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Thermal Stabilization of the Modified Ordinary Wave

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Thermal stabilization of the modified ordinary (MO) wave propagating across an external magnetic field in a two-stream electron plasma is studied. It is found that temperature (1) increases the threshold of the relative streaming velocity below which the wave is stable, (2) imposes an upper limit on the spectrum of unstable wave numbers, and (3) reduces the growth rate. Discussion is given on the physical mechanism of the instability.

INTRODUCTION

In recent papers,[1,2] a study is made of the perpendicular propagation of electromagnetic waves in a two-stream electron plasma (infinite ion mass) of zero temperature immersed in a constant and uniform magnetic field which is aligned with the direction of streaming. It is shown that the presence of relative streaming renders the linearly-polarized mode dependent on the magnetic field. For this reason, it is referred to as the modified ordinary (MO) wave. This mode is found to be unstable if the ratio of plasma frequency to gyrofrequency exceeds one half the ratio of light velocity to relative streaming velocity. It follows that when this condition is satisfied, the cold-plasma theory predicts that there is in the system an electromagnetic instability in addition to the familiar electrostatic two-stream (TS) instability. It is well known that in a warm plasma, thermal motion has a stabilizing effect on the TS instability.[3] The question naturally arises as to whether this is also true for the MO instability and if so, what is the nature of the thermal stabilization. We wish to investigate this aspect in the present paper.

PHYSICAL MECHANISM

A rigorous calculation of the MO wave in a warm plasma would involve the solution of the Vlasov equation and Maxwell's equations. In the presence of a magnetic field, this procedure is complicated. It is our contention that the thermal effect can be studied using the fluid equations, with a pressure term added to the momentum equation. The reason for this lies in the physical mechanism of the MO instability, which we now discuss.

The system under consideration is shown in Fig. 1. The external magnetic field B_0 and the stream velocities are along the y axis, and we study waves with propagation vector $\vec{k}$ along the x axis. Consider first the case of no magnetic field and zero temperature. This has been discussed by Momota,[4] and we reiterate it here for completeness. Suppose there is a perturbed magnetic field $B_z = B \sin kx$ as shown. This gives rise to a Lorentz force acting on the streams. The streams with directions corresponding to a current along the $+y$ axis are pushed toward

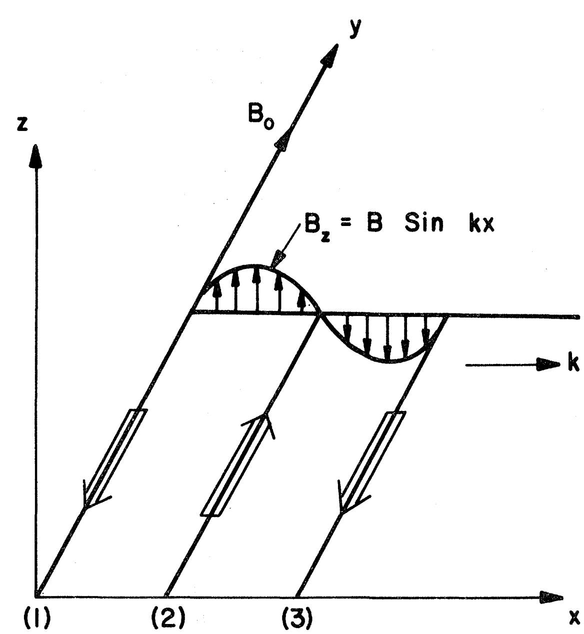

FIG. 1. Illustrating the physical mechanism of the instability. The Lorentz force due to a perturbed magnetic field B_z and streaming has the effect of pushing the streams with a plus current toward position (2), and those with a minus current toward positions (1) and (3). The resultant bunching of the currents enhances the initial magnetic perturbation. The presence of an external magnetic field B_0 along the y axis tends to inhibit this bunching process while thermal motion tends to diffuse away the bunched currents. Both effects are stabilizing.

position (2). Those corresponding to a current along the $-y$ axis are assembled at positions (1) and (3). The resultant bunching of the currents has the effect of strengthening the initial magnetic-field perturbation, and hence, an instability results.

Consider now the effect of an external magnetic field B_0. It tends to inhibit the motion of charged particles across the field lines or inhibit the bunching process described above. The effect is thus stabilizing. This is borne out by the requirement for stability given in I for a system composed of two counterstreaming electron plasmas:

$$\frac{\text{electron plasma frequency}}{\text{electron gyrofrequency}} < \frac{\text{light velocity}}{\text{streaming velocity}}$$

The above criterion shows that the instability is suppressed if the magnetic field is strong enough.

Let us now consider the effect of temperature. Temperature introduces random thermal motion of the charged particles, which tends to diffuse away the bunched currents so that the effect is again stabilizing. The important point to note is that diffusion is a macroscopic phenomenon and can be studied using the fluid equations. It is not a resonance effect, and using the Vlasov equation would enhance the complexity of the problem rather unnecessarily.

In the next section, we first study the case of no external magnetic field, using the fluid equations. The results are compared with those of Momota,[4] who has studied the nonmagnetized plasma using the Vlasov equation. It will be seen that the two approaches yield the same essential results, showing the correctness of the above physical argument. In the last section, we study the thermal stabilization of the MO wave in a magnetized plasma. This has not been investigated by Momota.

NONMAGNETIZED PLASMA

We use the equation of continuity, Maxwell's equations, and the momentum equation with a pressure term as our basic equations. We look for solutions of the form $\exp i(\omega t + kx)$ since we are concerned with perpendicular propagation. In the absence of an external magnetic field and neglecting collisions, the set of equations becomes:

$$i\omega\vec{v}_j = \frac{q_j}{m_j}\vec{E} + \frac{q_j}{m_j}u_j\vec{e}_y \times \vec{B} - v_{Tj}{}^2\frac{ikn_j}{N_j}\vec{e}_x \ , \qquad (1)$$

$$n_j + kN_jv_{jx} = 0 \ , \qquad (2)$$

$$-\vec{k}\times(\vec{k}\times\vec{E}) = \frac{\omega^2}{c^2}\vec{E} - \frac{4\pi i\omega}{c^2}\sum_j q_j(N_j\vec{v}_j + u_jn_j\vec{e}_y) \ , \qquad (3)$$

where q_j and m_j are the charge and mass of the particles which constitute the stream classified by the index j. The quantity $v_{Tj} = (KT_j/m_j)^{1/2}$ is the thermal velocity. The number density N_j and the velocity u_j of the particles in the unperturbed stream are assumed to be constant with respect to space and time. The number density n_j, velocity $\vec{v}_j$, electric field $\vec{E}$, and magnetic field $\vec{B}$ are small perturbations. $\vec{e}_x$ and $\vec{e}_y$ are unit vectors along the x and y axis, respectively. c is the velocity of light. Gaussian units are used.

As in I, we consider two identical counterstreaming electron plasmas (infinite ion mass), each with density $N/2$, moving with equal and opposite velocities $+u$ and $-u$, respectively. For this system, standard manipulations of Eqs. (1)–(3) yields the following dispersion relation for the linearly polarized mode with electric vector in the y axis:

$$\omega^2 - \omega_p{}^2 - k^2c^2 - k^2u^2\omega_p{}^2/(\omega^2 - k^2v_T{}^2) = 0 \ , \qquad (4)$$

where ω_p is electron plasma frequency and v_T is electron thermal velocity. The above can be written as

$$\omega^4 - \omega^2(k^2c^2 + k^2v_T{}^2 + \omega_p{}^2) + k^4v_T{}^2c^2 + k^2\omega_p{}^2(v_T{}^2 - u^2) = 0 \ . \qquad (5)$$

Equation (5) has a negative root for ω^2, corresponding to an instability, if

$$k^4v_T{}^2c^2 + k^2\omega_p{}^2(v_T{}^2 - u^2) < 0 \ . \qquad (6)$$

This can be expressed in two alternative forms:

$$u > v_T(1 + k^2c^2/\omega_p{}^2)^{1/2} \qquad (7)$$

or

$$k < (\omega_p/cv_T)(u^2 - v_T{}^2)^{1/2} \ . \qquad (8)$$

Equation (7) gives the minimum streaming velocity to excite a given wave number k, and (8) gives the maximum unstable wave number for a given streaming velocity. It is seen that if $v_T = 0$, the system is unstable for any value of u and for $0 < k < \infty$. The effect of temperature is to impose an upper bound on the spectrum of unstable wave numbers. It also reduces the growth rate, as can be seen from Eq. (5). Moreover, the minimum velocity required to trigger the electromagnetic instability is the thermal velocity. Since this is also the threshold for the electrostatic two-stream instability,[3] the two unstable modes tend to coexist together. As pointed out in I, however, the coherent radiation emitted by the electromagnetic mode is polarized while that caused by the TS instability is unpolarized.

The results of this section are arrived at by using the fluid equations. It is interesting to compare them with those based on the Vlasov equation, as obtained by Momota.[4] The instability criterion given by Eq. (8) is the same as that given by Eq. (51) in Momota's paper. Thus the macroscopic and the microscopic approaches lead to the same conclusion. This is not surprising in view of the physical picture described in the previous section. In the next section, we turn to the magnetized plasma, which is not studied by Momota.

MAGNETIZED PLASMA (MODIFIED ORDINARY WAVE)

With an external magnetic field B_0 in the direction of the y axis, the following equation replaces (1):

$$i\omega\vec{v}_j = \frac{q_j}{m_j}\vec{E} + \frac{q_j}{m_j c}(u_j\vec{e}_y \times \vec{B} + B_0\vec{v}_j \times \vec{e}_y) - v_{Tj}^2 ikn_j\vec{e}_x/N_j \quad . \tag{9}$$

Equations (9), (2), and (3) yield the following dispersion relation for the MO wave in a warm plasma:

$$\omega^2 - \omega_p^2 - k^2c^2 + \frac{k^2\omega_p^2u^2}{\Omega^2 - \omega^2 + k^2v_T^2} = 0 \quad , \tag{10}$$

where Ω is the electron gyrofrequency. Equation (10) can be written as

$$\omega^4 - C_1\omega^2 + C_2 = 0 \quad , \tag{11}$$

where $C_1 = \Omega^2 + \omega_p^2 + k^2c^2 + k^2v_T^2$, (12)

$$C_2 = k^4c^2v_T^2 + k^2(c^2\Omega^2 - u^2\omega_p^2 + \omega_p^2v_T^2) + \omega_p^2\Omega^2 \quad . \tag{13}$$

The instability criterion for the MO wave is given by $C_2 < 0$:

$$k^4c^2v_T^2 + k^2(c^2\Omega^2 - u^2\omega_p^2 + \omega_p^2v_T^2) + \omega_p^2\Omega^2 < 0 \tag{14}$$

or alternatively:

$$u^2 > v_T^2 + \frac{c^2\Omega^2}{\omega_p^2} + \frac{\Omega^2}{k^2} + \frac{k^2c^2}{\omega_p^2}v_T^2 . \tag{15}$$

At the instability-stability boundary, $C_2 = 0$. This is a quadratic equation in k^2, of which there are two solutions $k_\pm^2$:

$$k_\pm^2 = \frac{\omega_p^2}{2c^2}\left\{\frac{u^2}{v_T^2} - 1 - \frac{c^2\Omega^2}{v_T^2\omega_p^2} \pm \left[\left(\frac{u^2}{v_T^2} - 1 - \frac{c^2\Omega^2}{v_T^2\omega_p^2}\right)^2 - 4\frac{c^2\Omega^2}{v_T^2\omega_p^2}\right]^{\frac{1}{2}}\right\} . \tag{16}$$

If

$$\left(\frac{u^2}{v_T^2} - 1 - \frac{c^2\Omega^2}{v_T^2\omega_p^2}\right)^2 \gg \frac{4c^2\Omega^2}{v_T^2\omega_p^2} \quad ,$$

we have the following approximate expressions:

$$k_+^2 \simeq \frac{\omega_p^2(u^2 - v_T^2) - c^2\Omega^2}{c^2v_T^2} - \frac{\omega_p^2\Omega^2}{\omega_p^2(u^2 - v_T^2) - c^2\Omega^2} \quad , \tag{17}$$

$$k_-^2 \simeq \frac{\omega_p^2\Omega^2}{\omega_p^2(u^2 - v_T^2) - c^2\Omega^2} \quad . \tag{18}$$

It is clear from (15) that the range of unstable wave numbers is

$$k_- < k < k_+ \quad . \tag{19}$$

Moreover, from the requirement that k^2 be real, the minimum value of the streaming velocity required to excite the MO instability is obtained by setting the square bracket in (16) to zero. This yields

$$u_{\min} = v_T + c\Omega/\omega_p \quad . \tag{20}$$

At this value of u, we have

$$k_+ = k_- = (\omega_p\Omega/cv_T)^{1/2} \quad . \tag{21}$$

In the limit of the vanishing temperature,

$$k_+^2 \to \infty \quad , \tag{22}$$

$$k_-^2 \to \Omega^2/(u^2 - c^2\Omega^2/\omega_p^2) \quad , \tag{23}$$

which was the result obtained in I. Comparing (23) with (18), we see that $(k_-)_{T=0} < (k_-)_{T>0}$. In the limit of vanishing magnetic field, we recover the result of the previous section with k_+ given by Eq. (8) and $k_- = 0$.

It is seen from the above that the stabilizing effect of the magnetic field is to introduce a lower limit, while temperature imposes an upper bound on the spectrum of unstable wave numbers. The growth rate, of course, is reduced by both, as is evident from their contributions to the positive terms in C_2. If $C_1^2 \gg 4C_2$, the growth rate ω_i is approximately

$$\omega_i \simeq \left(-\frac{C_2}{C_1}\right)^{1/2}$$

$$= [k^2(u^2\omega_p^2 - v_T^2\omega_p^2 - c^2\Omega^2)$$

$$- \omega_p^{\,2}\Omega^2 - k^4c^2v_T^{\,2}]^{1/2}$$

$$\times [\omega_p^{\,2} + \Omega^2 + k^2(v_T^{\,2} + c^2)]^{-1/2} \ . \tag{24}$$

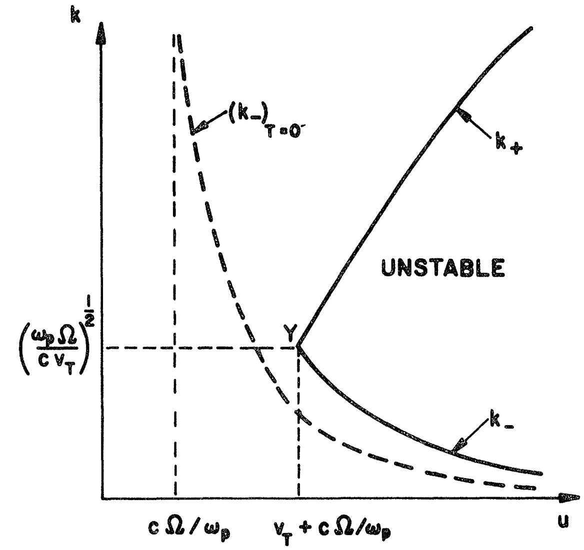

FIG. 2. Stability boundaries of the MO instability in a warm plasma. The region bounded by the curves k_+ and k_- [Eq. (16)] are unstable. The broken curve is the minimum unstable wave number for the case of cold plasma [Eq. (23)].

Figure 2 illustrates the thermal stabilization of the MO instability. The point marked Y is the intersection of the two lines k_+ and k_-. Its coordinates are

$$(v_T + c\Omega/\omega_p, \ (\omega_p\Omega/cv_T)^{1/2}) \ .$$

In the limits of zero temperature or zero magnetic field, this point becomes $(c\Omega/\omega_p, \infty)$, and $(v_T, 0)$, respectively. The dependence of the growth rate on temperature and wave number is illustrated in Fig. 3.

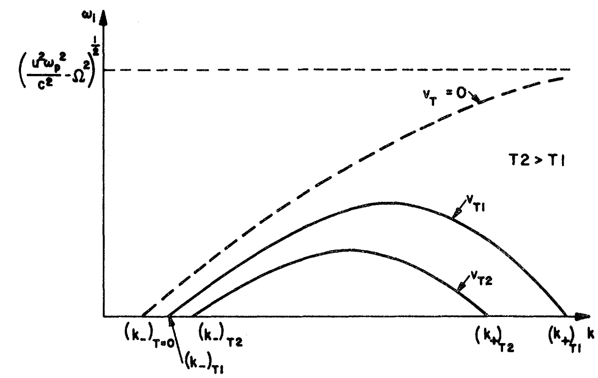

FIG. 3. Illustrating the dependence of the growth rate ω_i on temperature and wave number. $(k_\pm)_{T>0}$ and $(k_-)_{T=0}$ are given by Eqs. (16) and (23), respectively.

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