutions (13) for the macroscopic variables and invalidating the main thesis of this paper—that the correlations are negligible. To find what happens then, Eqs. (10)–(12) must all be solved simultaneously. We leave this problem for a future paper.

A physical understanding of Eq. (22) appears in Sec. IV. In the meantime, if we divide it by r^2, we obtain the rate at which $G-\mathrm{Re}R$ grows to a significant level, namely, a relaxation rate times $(2\bar{n}+\eta z+1)/r^2$. For the cases of negligible correlation considered in this paper, we have $r^2 \gg \bar{n}$, ηz. Thus $G-\mathrm{Re}R$ grows much more slowly than the rate at which our other variables approach their pseudo-steady states. We have a considerable regime in time after achievement of the pseudo-steady state and before it is appreciably changed by the growth of $G-\mathrm{Re}R$. For lasers it may take anywhere from 1 to 10^6 sec for the correlations to become significant. For the hydrogen maser it takes about a week.

IV. CLASSICAL INTERPRETATION OF CORRELATIONS

In our previous study[12] of the macroscopic equations we introduced separate variables for the phases and amplitudes of the field and polarization:

$$A=re^{i\theta},$$
$$M=i\mu e^{i(\theta-\varphi)}.$$

Here r, μ, θ, and φ are real and have the steady-state solutions on tune

$$r^2=(\gamma_1\gamma_2/4b^2)(z-1),$$
$$\mu=(\beta/b)r,$$
$$\varphi=0,$$
$$\theta \text{ arbitrary}.$$

If we choose zero for the arbitrary value of θ and then consider small fluctuations of A and M, we would relate them to fluctuations in r, μ, φ, and θ by

$$\delta a = \delta r + ir\delta\theta\,, \tag{23a}$$

$$\delta m = i\delta\mu - \mu\delta\theta + \mu\delta\varphi\,. \tag{23b}$$

Instead, however, we shall use Eqs. (23) to define new Hermitian operators δr, $\delta\mu$, $\delta\varphi$, and $\delta\theta$ from the residual operators δa and δm. Then Eqs. (23) are exact and unambiguous since they are merely a decomposition into Hermitian and anti-Hermitian parts plus removal of some scalar factors. Of course, $\delta\theta$ cannot then be strictly interpreted as phase, but we thereby avoid the problem of trying to define a quantum-mechanical phase operator.[26] We shall continue to think of the new operators as classically representing fluctuations of amplitude and phase, but we must remember that it is meaningful to do so only if the fluctuations are small. The four new operators together with δw obey the commutation relations

$$\begin{aligned}
&[\delta r,\delta\theta] = [\delta r,\delta\varphi] = i/2r\,,\\
&[\delta\mu,\delta\varphi] = (i/2\mu)(W+\delta w)\,,\\
&[\delta w,\delta\varphi] = -(2i/\mu)(\mathrm{Im}M+\delta\mu)\,,\\
&[\delta w,\delta\mu] = 2i\,\mathrm{Re}M - 2i\mu(\delta\theta-\delta\varphi)\,.
\end{aligned}$$

All other commutators vanish.

Using Eqs. (23) in the definitions (8), we can express our correlation variables in terms of these Hermitian residual operators. Solving for the expectation value of product pairs of the new operators, we then obtain the following list; for noncommuting pairs we have written the symmetrized form (curly brackets denote the anticommutator):

$$\langle\delta r\delta r\rangle = \tfrac{1}{2}(G+\mathrm{Re}R+\tfrac{1}{2})\,, \tag{24a}$$

$$\langle\delta r\delta\mu\rangle = \tfrac{1}{2}\,\mathrm{Im}(E+F)\,, \tag{24b}$$

$$\langle\delta r\delta w\rangle = \mathrm{Re}D\,, \tag{24c}$$

$$\langle\delta\mu\delta\mu\rangle = \tfrac{1}{2}(C-\mathrm{Re}H)+\tfrac{1}{4}N-\tfrac{1}{2}J+\tfrac{1}{2}\,\mathrm{Re}L\,, \tag{24d}$$

$$\tfrac{1}{2}\langle\{\delta\mu,\delta w\}\rangle = \mathrm{Im}B-\mathrm{Im}K\,, \tag{24e}$$

$$\langle\delta w\delta w\rangle = V+N-U\,, \tag{24f}$$

$$\tfrac{1}{2}\langle\{\delta r,\delta\theta\}\rangle = (2r)^{-1}\,\mathrm{Im}R\,, \tag{24g}$$

$$\langle\delta\mu\delta\theta\rangle = (2r)^{-1}\,\mathrm{Re}(E-F)\,, \tag{24h}$$

$$\langle\delta w\delta\theta\rangle = r^{-1}\,\mathrm{Im}D\,, \tag{24i}$$

$$\tfrac{1}{2}\langle\{\delta r,\delta\varphi\}\rangle = (2\mu)^{-1}\,\mathrm{Re}(E+F)+(2r)^{-1}\,\mathrm{Im}R\,, \tag{24j}$$

$$\tfrac{1}{2}\langle\{\delta\mu,\delta\varphi\}\rangle = (2\mu)^{-1}\,\mathrm{Im}H+(2r)^{-1}\,\mathrm{Re}(E-F) - (2\mu)^{-1}\,\mathrm{Im}L\,, \tag{24k}$$

$$\tfrac{1}{2}\langle\{\delta w,\delta\varphi\}\rangle = \mu^{-1}\,\mathrm{Re}B+r^{-1}\,\mathrm{Im}D-\mu^{-1}\,\mathrm{Re}K\,, \tag{24l}$$

$$\langle\delta\theta\delta\theta\rangle = (2r^2)^{-1}(G-\mathrm{Re}R+\tfrac{1}{2})\,, \tag{24m}$$

$$\begin{aligned}
\langle\delta\theta\delta\varphi\rangle &= (2r^2)^{-1}(G-\mathrm{Re}R+\tfrac{1}{2}) - (2\mu r)^{-1}\,\mathrm{Im}(E-F)\\
&= (2r^2)^{-1}(P+\tfrac{1}{2})\,,
\end{aligned} \tag{24n}$$

$$\begin{aligned}
\langle\delta\varphi\delta\varphi\rangle &= (2r^2)^{-1}(G-\mathrm{Re}R+\tfrac{1}{2})-(\mu r)^{-1}\,\mathrm{Im}(E-F)\\
&\quad+(2\mu^2)^{-1}(C+\mathrm{Re}H)+(4\mu^2)^{-1}N - (2\mu^2)^{-1}(J+\mathrm{Re}L)\\
&= (2r^2)^{-1}(Q+\tfrac{1}{2})+(4\mu^2)^{-1} \times(N-2J-2\,\mathrm{Re}L)\,.
\end{aligned} \tag{24o}$$

We can now restate the results of Sec. III. We first found from Eqs. (14a)–(14f) that ImR, ImD, ReE, ReF, ReB, and ImH were all zero. Comparing with Eqs. (24g)–(24l), the classical interpretation is that the phase fluctuations are not correlated with the amplitude fluctuations. This is true only when the cavity is exactly tuned to the atomic resonance frequency. Similarly the correlations we found from Eqs. (14g)–(14l) are interpreted as amplitude correlations (24b), (24c), and (24e) or amplitude fluctuations (24a), (24d), and (24f). Thus the relative amplitude fluctuations of the maser field are given by [see (15a)]

$$\langle\delta r\delta r\rangle/r^2 = \tfrac{1}{2}g+\tfrac{1}{4}r^{-2}\,.$$

For a pure coherent field, g vanishes, and hence the second term above represents the unavoidable fluctuations due to the quantum-mechanical nature of the field (vacuum fluctuations). The correlation coefficient g, given in Eq. (19), contains the additional fluctuations due to the interaction with the atoms.

The combinations P and Q defined by (20), and used to solve (14m)–(14o) are now seen to be related to the phase correlation and the fluctuations of the relative phase φ, respectively (24n) and (24o). They are small like the other correlations. The culprit which grows with time is the phase fluctuations of the field:

$$\langle\delta\theta\delta\theta\rangle = \beta\gamma_2{}^2(\beta+\gamma_2)^{-2}(\bar{n}+\tfrac{1}{2}\eta z+\tfrac{1}{2})t/r^2+\tfrac{1}{4}r^{-2}\,. \tag{25}$$

Such behavior, analogous to Brownian motion,[27] is well known in the noise theory of self-oscillators and is often loosely referred to as a random walk of phase. In our earlier paper[12] we derived the $\bar{n}$ term due to thermal noise of the cavity by introducing a classical Langevin noise source into the macroscopic equation (10a). In (25) we also have the spontaneous-emission contribution [proportional to $\tfrac{1}{2}(\eta z+1)$] and the vacuum fluctuations. The latter do not increase with time, and so are soon swamped by the other contributions.

Equation (25) tells us that the time-increasing solution found in Eq. (22) is not as bad as we first thought. It is just the phase that is getting muddy; the maser still continues to oscillate with the same amplitude. When $\langle\delta\theta\delta\theta\rangle$ becomes comparable to unity, our solutions will be changed since $\langle\delta\theta\delta\theta\rangle$ contributes to (10c). Also, the interpretation of $\delta\theta$ as phase will become

[26] L. Susskind and J. Glogower, Physics 1, 49 (1964).

[27] M. C. Wang and G. E. Uhlenbeck, Rev. Mod. Phys. 17, 323 (1945).

questionable. Still, increasing incoherence is the correct interpretation. Of course, if we measure the phase of a maser, we will always find that it has one. The meaning of Eq. (25) is that if we measure the phases of an ensemble of independent, but identical masers now, and then remeasure them later, they will have wandered away from each other.

If, as others have done, we assume the phase fluctuations have a Gaussian distribution, then the autocorrelation function for the field becomes $\langle e^{i\delta\theta}\rangle = \exp(-\frac{1}{2}\langle\delta\theta\delta\theta\rangle)$ and the maser output has a Lorentzian line shape with a half-width at half-power of

$$\Delta\omega = \langle\delta\theta\delta\theta\rangle/2t = \beta\gamma_2{}^2(\beta+\gamma_2)^{-2}(\bar{n}+\tfrac{1}{2}\eta z+\tfrac{1}{2})/2r^2. \quad (26)$$

This linewidth agrees with that calculated by Haken[28] and by Lax,[29] both of whom used quantum-mechanical Langevin noise sources in the Heisenberg equations of motion. For the laser case of large γ_2, the linewidth has also been calculated by Sauermann[30] and by Scully and Lamb.[31] However, there seems to be some disagreement on the interpretation of the individual terms in the linewidth. Lax and Sauermann call $\frac{1}{2}\eta z$ the spontaneous-emission term. The $\frac{1}{2}$, which in their work came with the $\bar{n}$, was considered due to the vacuum fluctuations of the field. We interpret $\frac{1}{2}\eta z+\frac{1}{2}$ together as the spontaneous-emission part, since it is proportional to the steady-state number of atoms in the upper state. Our vacuum fluctuations would be the extra term in (25) which does not contribute to the linewidth. In any event, we have a derivation for the phase fluctuations which is independent of the assumptions used in the Langevin technique.

V. SUMMARY AND DISCUSSION

We have shown that the semiclassical approximation in the theory of a simple model for masers or lasers is very good whenever the photon number generated by stimulated emission is large. We have also shown how corrections to the semiclassical theory can be found quantitatively by calculating correlations or second moments. These same calculations give expressions for quantities interpretable as phase and amplitude fluctuations when the semiclassical approximation is good.

The Langevin noise source theories of Haken[28] and of Lax,[29] which were successfully used to calculate the phase fluctuations, could also be used to calculate the correlations if sufficient labor were expended in that direction. With suitable assumptions about the nature of the noise sources, the Langevin technique also gives the spectral distribution of fluctuations, whereas our moment method gives only the total fluctuations. Another approach[11,24,32] has been the equation of motion for the density matrix, with terms added to simulate dissipation. This equation then resembles the Fokker-Planck equation of statistical mechanics. If it can be solved, it gives the complete statistical distribution of the variables, but so far headway has been made only with the case $\gamma_1,\gamma_2 \gg \beta$.[24,31] Our second-moment equations can be readily deduced from the density-matrix equation. Both the Langevin and Fokker-Planck methods thus can yield more information than our approach. However, the second moments are adequate for the purposes of this paper, require fewer assumptions, and can be solved for arbitrary relaxations.

Correlations in the case of a maser amplifier have also been investigated. For a linear amplifier the E and MA terms in (10c) are dropped as part of a small signal approximation. The resulting solution of the macroscopic equations (10) for the amplifier has been given previously.[12] The correlations are found from Eqs. (11) just as for the maser oscillator, but with less work. The result is that the normalized correlation d is of the order of $(\bar{n}+1)/W$. Hence the correlation term ibD in Eq. (10b) can be neglected. The problem of increasing phase fluctuations does not occur, since the phase of the maser amplifier is determined by the phase of the signal that it is amplifying.

[28] H. Haken, Z. Physik **190**, 327 (1966).

[29] M. Lax, in *Physics of Quantum Electronics*, edited by P. L. Kelley, B. Lax, and P. E. Tannenwald (McGraw-Hill Book Co., New York, 1966), pp. 735–747.

[30] H. Sauermann, Z. Physik **189**, 312 (1966).

[31] M. O. Scully and W. E. Lamb, Phys. Rev. **159**, 208 (1967).

[32] W. Weidlich, H. Risken, and H. Haken, Z. Physik **204**, 223 (1967); H. Haken, H. Risken, and W. Weidlich, *ibid.* **206**, 355 (1967).