Lindhard Dielectric Functions with a Finite Electron Lifetime*

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The finite-electron-lifetime expressions for the longitudinal and transverse dielectric functions of a free-electron gas obtained by Lindhard in the self-consistent-field approximation are examined. It is shown that these results are incorrect, and appropriate expressions are developed for the case where the system can be characterized by a single relaxation time.

THE free-electron-gas dielectric response functions, both transverse and longitudinal, were first derived by Lindhard[1] in the random-phase or self-consistent-field (SCF) approximation. This calculation was done in detail for the electron lifetime $\tau=\infty$, but a scheme for incorporating a finite τ was indicated. It is the purpose of the present paper both to demonstrate that the finite-τ generalizations are incorrect and to develop correct expressions. We consider only the case where the system can be characterized by a single τ (as in s-wave scattering), and do not treat more complex scattering situations.

TRANSVERSE DIELECTRIC FUNCTION

Since the transverse dielectric constant has been less extensively discussed in the literature than the longitudinal, we present here a brief derivation thereof, using the self-consistent-field technique of Ehrenreich and Cohen.[2]

Consider a transverse electromagnetic wave propagating in the z direction with the electric field in the x direction. We can then write the single-particle Hamiltonian for particles of charge $-e$ and mass m as

$$H=H_0+H_1, \tag{1}$$

where H_0 is the usual kinetic energy term and, to lowest order in the vector potential $\mathbf{A}$,

$$H_1=(e/mc)A_x p_x, \tag{2}$$

where c is the velocity of light, p_x is the x component of the canonical momentum, and A_x is the x component of the vector potential. Wave functions $|\mathbf{k}\rangle=V^{-1/2}\times\exp(i\mathbf{k}\cdot\mathbf{x})$, with V the volume of the system, yield

$$H_0|\mathbf{k}\rangle=\epsilon_{\mathbf{k}}|\mathbf{k}\rangle, \tag{3}$$

where $\epsilon_{\mathbf{k}}=\hbar^2k^2/2m$.

To lowest order in the perturbation, the equation of motion for the single-particle density matrix operator ρ is

$$i\hbar(\partial\rho/\partial t)=[H,\rho]\simeq[H_0,\rho_1]+[H_1,\rho_0]. \tag{4}$$

We have written

$$\rho=\rho_0+\rho_1, \tag{5}$$

where ρ_1 is the part of ρ associated with the perturbation,

$$\rho_0|\mathbf{k}\rangle=f_0(\epsilon_{\mathbf{k}})|\mathbf{k}\rangle, \tag{6}$$

and $f_0(\epsilon_{\mathbf{k}})$ is the equilibrium statistical distribution function. If we take matrix elements between the states $|\mathbf{k}\rangle$ and $|\mathbf{k}+\mathbf{q}\rangle$, Eq. (4) becomes

$$i\hbar(\partial/\partial t)\langle\mathbf{k}|\rho_1|\mathbf{k}+\mathbf{q}\rangle=(\epsilon_{\mathbf{k}}-\epsilon_{\mathbf{k}+\mathbf{q}})\langle\mathbf{k}|\rho_1|\mathbf{k}+\mathbf{q}\rangle+[f_0(\epsilon_{\mathbf{k}+\mathbf{q}})-f_0(\epsilon_{\mathbf{k}})]H_{\mathbf{q}}(t), \tag{7}$$

where

$$H_{\mathbf{q}}(t)\equiv\langle\mathbf{k}|H_1|\mathbf{k}+\mathbf{q}\rangle. \tag{8}$$

We now assume that the perturbation has a time dependence $\exp[-i(\omega+is)t]$, where, in the usual case, $s\to 0$ serves to switch the perturbation adiabatically on in the remote past. With this time dependence, Eq. (7) can be written

$$\langle\mathbf{k}|\rho_1|\mathbf{k}+\mathbf{q}\rangle=\frac{f_0(\epsilon_{\mathbf{k}+\mathbf{q}})-f_0(\epsilon_{\mathbf{k}})}{\epsilon_{\mathbf{k}+\mathbf{q}}-\epsilon_{\mathbf{k}}+\hbar(\omega+is)}H_{\mathbf{q}}. \tag{9}$$

Note that one factor of $(\omega+is)$ has appeared from the time derivative of the density matrix.

Since the x component of the one-particle current density operator is

$$j_{x\,\mathrm{op}}=-\tfrac{1}{2}e[\delta(\mathbf{x}-\mathbf{x}_e)v_x+v_x\delta(\mathbf{x}-\mathbf{x}_e)], \tag{10}$$

where v_x is the operator representing the x component of the velocity and $\mathbf{x}_e$ is the position operator, the current density for the system is

$$j_x=\mathrm{Tr}\{\rho j_{x\,\mathrm{op}}\}. \tag{11}$$

Now $\mathbf{v}=(\mathbf{p}+e\mathbf{A}/c)/m$ and the Fourier transform of A_x can be written

$$A_x=\sum_{\mathbf{q}}A_{\mathbf{q}}e^{-i\mathbf{q}\cdot\mathbf{x}}\delta_{q_x,0}\delta_{q_y,0}. \tag{12}$$

It is then a straightforward exercise to show that the Fourier transform $j_x(\mathbf{q})$ of the current density becomes

$$j_x(\mathbf{q})=-\left(\frac{e\hbar}{m}\right)^2\frac{1}{cV}\sum_{\mathbf{k}}k_x^2\frac{f_0(\epsilon_{\mathbf{k}+\mathbf{q}})-f_0(\epsilon_{\mathbf{k}})}{\epsilon_{\mathbf{k}+\mathbf{q}}-\epsilon_{\mathbf{k}}+\hbar(\omega+is)}\times A_{\mathbf{q}}\delta_{q_x,0}\delta_{q_y,0}-\frac{ne^2}{mc}A_{\mathbf{q}}\delta_{q_x,0}\delta_{q_y,0}, \tag{13}$$

where n is the mean electron density.

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[1] J. Lindhard, Kgl. Danske Videnskab. Selskab, Mat.-Fys. Medd. 28, No. 8 (1954).

[2] H. Ehrenreich and M. H. Cohen, Phys. Rev. 115, 786 (1959).

The Fourier-transformed electric field is given by

$$E_x(\mathbf{q})=-\frac{1}{c}\frac{\partial A_q}{\partial t},$$

and, since the time dependence of ρ_1 is also that of $\mathbf{A}$, we have

$$E_x(\mathbf{q})=i(\omega+is)A_q/c. \tag{14}$$

Note that a second factor of $(\omega+is)$ has emerged from the $\mathbf{A}$-to-$\mathbf{E}$ conversion. Since $\mathbf{q}$ has only a z component, we can use Eqs. (13) and (14), the definition of the transverse conductivity σ_t,

$$j_x(q_z)=\sigma_t(q_z,\omega)E_x(q_z), \tag{15}$$

and the definition of the transverse dielectric function ϵ_t,

$$\epsilon_t(q_z,\omega)=1+4\pi i\sigma_t(q_z,\omega)/\omega, \tag{16}$$

to write

$$\epsilon_t(q_z,\omega)=1-\frac{4\pi}{\omega(\omega+is)}\left[\left(\frac{e\hbar}{m}\right)^2\frac{1}{V}\sum_{\mathbf{k}}k_x^2 \times\frac{f_0(\epsilon_{\mathbf{k}+q_z\hat{k}})-f_0(\epsilon_{\mathbf{k}})}{\epsilon_{\mathbf{k}+q_z\hat{k}}-\epsilon_{\mathbf{k}}+\hbar(\omega+is)}+\frac{ne^2}{m}\right], \tag{17}$$

where $\hat{k}$ is a unit vector in the z direction. Allowing $\mathbf{q}$ an arbitrary direction rather than being confined to the z direction, our expression for ϵ_t becomes finally[3]

$$\epsilon_t(\mathbf{q},\omega)=1-\frac{\omega_p^2}{\omega(\omega+is)}\left[\frac{\hbar^2}{2mn}\frac{1}{V} \times\sum_{\mathbf{k}}\frac{[\mathbf{k}^2-(\mathbf{k}\cdot\mathbf{q})^2/q^2][f_0(\epsilon_{\mathbf{k}+\mathbf{q}})-f_0(\epsilon_{\mathbf{k}})]}{\epsilon_{\mathbf{k}+\mathbf{q}}-\epsilon_{\mathbf{k}}+\hbar(\omega+is)}+1\right], \tag{18}$$

where $\omega_p=(4\pi ne^2/m)^{1/2}$ is the plasma frequency.

This derivation has been based upon an interpretation of s as an infinitesimal with the $s\to 0$ limit tacitly assumed. To incorporate a finite electron lifetime we now identify s as τ^{-1} where τ is the electron lifetime. Our expression for $\epsilon_t(\mathbf{q},\omega,\tau)$ with τ finite differs from that of Lindhard[1] in that the factor $\omega(\omega+i/\tau)$ appears instead of ω^2 outside the bracket in Eq. (18). The presence of this factor is clearly warranted as is evident from the above derivation.

Writing the Fermi wave vector as k_F, the Fermi velocity as v_F, and using the notation of Lindhard,[1]

$$\begin{aligned} z&=q/2k_F,\\ \omega'&=\omega+i/\tau,\\ u'&=\omega'/qv_F, \end{aligned} \tag{19}$$

we can write, after integrating,

$$\epsilon_t(\mathbf{q},\omega,\tau)=1-(\omega_p^2/\omega\omega')f_t, \tag{20}$$

where

$$f_t=\tfrac{3}{8}(z^2+3u'^2+1)-\frac{3}{32z}\left[[1-(z-u')^2]^2\ln\left(\frac{z-u'+1}{z-u'-1}\right) +[1-(z+u')^2]^2\ln\left(\frac{z+u'+1}{z+u'-1}\right)\right]. \tag{21}$$

The correction to the transverse dielectric constant deduced above then involves the replacement of the factor ω^{-2} of Ref. 1 by the factor $(\omega\omega')^{-1}$ in the multiplier of f_t.

If we consider $q\ll k_f$ or $z\to 0$, Eq. (20) can be readily shown to reduce to the transverse dielectric function as obtained from a simple relaxation-time solution of the Boltzmann equation:

$$\epsilon_t(\mathbf{q},\omega,\tau)_B=1+\frac{\omega_p^2}{\omega\omega'}\frac{3}{2}\frac{1}{(ql')^3} \times\{ql'-[(ql')^2+1]\tan^{-1}(ql')\}, \tag{22}$$

where

$$l'=v_F\tau/(1-i\omega\tau). \tag{23}$$

It should be noted that the Boltzmann limit appears in the correct form only with the correction to the Lindhard expression. Since the $z\to 0$ limit of the self-consistent-field dielectric function corresponds to the region of validity of the Boltzmann equation result, and since the incorporation of a finite electron lifetime into the Boltzmann equation is straightforward, the fact that the corrected ϵ_t of Eq. (20) reduces in the Boltzmann limit to the correct form is convincing evidence of the validity of the correction obtained herein.

To make this point in a somewhat more dramatic way, let us consider the following limit for the case of finite τ. Suppose $\omega=0$. Then from the Lindhard expression $\sigma_t(\mathbf{q},0,\tau)=\infty$, whereas from Eq. (20) $\sigma_t(\mathbf{q},0,\tau)$ is finite. Now consider $q\to 0$. Then from Eq. (20)

$$\lim_{q\to 0}\sigma_t(\mathbf{q},0,\tau)=ne^2\tau/m,$$

the correct result. This limit is also obtained if, in Eq. (20), we first let $q\to 0$ and then take $\omega\to 0$. The reason that the Lindhard expression fails in these limits is that for $q\to 0$ it does not reduce to the correct classical limit, $\epsilon(\omega,\tau)_{\text{classical}}=1-\omega_p^2/\omega\omega'$, whereas Eq. (20) does.[4]

LONGITUDINAL DIELECTRIC CONSTANT

In the derivation of the transverse dielectric function above, the appropriate finite-lifetime expression arose

[3] This equation is the analog of Eq. (3.14) of Ref. 1. Note, however, that there is a factor of $\frac{1}{2}$ missing in the first term within the curly bracket in Eq. (3.14) and also that the factor $(\omega+is)$ appearing outside the summation in Eq. (23) below is just given as ω in Eq. (3.14) of Ref. 1.

[4] The expression here given for $\epsilon(\omega,\tau)_{\text{classical}}$ is valid, for free electrons, within the framework of the SCF approximation (or random-phase approximation) when $\mathbf{q}\to 0$. This is discussed, along with some other points concerning the incorporation of lifetime effects into dielectric functions, by H. Ehrenreich and H. R. Philipp, Phys. Rev. **128**, 1622 (1962). See also H. Ehrenreich, in *The Optical Properties of Solids*, edited by J. Tauc (Academic Press Inc., New York, 1966), p. 106.

simply from retaining all factors of $(\omega+is)$ when they appeared. Such is not the case for the longitudinal dielectric constant. The problem arises in this case from the fact that, when longitudinal fields are present, the electron gas relaxes to the state of nonuniform density which is induced by the longitudinal field rather than to the uniform equilibrium state characterized by $f_0(\epsilon_k)$ of Eq. (6).

We are herein interested in the longitudinal dielectric function in the self-consistent-field approximation. Since, however, we shall be drawing analogies with the longitudinal dielectric function formalism associated with the Boltzmann equation, we summarize the Boltzmann equation results here. If you derive the longitudinal dielectric constant by making the simple-relaxation-time approximation in the Boltzmann equation (that is, you compel the system to relax to the state of uniform density), the result is $\epsilon_w(\mathbf{q},\omega,\tau)_B$,

$$\epsilon_w(\mathbf{q},\omega,\tau)_B=1-\frac{3\omega_p^2}{(ql')^3\omega(\omega+i/\tau)}[(ql')-\tan^{-1}(ql')]. \quad (24)$$

Kittel[5] has shown that the longitudinal dielectric function which takes account of relaxation to the state of nonuniform density in the Boltzmann equation approximation is given by

$$\epsilon_l(\mathbf{q},\omega,\tau)_B=1+\frac{(\epsilon_{w\,B}-1)}{1-i(\epsilon_{w\,B}-1)(ql')^2(\omega+i/\tau)/(3\omega_p^2\tau)}, \quad (25)$$

where $\epsilon_{w\,B}$ is given by Eq. (24) and l' by Eq. (23). In the limit $\tau\rightarrow\infty$, $\epsilon_{l\,B}=\epsilon_{w\,B}$. We note also the following limiting behavior of $\epsilon_{l\,B}$ and $\sigma_{l\,B}$, the latter defined for the longitudinal case by

$$\epsilon_l(\mathbf{q},\omega,\tau)_B=1+4\pi i\sigma_l(\mathbf{q},\omega,\tau)_B/\omega. \quad (26)$$

(i) $$\lim_{\omega\rightarrow0}\sigma_{l\,B}(0,\omega,\tau)=ne^2\tau/m, \quad (27)$$

the expected dc conductivity.

(ii) $$\lim_{\omega\rightarrow0}\epsilon_{l\,B}(\mathbf{q},\omega,\tau)=1+3\omega_p^2/q^2v_F^2, \quad (28)$$

the Fermi-Thomas screening result.

The usual expression for the self-consistent-field longitudinal dielectric constant, here denoted $\epsilon_l'(\mathbf{q},\omega)$, is[6]

$$\epsilon_l'(\mathbf{q},\omega)=1-\lim_{s\rightarrow0}\frac{4\pi e^2}{q^2V}\sum_{\mathbf{k}}\frac{f_0(\epsilon_{\mathbf{k}+\mathbf{q}})-f_0(\epsilon_{\mathbf{k}})}{\epsilon_{\mathbf{k}+\mathbf{q}}-\epsilon_{\mathbf{k}}+\hbar(\omega+is)}, \quad (29)$$

using the time dependence described above. According to Lindhard,[1] the appropriate finite-lifetime generalization of $\epsilon_l'(\mathbf{q},\omega)$ is obtained by putting $s=\tau^{-1}$, which leads, after integrating, to

$$\epsilon_l'(\mathbf{q},\omega,\tau)=1+(3\omega_p^2/q^2v_F^2)f_l, \quad (30)$$

where

$$f_l=\tfrac{1}{2}+\frac{1}{8z}\left[[1-(z-u')^2]\ln\left(\frac{z-u'+1}{z-u'-1}\right)+[1-(z+u')^2]\ln\left(\frac{z+u'+1}{z+u'-1}\right)\right], \quad (31)$$

using the notation of Eqs. (19). Since $\epsilon_l'=1+4\pi i\sigma_l'/\omega$, we find

$$\lim_{\omega\rightarrow0}\sigma_l'(0,\omega,\tau)=-i\omega ne^2\tau^2/m\rightarrow0, \quad (32)$$

that is, the correct dc conductivity is not obtained.[7] In addition,

$$\lim_{q\rightarrow0}\epsilon_l'(\mathbf{q},0,\tau)=1+\omega_p^2\tau^2, \quad (33)$$

which does not agree with the Fermi-Thomas expression. Hence we can conclude that $\epsilon_l'(\mathbf{q},\omega,\tau)$ is not a correct dielectric function for finite τ.

In an attempt to develop an appropriate dielectric function for finite τ we proceed as follows. First we will obtain the self-consistent-field analog of $\epsilon_{w\,B}$. Recognizing then that Eq. (25) provides a prescription for constructing a dielectric function that includes relaxation to the nonuniform state (ϵ_l) from one which does not (ϵ_w), we will then tentatively conclude that the appropriate finite-τ generalization of the Lindhard dielectric function is of the form of Eq. (25), with ϵ_w being the self-consistent-field analog of $\epsilon_{w\,B}$.

In the limit $z\rightarrow0$, $\epsilon_l'(\mathbf{q},\omega,\tau)$, Eq. (30), should reduce to $\epsilon_w(\mathbf{q},\omega,\tau)_B$, Eq. (24).[8] But

$$\lim_{z\rightarrow0}\epsilon_l'(\mathbf{q},\omega,\tau)=1-\frac{3\omega_p^2}{(ql')^3(\omega+i/\tau)^2}\times[(ql')-\tan^{-1}(ql')], \quad (34)$$

which is not equal to $\epsilon_{w\,B}$. The second terms on the right-hand sides of Eqs. (24) and (28) differ by a factor of $(\omega+i/\tau)/\omega$. We define $\epsilon_w(\mathbf{q},\omega,\tau)$ to be that modifica-

[5] C. Kittel, *Quantum Theory of Solids* (Wiley-Interscience, Inc., New York, 1963), pp. 326–332. In Kittel's Eq. (33), σ_{zz} is the longitudinal conductivity obtained when the electron gas relaxes to the equilibrium state of uniform density, while σ' takes account of relaxation to the state of nonuniform density. Going over to our notation, we therefore write σ_{zz} as $\sigma_{w\,B}$ and σ' as $\sigma_{l\,B}$ so that this equation becomes $\sigma_{w\,B}=\sigma_{l\,B}[1+i(\sigma_{w\,B}/\sigma_0)(av_F/3c_s)]^{-1}$. From Kittel's Eq. (32), $c_s=\omega/q$; also, $\sigma_0=ne^2\tau/m=\omega_p^2\tau/4\pi$ and $a=v_F\tau/(1-i\omega\tau)$. Noting that $\sigma_{w\,B}=(\omega/4\pi i)(\epsilon_{w\,B}-1)$ and $\sigma_{l\,B}=(\omega/4\pi i)\times(\epsilon_{l\,B}-1)$, one can readily solve for $\epsilon_{l\,B}$ and arrive at Eq. (25) below.

[6] See, for example, Ref. 2.

[7] When $\mathbf{q}\rightarrow0$, with ω finite, the longitudinal and transverse dielectric functions become equal. See, for example, the latter reference in Ref. 4, especially pp. 124–128. Thus, as $\mathbf{q}\rightarrow0$, the longitudinal dielectric function must reduce to $\epsilon(\omega,\tau)_{\text{classical}}=1-\omega_p^2/(\omega\omega')$ (see Ref. 4), a limit not obtained from Eq. (30).

[8] The insertion of a finite lifetime into $\epsilon_l'(\mathbf{q},\omega)$ corresponds to a simple relaxation-time approximation. This approximation with the Boltzmann equation yields $\epsilon_w(\mathbf{q},\omega,\tau)_B$. Thus we should have $\lim_{z\rightarrow0}\epsilon_l'(\mathbf{q},\omega,\tau)=\epsilon_w(\mathbf{q},\omega,\tau)_B$.

tion of $\epsilon_l'(\mathbf{q},\omega,\tau)$ which will yield, in the limit $z\to 0$, $\epsilon_w(\mathbf{q},\omega,\tau)_B$. Thus,

$$(\epsilon_w-1)=\frac{(\omega+i/\tau)}{\omega}(\epsilon_l'-1),$$

so that

$$\epsilon_w(\mathbf{q},\omega,\tau)=1+\frac{3\omega_p^2(\omega+i/\tau)}{q^2v_F^2\omega}f_l. \tag{35}$$

As outlined above we then use the form of Eq. (25) to write a tentative longitudinal dielectric function $\epsilon_l(\mathbf{q},\omega,\tau)_{\text{ten}}$,

$$\epsilon_l(\mathbf{q},\omega,\tau)_{\text{ten}}=1+\frac{(\epsilon_w-1)}{1-i(\epsilon_w-1)(ql')^2(\omega+i/\tau)/(3\omega_p^2\tau)}, \tag{36}$$

where ϵ_w is given by Eq. (35).

With $\epsilon_{l\text{ ten}}=1+4\pi i\sigma_{l\text{ ten}}/\omega$, we find

$$\lim_{\omega\to 0}\sigma_l(0,\omega,\tau)=ne^2\tau/m.$$

Thus $\epsilon_{l\text{ ten}}$ does yield the correct dc conductivity. In the limit $\omega=0$ and $q\to 0$, $\epsilon_{l\text{ ten}}$ should reduce to the Fermi-Thomas result, given in the right-hand side of Eq. (28), whether τ is finite or infinite. But,

$$\epsilon_l(\mathbf{q},0,\tau)_{\text{ten}}=1+3\omega_p^2/(q^2v_F^2), \tag{37}$$

that is, $\epsilon_l(\mathbf{q},0,\tau)_{\text{ten}}$ is given by the Fermi-Thomas expression for *all* q rather than just in the limit $\mathbf{q}\to 0$. [Note that the correct Fermi-Thomas screening is contained in ϵ_l' of Eq. (29) but not in $\epsilon_w(\mathbf{q},\omega,\tau)$, Eq. (35).[9]]

That Eq. (37) is true for all $\mathbf{q}$ is indeed unfortunate, since, in the limit $\omega\to 0$, the longitudinal dielectric constant should contain the singularity (or the rapid change when τ is finite) for $q\sim 2k_F$ associated with the Friedel oscillations. Let us put that in alternative terms. If we consider the Lindhard finite-τ result, Eq. (30), then, when $\omega\to 0$, we obtain

$$\epsilon_l'(\mathbf{q},0,\tau)=1+\frac{3\omega_p^2}{q^2v_F^2}f_l|_{\omega=0}. \tag{38}$$

When $q=2k_F$, this expression has a singularity when $\tau\to\infty$ and it is rapidly changing when τ is finite. That is, for all values of τ such that the concept of a Fermi surface is meaningful,[10] $|u'|\ll 1$ for $\omega=0$ and $q\sim 2k_F$ and thus the basic structure leading to the Friedel oscillations is contained in Eq. (38) even for finite τ. This suggests that Eq. (38) is essentially correct for large $\mathbf{q}$, i.e., $q\gtrsim k_F$. However, Eq. (38) is not valid for small $\mathbf{q}$, as is clear from Eq. (33).

[9] This indicates clearly the fact that making a simple relaxation-time approximation when longitudinal fields are present is invalid in principle.

[10] If the concept of a Fermi surface is to be meaningful, we must have $\epsilon_F\tau\gg\hbar$, where ϵ_F is the Fermi energy. This means $\hbar k_F^2\tau/2m\gg 1$ or $k_Fv_F\tau\gg 1$ or $k_Fl\gg 1$.

We are thus faced with the dilemma that the Lindhard finite-τ result seems to be valid for large $\mathbf{q}$ only and $\epsilon_{l\text{ ten}}$, Eq. (36) together with the ϵ_w of Eq. (35), is valid only for small $\mathbf{q}$.

In seeking to eliminate the low-q restriction on Eq. (36), it seems reasonable to attempt a modification of the denominator of the second term on the right-hand side. The second term in this denominator arises explicitly from the inclusion of the relaxation to the state of nonuniform density. Since such collective effects are most pronounced for small $\mathbf{q}$, we suggest that the denominator factor (ϵ_w-1) be replaced by this factor in the limit $z\to 0$, i.e., the Boltzmann-equation result $(\epsilon_{w\,B}-1)$. This then means that the denominator becomes identical to that in Eq. (25). Thus our tentative expression for $\epsilon_l(\mathbf{q},\omega,\tau)$ becomes

$$\epsilon_l(\mathbf{q},\omega,\tau)=1+\frac{(\epsilon_w-1)}{1-i(\epsilon_{w\,B}-1)(ql')^2(\omega+i/\tau)/(3\omega_p^2\tau)}$$

$$=1+\frac{(\epsilon_w-1)}{1+i[ql'-\tan^{-1}(ql')]/(ql'\omega\tau)}, \tag{39}$$

with ϵ_w given by Eq. (35). Note that this modification has no effect on the $q\to 0$ limit of ϵ_l.

In the limit $\omega\to 0$, Eq. (39) with ϵ_w given by Eq. (35) is

$$\epsilon_l(\mathbf{q},0,\tau)=1+\frac{[3\omega_p^2/(q^2v_F^2)]f_l|_{\omega=0}}{[ql-\tan^{-1}(ql)]/(ql)}, \tag{40}$$

where $l=v_F\tau$. For $q\gtrsim k_F$, $ql\gg 1$ as discussed above, and so

$$\epsilon_l(|\mathbf{q}|\gtrsim k_F,0,\tau)\simeq 1+\frac{3\omega_p^2}{q^2v_F^2}f_l|_{\omega=0}.$$

Thus the structure in ϵ_l describing the Friedel oscillations is contained in Eq. (39). In addition, for $q\ll k_F$, Eq. (40) reduces to the Fermi-Thomas screening result, as it should. We then conclude that the appropriate self-consistent-field longitudinal dielectric function for finite τ is given by Eq. (39), with ϵ_w given by Eq. (35).

In summary, we list some of the properties possessed by this new dielectric function.

(i) When $z\to 0$, $\epsilon_l(\mathbf{q},\omega,\tau)$ reduces to that longitudinal dielectric function obtained from the Boltzmann equation when relaxation to the state of nonuniform density is included.

(ii) For $q\to 0$, $\epsilon_l(\mathbf{q},\omega,\tau)\to 1-\omega_p^2/[\omega(\omega+i/\tau)]$, the classical result. Thus the correct dc conductivity is obtained.

(iii) $\epsilon_l(|\mathbf{q}|\gtrsim k_F,0,\tau)$ contains the structure associated with the Friedel oscillations.

(iv) $\lim_{q\to 0}[\lim_{\omega\to 0}\epsilon_l(\mathbf{q},\omega,\tau)]=1+3\omega_p^2/(q^2v_F^2)$, the Fermi-Thomas screening result.

(v) When $\tau\to\infty$, $\epsilon_l(\mathbf{q},\omega,\tau)$ becomes the Lindard infinite-τ expression.

DISCUSSION

To illustrate the difference between the Lindhard finite-τ results and the corrected equations we have plotted in Figs. 1–4 the real and imaginary parts of the longitudinal and transverse dielectric functions for fixed ω as a function of q. These curves have been determined for $v_F=0.85\times10^8$ cm/sec, $\omega_p=6.61\times10^{15}$/sec, and an effective mass equal to the electronic mass, parameters representative of potassium. We have used $\tau=10^3/\omega_p$ corresponding to a moderately pure metal, and have chosen frequencies such that $\omega\tau=10^{-1}$ and 10.

As is evident from the discussion above, the difference between the Lindhard expressions and those developed here will be most pronounced when $\omega\tau\lesssim1$. For the $\mathrm{Re}\epsilon_l$, Fig. 1, the discrepancy is particularly significant. The Lindhard expression is seen to have an incorrect sign for small q when $\omega\tau<1$. This is due to incorrect limiting behavior as $q\to0$. In the limit of $q\to0$, the longitudinal dielectric constant should approach the classical value of $1-\omega_p^2/[\omega(\omega+i/\tau)]$, which the correct expression does. However, the Lindhard expression approaches $1-\omega_p^2/[\omega+i/\tau]^2$. This difference results in an incorrect sign when $\omega\tau<1$. For $q\to\infty$, both expressions give $\epsilon_l\to1$.

The $\mathrm{Im}\epsilon_l$ is shown in Fig. 2. Again, the Lindhard and the corrected results differ markedly when $\omega\tau=10^{-1}$, whereas the $\omega\tau=10$ results agree reasonably well. The factor-of-2 difference between the different expressions for $\mathrm{Im}\epsilon_l$ when $\omega\tau=10$ and $q\to0$ is again a consequence of the incorrect limiting behavior of the Lindhard result as $q\to0$. This factor of 2 occurs independent of the frequency for $\omega\tau\gg1$.

For the real part of the transverse dielectric function, plotted in Fig. 3, the difference between the Lindhard result and the corrected result is minor for $\omega\tau\gtrsim10$. In both limits, $q\to0$ and $q\to\infty$, the Lindhard expression reduces to $1-\omega_p^2/\omega^2$ and the corrected expression to $1-\omega_p^2/[\omega(\omega+i/\tau)]$. Thus, for a given curve, the $q\to0$ and the $q\to\infty$ values are the same, since ω is fixed. This is also true for the $\mathrm{Im}\epsilon_t$ shown in Fig. 4. However, in this case the incorrect limiting behavior leads to significant discrepancies between the Lindhard expression and the corrected expression even for large ω, since, in the limits $q\to0$ and $q\to\infty$, the imaginary part of the transverse dielectric function as given by Lindhard becomes 0, whereas the corrected equation gives $(\omega_p\tau)^2/\{\omega\tau[(\omega\tau)^2+1]\}$.

The question as to the general validity of the above-developed dielectric functions is not easily answered, particularly for the longitudinal dielectric function. In the case of the transverse dielectric function the finite-τ expression follows directly from the self-consistent-field approximation but such is clearly not so for the longitudinal dielectric function. The argument

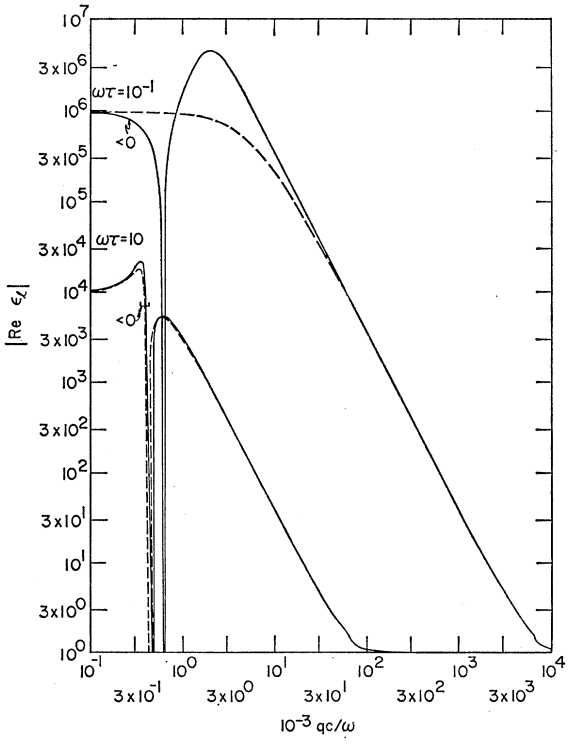

FIG. 1. The absolute value of the real part of the longitudinal dielectric function as a function of q/ω. The solid lines are the corrected results, Eq. (39) using ϵ_w of Eq. (35), and the dashed lines are the Lindhard results, Eq. (30). Regions where $\mathrm{Re}\epsilon_l<0$ are labeled <0. The curves were obtained using parameters representing potassium (see text).

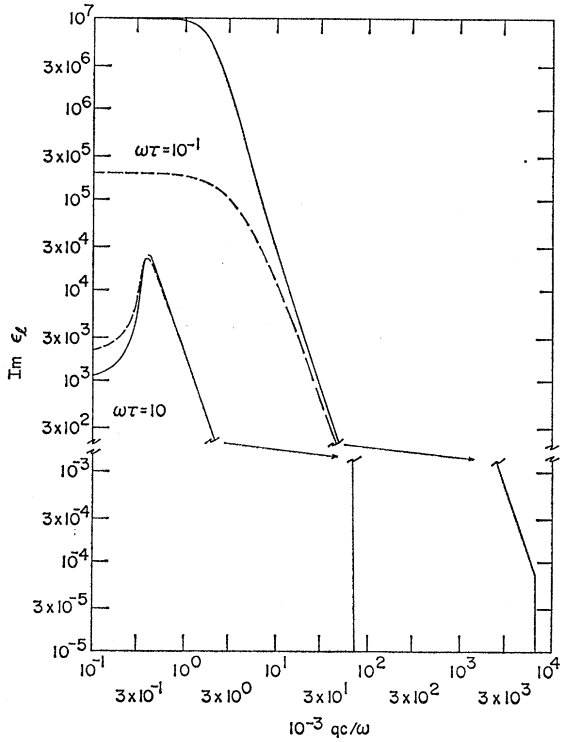

FIG. 2. The imaginary part of the longitudinal dielectric function as a function of q/ω. The solid lines are the corrected results and the dashed lines are the Lindhard results. The curves were obtained using parameters representing potassium (see text).

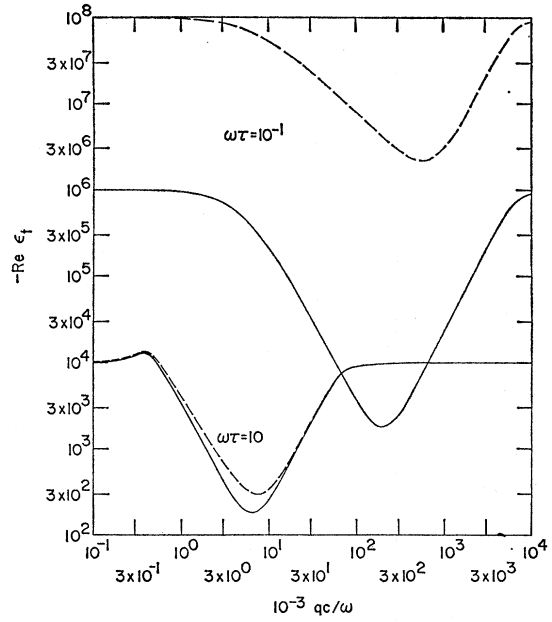

FIG. 3. The real part of the transverse dielectric function as a function of q/ω. The solid lines are the corrected results, Eq. (20), and the dashed lines are the Lindhard results. The curves were obtained using parameters representing potassium (see text).

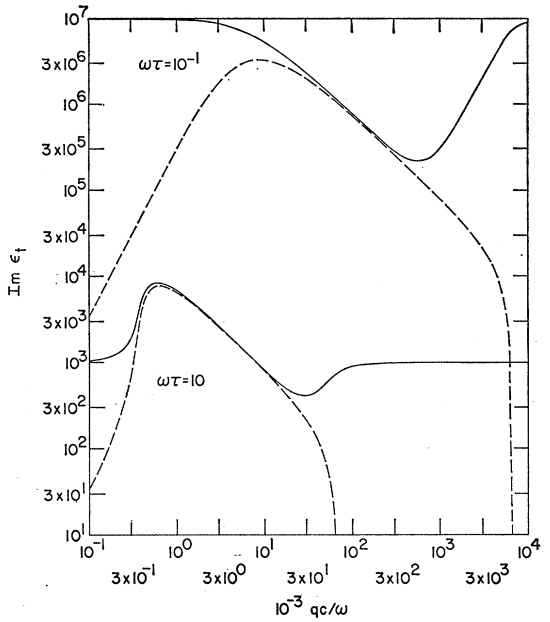

FIG. 4. The imaginary part of the transverse dielectric function as a function of q/ω. The solid lines are the corrected results and the dashed lines are the Lindhard results. The curves were obtained using parameters representing potassium (see text).

above leading to the longitudinal dielectric function Eq. (39) is phenomenological in nature. Since, however, we are attempting to incorporate a phenomenological concept, that of the relaxation time, this could perhaps have been anticipated. The fact that a number of limits, embodying a broad variety of physical effects, are satisfied does engender confidence in the essential validity of the expressions developed herein when the scattering can be characterized by a single relaxation time. To what extent is characterizing the electron gas by a single relaxation time justifiable? This is the problem we comment upon now.

If we take the temperature to be sufficiently low, the dominant scattering mechanism will be elastic scattering of the electrons by impurities and lattice defects.[11,12] Suppose we then consider impurity scattering under the conditions $\omega\tau\ll1$ and $ql\ll1$, where l, the mean free path, is defined by $l=v_F\tau$. In this hydrodynamic regime a single effective electron lifetime indeed characterizes the system.[13] It is from this regime that both ϵ_t and ϵ_l [Eqs. (20) and (39)] reduce to the classical dielectric constant as $q\to0$. Thus, the effective lifetime is that which appears in the dc conductivity and whose existence leads to the Wiedemann-Franz law.

When $\omega\tau\gg1$, the effects of collisions cease to be important in determining the properties of the system. Thus we would expect the expressions we developed for ϵ_l and ϵ_t to adequately take account of lifetime effects under these conditions, since any treatment of lifetime effects which embodies correct limiting behavior would probably be legitimate in this regime. The phenomenological character of τ must be kept in mind, however, and thus we must allow for the possibility that τ may be frequency-dependent.[14]

The considerations of the previous paragraph are also valid when $ql\gg1$. Under such conditions the effects of scattering are minor and a single-relaxation-time approximation appears to be valid, not in a rigorous sense, perhaps, but in the sense of a phenomenological scheme to incorporate the minor effects associated with scattering. Thus, the only regimes where the single-relaxation-time approximation seems questionable for impurity scattering are those for which $ql\sim1$ or $\omega\tau\sim1$. There is, however, strong evidence that these regimes may also be adequately characterized by a single relaxation time. We discuss this evidence now.

[11] D. Pines and P. Nozieres, *The Theory of Quantum Liquids, I: Normal Fermi Liquids* (W. A. Benjamin, Inc., New York, 1966), p. 188.

[12] It is important to recognize here the distinction between the elastic scattering due to the impurities and the (far slower) inelastic scattering due to the electron-phonon interaction. It is the latter process that tends to restore the ground state. See J. L. Warren and R. A. Ferrell, Phys. Rev. **117**, 1252 (1960); A. B. Pippard, in *Low-Temperature Physics*, edited by C. DeWitt, B. Dreyfus, and P.-G. DeGennes (Gordon and Breach, Science Publishers, Inc., New York, 1962), p. 44.

[13] Reference 11, p. 192. Note that in this case the scattering need not be isotropic for the single-relaxation-time approximation to be valid.

[14] Phase-space considerations suggest that τ should indeed be frequency-dependent when $\hbar\omega\gtrsim\epsilon_F$, the Fermi energy. In addition, it has been suggested that electron-electron interactions give rise to a frequency dependence of the lifetime. See, R. N. Gurzhi, Zh. Eksperim. i Teor. Fiz. **35**, 965 (1958) [English transl.: Soviet Phys.—JETP **8**, 673 (1959)].

Suppose we consider, for the moment, the Boltzmann equation. Let us define $W_{\mathbf{k}\mathbf{k}'}$ to be the transition rate for scattering from electron state $\mathbf{k}$ to electron state $\mathbf{k}'$. Since we are interested here in elastic scattering (we are still considering impurity effects), we write

$$W_{\mathbf{k}\mathbf{k}'}=W_0+W_1\cos\alpha,$$

where α is the angle between $\mathbf{k}$ and $\mathbf{k}'$. $W_{\mathbf{k}\mathbf{k}'}$ is written in this way to allow explicitly for the inclusion of both s- and p-wave scattering. Defining

$$\tau_0^{-1}=\sum_{\mathbf{k}'} W_{\mathbf{k}\mathbf{k}'} \quad \text{and} \quad \tau_1^{-1}=\sum_{\mathbf{k}'} W_{\mathbf{k}\mathbf{k}'}\cos\alpha,$$

it can then be shown[15] that in the hydrodynamic regime the effective relaxation time τ_{eff} is given by

$$\tau_{\text{eff}}^{-1}=\tau_0^{-1}-\tau_1^{-1},$$

which is just the conclusion reached in Ref. 13. In addition, it has been demonstrated[15] that if both s- and p-wave scattering are included outside the hydrodynamic regime, then the theoretical surface impedance for the anomalous skin effect when the electromagnetic field is incident normal to the surface can be accurately characterized by the relaxation time τ_{eff} if $\tau_0/\tau_1<0.4$.[16] Since the surface-impedance calculation is sensitive to conditions for which $ql\sim 1$ and $\omega\tau\sim 1$, this indicates that for impurity scattering the transverse dielectric constant as determined from the Boltzmann equation can be characterized by a single effective relaxation time for all conditions where the Boltzmann equation is valid. Now we have above extended the calculation of the transverse dielectric constant into regions where the Boltzmann equation is not valid, i.e., $q\gtrsim k_F$. However, since the peculiarly quantal properties which occur in the transverse dielectric constant developed above occur for just such values of q and thus are insensitive to scattering (see Ref. 10), we can then conclude that the use of a single effective relaxation time in the transverse dielectric constant is indeed a very good approximation when impurity scattering is dominant.[17]

For the longitudinal dielectric constant the situation is similar. Inclusion of p-wave scattering in the solution of the Boltzmann equation leads to a correction term, proportional to τ_0/τ_1, added to the denominator of the second term on the right-hand side of Eq. (25).[15] We have made a numerical comparison of Eq. (25), using $\tau=\tau_{\text{eff}}$, and the equation including p-wave scattering for a wide range of values of τ_0 and τ_1. As might have been anticipated, the only conditions for which there is an appreciable ($\gtrsim 1\%$), physically meaningful difference between the two are $\omega\tau\sim 1$ and, simultaneously, $ql\sim 1$. When $\tau_0/\tau_1=0.4$ these differences are $\sim 10\%$. The discrepancy drops to $\sim 7\%$ when $\tau_0/\tau_1=0.3$ and to $\sim 3\%$ when $\tau_0/\tau_1=0.2$. Since these differences are not large and occur only over limited ranges of q and ω, this indicates that when conditions are such that the Boltzmann equation is valid, the single-relaxation-time approximation is excellent. Since, as for the transverse dielectric constant, the quantum-mechanical extension to the Boltzmann equation results occur in a regime where scattering is a minor effect ($q\lesssim k_F$), this indicates that the approximation whereby a single phenomenological relaxation time is used to represent the effects of scattering in the longitudinal dielectric constant is very good when the scattering is by impurities.[18]

When the temperature is no longer very low, the major cause of scattering is the electron-phonon interaction.[11] In general, when phonon scattering is dominant, the relaxation-time approximation ceases to be valid, and our expressions for ϵ_l and ϵ_t are of dubious validity. However, if the temperature is such that $T\gtrsim\Theta_D$, where Θ_D is the Debye temperature, the conditions for which the scattering is essentially elastic are restored.[19,20] Thus again, for high temperatures, we assert the validity of our expressions for ϵ_l and ϵ_t.[21]

In summary, we have developed finite-electron-lifetime generalizations of the Lindhard dielectric functions for a free-electron gas. These new functions satisfy all the well-known limiting conditions and, when the dominant scattering is due to impurities or when the temperature is above the Debye temperature, are probably valid for all physically meaningful values of $\mathbf{q}$ and ω.

ACKNOWLEDGMENT

We wish to acknowledge helpful conversations with Professor S. H. Liu and also to thank him for making available to us some unpublished calculations.

[15] S. H. Liu (private communication).

[16] For a screened Coulomb interaction, we would expect $\tau_0/\tau_1<0.2$.

[17] Figures 3 and 4 apparently contradict this statement; the transverse dielectric constant does, in fact, depend upon τ for large q. As shown in Fig. 4, the imaginary part of the Lindhard transverse dielectric function decreases sharply to zero at $q\sim 2k_F$, while the imaginary part of the corrected transverse dielectric function approaches a nonzero value, which depends upon τ, as $q\to\infty$. There is also significant disagreement as $q\to 0$. Figure 3 shows a large discrepancy between the real parts of the Lindhard and corrected dielectric constants for all q, when $\omega\tau=10^{-1}$. For $q\ll k_F$, where the Boltzmann equation is valid, the Lindhard dielectric constant cannot be correct since it does not agree with the dielectric function obtained from a solution of the Boltzmann equation. When $q\lesssim k_F$, arguments based on the Boltzmann equation cannot be applied. It might be argued that the Lindhard dielectric constant is preferable for $q\lesssim k_F$ because for such conditions it is insensitive to the value of τ. We feel, however, that this is an irrelevant question since we know of no physically observable effects that are influenced by the value of the transverse dielectric constant when $q\gtrsim k_F$. The corrected dielectric constant can thus be legitimately used for all q, and the details by which scattering is included cease to be relevant when $q\lesssim k_F$.

[18] Here it must be remembered that the presence of a single τ does not imply a simple relaxation-time approximation.

[19] A. B. Pippard (Ref. 12).

[20] A. C. Smith, J. F. Janak, and R. B. Adler, *Electronic Conduction in Solids* (McGraw-Hill Book Co., New York, 1967), p. 176.

[21] This conclusion is only valid, clearly, if $kT\ll\epsilon_F$, the Fermi energy.