[9]P. W. Anderson, Phys. Rev. Letters 21, 13 (1968).
[10]C. Edmiston and K. Rudenberg, Rev. Mod. Phys. 35, 457 (1963).
[11]M. Cohen and V. Heine, Phys. Rev. 122, 1821 (1961).
[12]J. Linderberg and Y. Öhrn, Proc. Roy. Soc. (London) A285, 445 (1965); Phys. Rev. 139, A1063 (1965).

PHYSICAL REVIEW VOLUME 181, NUMBER 1 5 MAY 1969

Transition Operators in Radiative Damping Theory

R. H. Lehmberg
U. S. Naval Air Development Center, Johnsville, Pennsylvania
(Received 18 November 1968)

The application of transition operators $\mathcal{P}_{m,m'}(t) = \exp(iHt)|m\rangle\langle m'|\exp(-iHt)$ is studied for the problem of an atomic system S with eigenstates $\{m\}$ interacting with one or more damping reservoirs R. The average value of these operators gives the reduced density matrix $\rho_{m',m}^{(S)}(t)$ for S. If R consists of broad-band distributions of harmonic oscillators, (e.g., radiative damping), then damped equations of motion can be derived for all $\mathcal{P}_{m,m'}(t)$, even if S is a multilevel system. One need not specify the initial states of R, nor restrict the treatment to second order in the S-R coupling. The formalism is illustrated for the case where S consists of (i) a four-level atom in a resonant cavity (with broad-band modes also present), and (ii) a collection of atoms that can be treated as a multilevel spin system. Density-matrix equations are obtained for the case where no damping radiation is present initially. In (ii), the formalism is used to derive a two-time correlation function without the aid of the fluctuation-regression theorem.

I. INTRODUCTION

Dissipation is ordinarily treated quantum mechanically by coupling the appropriate atomic or spin system S to a loss mechanism or damping reservoir R. Most treatments assume that (i) R has a broad continuum of modes coupled more or less uniformly to S; (ii) the initial density operator can be written as $\rho^{(S)}(0)\rho^{(R)}(0)$, where $\rho^{(S)}(0)$ describes the initial states of S, and $\rho^{(R)}(0)$ is a thermal equilibrium distribution for R; (iii) R is only slightly affected by its interaction with S. The usual procedure is to write equations of motion for the reduced density operator $\rho^{(S)}(t)$[1,2] (or for reservoir-averaged amplitude operators[3]) to second order in the coupling constant. Assumption (iii) is then implemented by an approximation equivalent to replacing the actual density operator $\rho(t')$ in the second-order terms by the factored expression $\rho^{(S)}(t')\rho^{(R)}(0)$.

If R is a collection of harmonic oscillators, then one can derive damped equations of motion for the amplitude operators of S, without explicitly using assumptions (ii) or (iii). Only the unperturbed reservoir coordinates appear in these equations, the perturbation due to S being entirely absorbed in the damping constant and frequency shift. (A well-known example of this would be the case where S is itself a harmonic oscillator.[4]) If similar damped equations could be derived for the reduced density-matrix elements of a multilevel atom, it would provide a convenient and nearly exact starting point for studying its interaction with a known radiation field. These can, in principle, be obtained from the amplitude operator equations[5,6]; however, the procedure is rather tedious to apply to multilevel atoms.

In this article, we define a set of operators that enable one to obtain the reduced density-matrix equations by a straightforward, yet rigorous, derivation. Two examples will be presented. In the first, the system S consists of a four-level atom in a resonant cavity; in the second, S is a collection of atoms that can be treated as a multilevel spin system. We show in the second example that the formalism can be used to calculate two time correlation functions, without using the fluctuation-regression theorem[6-8] required in an ordinary density operator treatment.

If H and $\rho(t)$ are the complete Hamiltonian and density operator, respectively, and S is described by basis states $|m\rangle$, then the components of the reduced density operator can be written as

$$\rho_{l,m}^{(S)}(t)=\langle l\,|\,\mathrm{Tr}_R\rho(t)\,|m\rangle=\mathrm{Tr}\{\rho(0)\mathcal{P}_{m,l}(t)\}$$
$$\equiv\langle\mathcal{P}_{m,l}(t)\rangle\,, \qquad (1.1)$$

where Tr_R denotes a trace only over reservoir coordinates, and ($\hbar\equiv1$),

$$\mathcal{P}_{m,l}(t)\equiv|m,t\rangle\langle l,t|=e^{iHt}|m\rangle\langle l|e^{-iHt}\,, \qquad (1.2)$$

could be referred to as a generalized projection operator or a transition operator. In the formalism presented here, we calculate the dynamical properties of the damped system from $\mathcal{P}_{m,l}(t)$ rather than $\rho(t)$.

II. FOUR-LEVEL SYSTEM

As a first illustration, we consider the four level system treated by Scully and Lamb.[2] An atom with energy states a,b,c,d is located in a cavity of resonant frequency $\omega\approx\epsilon_{ab}$, where $\epsilon_{ab}=\epsilon_a-\epsilon b$ is the energy of the ab transition. Level a can also decay to the lower state c, emitting a photon into a broad band of closely spaced modes characterized by frequencies ω_S. Similarly, b can decay to level d, giving broad-band radiation characterized by ω_σ. For simplicity, we assume that these bands do not overlap; i.e., $\omega\neq\omega_S\neq\omega_\sigma$.

In the rotating wave approximation[4] (RWA), the Hamiltonian of the entire system is

$$H=H_S+H_R+V_{SR}\,, \qquad (2.1)$$

where

$$H_S=\sum_{\alpha=a}^{d}\epsilon_\alpha A_\alpha^\dagger A_\alpha+\omega a^\dagger a+gA_a^\dagger A_b a+\mathrm{H.c.}$$
$$=\epsilon_c A_c^\dagger A_c+\epsilon_d A_d^\dagger A_d+\mathfrak{K}\,, \qquad (2.2a)$$

$$H_R=\sum_S\omega_S a_S^\dagger a_S+\sum_\sigma\omega_\sigma a_\sigma^\dagger a_\sigma\,, \qquad (2.2b)$$

$$V_{SR}=\sum_S g_S A_a^\dagger A_c a_S+\mathrm{H.c.}$$
$$+\sum_\sigma g_\sigma A_b^\dagger A_d a_\sigma+\mathrm{H.c.} \qquad (2.2c)$$

(H.c. refers to Hermitian conjugate). The coupling constants g, g_S, and g_σ are taken to be real, and the atomic and radiation operators satisfy the usual relations

$$[A_\alpha,A_\beta]_+=0,\quad [A_\alpha,A_\beta^\dagger]_+=\delta_{\alpha,\beta}, \qquad (2.3a)$$

$$[a_i,a_j]=0,\quad [a_i,a_j^\dagger]=\delta_{i,j}\,. \qquad (2.3b)$$

If $|\alpha,n\rangle$ describes the atom in its αth state, and the cavity mode with n photons, then the reduced density matrix for the atom-cavity system can be written as

$$\rho^{(S)}_{\alpha,n;\beta,n'}(t)=\mathrm{Tr}\{\rho(0)\mathcal{P}_{\beta,n';\alpha,n}(t)\}$$
$$\equiv\langle\mathcal{P}_{\beta,n';\alpha,n}(t)\rangle\,, \qquad (2.4)$$

where

$$\mathcal{P}_{\beta,n';\alpha,n}(t)=e^{iHt}|\beta,n'\rangle\langle\alpha,n|e^{-iHt}\,. \qquad (2.5)$$

Scully and Lamb derive damped equations of motion for $\rho^{(S)}_{\alpha,n;\beta,n'}(t)$ by applying Wigner-Weisskopf techniques in the conventional density-matrix formalism, and they use the factorization discussed in connection with postulate (iii). Here, their results will be derived from the transition operator formalism, without this restriction. From Eqs. (2.1)–(5), we obtain

$$\dot{\mathcal{P}}_{a,n';b,n+1}-i[\mathfrak{K},\mathcal{P}_{a,n';b,n+1}]$$
$$=i[H_R+V_{SR},\mathcal{P}_{a,n';b,n+1}]$$
$$=i\sum_S g_S a_S^\dagger\mathcal{P}_{c,n';b,n+1}$$
$$-i\sum_\sigma g_\sigma\mathcal{P}_{a,n';d,n+1}a_\sigma\,, \qquad (2.6)$$

$$\dot{\mathcal{P}}_{a,n';a,n}-i[\mathfrak{K},\mathcal{P}_{a,n';a,n}]$$
$$=i\sum_S g_S(a_S^\dagger\mathcal{P}_{c,n';a,n}-\mathcal{P}_{a,n';c,n}a_S)\,, \qquad (2.7)$$

$$\dot{\mathcal{P}}_{c,n';c,n}-i(n'-n)\omega\mathcal{P}_{c,n';c,n}$$
$$=-i\sum_S g_S(a_S^\dagger\mathcal{P}_{c,n';a,n}-\mathcal{P}_{a,n';c,n}a_S)\,, \qquad (2.8)$$

while similar equations can be written for $\dot{\mathcal{P}}_{b,n'+1;b,n+1}$ and $\dot{\mathcal{P}}_{d,n'+1;d,n+1}$. The a_S operator satisfies

$$\dot{a}_S=i[H,a_S]=-i\omega_S a_S-ig_S A_c^\dagger A_a\,, \qquad (2.9)$$

with the formal solution

$$a_S(t)=a_S^{(0)}(t)$$
$$-ig_S\int_0^t dt'e^{-i\omega_S(t-t')}A_c^\dagger(t')A_a(t')\,, \qquad (2.10)$$

where $a_S^{(0)}(t)=a_S(0)\exp(-i\omega_S t)$. Similarly,

$$a_\sigma(t)=a_\sigma^{(0)}(t)$$
$$-ig_\sigma\int_0^t dt'e^{-i\omega_\sigma(t-t')}A_d^\dagger(t')A_b(t')\,, \qquad (2.11)$$

where $a_\sigma^{(0)}(t)=a_\sigma(0)\exp(-i\omega_\sigma t)$.

If (2.6) is evaluated at time t, then according to (2.10), the first term on the right-hand side can be written

$$i\sum_S g_S a_S^{(0)\dagger}(t)\mathcal{P}_{c,n';b,n+1}(t)$$

$$-\sum_s g_s^2 \int_0^t dt' e^{i\omega_s(t-t')} \times A_a^\dagger(t') A_c(t') \mathcal{P}_{c,n';b,n+1}(t). \quad (2.12)$$

We make the usual assumption that the modes are closely spaced, with density $\rho(\omega_s)$, such that

$$\sum_s g_s^2 \to \int_0^\infty d\omega_s \rho(\omega_s) g^2(\omega_s). \quad (2.13)$$

The most important contributions to the integral in expression (2.12) are those for which

$$t - t' \leq \delta t \equiv 2\pi/\Delta\omega_s, \quad (2.14)$$

where $\Delta\omega_s$ is the effective bandwidth of $\rho(\omega_s) g^2(\omega_s)$. Since $\Delta\omega_s$ is typically on the order of atomic frequency $\epsilon_{ac} = \epsilon_a - \epsilon_c$, the correlation time δt will be only a few cycles in duration; in any case, we assume that δt is short compared with the time required for an appreciable secular change in the atomic states. To a high degree of accuracy, one can therefore substitute the lowest approximation

$$A_a^\dagger(t') A_c(t') \simeq A_a^\dagger(t) A_c(t) \exp[i\epsilon_{ac}(t'-t)] \quad (2.15)$$

in expression (2.12). The result is

$$i\sum_s g_s a_s^{(0)\dagger}(t) \mathcal{P}_{c,n';b,n+1}(t) - [\tfrac{1}{2}\gamma_a(t) + i\Omega_a(t)] \mathcal{P}_{a,n';b,n+1}(t), \quad (2.16)$$

where

$$\gamma_a(t) = 2\int_0^\infty dw' \rho(\omega') g^2(\omega') \frac{\sin(\omega' - \epsilon_{ac})t}{\omega' - \epsilon_{ac}}, \quad (2.17a)$$

$$\xrightarrow{t \gg \epsilon_{ac}^{-1}} 2\pi\rho(\epsilon_{ac}) g^2(\epsilon_{ac}) \equiv \gamma_a, \quad (2.17b)$$

$$\Omega_a(t) = \int_0^\infty d\omega' \rho(\omega') g^2(\omega') \frac{1 - \cos(\omega' - \epsilon_{ac})t}{\omega' - \epsilon_{ac}}, \quad (2.18a)$$

$$\xrightarrow{t \gg \epsilon_{ac}^{-1}} P\int_0^\infty dw' \frac{\rho(\omega') g^2(\omega')}{\omega' - \epsilon_{ac}} \equiv \Omega_a. \quad (2.18b)$$

We can treat the second term on the right-hand side of (2.6) by substituting (2.11) and carrying out a procedure analogous to the one above. The final result is

$$\dot{\mathcal{P}}_{a,n';b,n+1}(t) - i[\mathcal{H}(t), \mathcal{P}_{a,n';b,n+1}(t)] = -[\tfrac{1}{2}(\gamma_a + \gamma_b) + i(\Omega_a - \Omega_b)] \mathcal{P}_{a,n';b,n+1}(t) + i\sum_s g_s a_s^{(0)\dagger}(t) \mathcal{P}_{c,n';b,n+1}(t) - i\sum_\sigma g_\sigma \mathcal{P}_{a,n';d,n+1}(t) a_\sigma^{(0)}(t), \quad (2.19)$$

where γ_b and Ω_b have the same form as (2.17) and (2.18), respectively, with ϵ_{ac} replaced by ϵ_{bd}.

So far, nothing has been said about the initial state of the damping radiation or the atom-cavity system. If the initial density operator is $\rho(0)$, then the expression for $\dot{\rho}_{b,n+1;a,n'}$ follows from substituting (2.19) into (2.4), and noting that

$$\mathrm{Tr}\{\rho(0)[\mathcal{H}(t), \mathcal{P}_{a,n';b,n+1}(t)]\} = \mathrm{Tr}\{e^{-iHt}\rho(0)e^{+iHt}[\mathcal{H}, \mathcal{P}_{a,n';b,n+1}]\} = [\rho^{(S)}(t), \mathcal{H}]_{b,n+1;a,n'}. \quad (2.20)$$

If

$$\rho(0) = \rho^{(S)}(0)|0\rangle\langle 0|, \quad (2.21)$$

where $|0\rangle$ is the vacuum state for all s and σ photons, then

$$\dot{\rho}^{(S)}_{b,n+1;a,n'}(t) = i[\rho^{(S)}(t), \mathcal{H}]_{b,n+1;a,n'} - [\tfrac{1}{2}(\gamma_a + \gamma_b) + i(\Omega_a - \Omega_b)]\rho^{(S)}_{b,n+1;a,n'}(t), \quad (2.22)$$

which reduces to the expression derived by Scully and Lamb when Ω_a and Ω_b are ignored. Similar considerations applied to Eqs. (2.7) and (2.8) lead to the results

$$\dot{\rho}^{(S)}_{a,n;a,n'}(t) = i[\rho^{(S)}(t), \mathcal{H}]_{a,n;a,n'} - \gamma_a \rho^{(S)}_{a,n;a,n'}(t), \quad (2.23)$$

$$\dot{\rho}^{(S)}_{c,n;c,n'}(t) = i(n' - n)\omega\rho^{(S)}_{c,n;c,n'}(t) + \gamma_a \rho^{(S)}_{a,n;a,n'}(t), \quad (2.24)$$

again, in agreement with Scully and Lamb.[2]

III. COUPLED TWO-LEVEL SYSTEMS

We now consider the problem discussed by Dicke[9]; i.e., a system of N identical two-level atoms (or spins), coupled only by the radiation from their transitions. As in the preceding example, the radiation is assumed to be distributed in a broad band of closely spaced modes whose frequencies ω_k overlap the atomic resonant frequency $\epsilon = \epsilon_2 - \epsilon_1$. For simplicity, we restrict the atoms to a region small in comparison to the average wavelength $2\pi c/\epsilon$.

The complete RWA Hamiltonian can be written as

$$H = \epsilon R_3 + \sum_k \omega_k a_k^\dagger a_k + \sum_k g_k (R_+ a_k + a_k^\dagger R_-). \quad (3.1)$$

The system operators are defined as

$$R_- = \sum_{\alpha=1}^{N} \sigma_\alpha, \qquad R_+ = R_-{}^\dagger, \tag{3.2a}$$

$$R_3 = \sum_{\alpha=1}^{N} (Q_\alpha - \tfrac{1}{2}), \quad Q_\alpha = \sigma_\alpha{}^\dagger \sigma_\alpha, \tag{3.2b}$$

where $\sigma_\alpha{}^\dagger$ and σ_α are raising and lowering operators for the αth atom, satisfying the relations

$$[\sigma_\alpha, \sigma_\alpha{}^\dagger]_+ = 1, \quad \sigma_\alpha{}^2 = (\sigma_\alpha{}^\dagger)^2 = 0, \tag{3.3a}$$

$$[\sigma_\alpha, \sigma_\beta] = [\sigma_\alpha, \sigma_\beta{}^\dagger] = 0, \quad \alpha \neq \beta; \tag{3.3b}$$

hence

$$[R_3, R_\pm] = \pm R_\pm, \tag{3.4a}$$

$$[R_\pm, R_\mp] = \pm 2R_3 . \tag{3.4b}$$

Since the σ_α operators satisfy N-coupled nonlinear equations of motion, they show little promise as a starting point for a solution. We therefore begin by examining the dynamical equations for R_- and R_3. From Eqs. (3.1), (3.4), and (2.3b),

$$\dot{R}_-(t) = i[H(t), R_-(t)]$$
$$= -i\epsilon R_-(t) + 2i\textstyle\sum_k g_k R_3(t) a_k(t), \tag{3.5}$$

$$\dot{R}_3(t) = -i\textstyle\sum_k g_k [R_+(t) a_k(t) - a_k{}^\dagger R_-(t)], \tag{3.6}$$

$$\dot{a}_k(t) = -i\omega_k a_k(t) - i g_k R_-(t), \tag{3.7}$$

and if $a_k{}^{(0)}(t) = a_k(0)\exp(-i\omega_k t)$,

$$a_k(t) = a_k{}^{(0)}(t) - i g_k \int_0^t dt' \times e^{-i\omega_k(t-t')} R_-(t'). \tag{3.8}$$

Substituting (3.8) into (3.5), and replacing the k summation by an integral, as in (2.13), we obtain

$$\dot{R}_-(t) = -i\epsilon R_-(t) + 2i\textstyle\sum_k g_k R_3(t) a_k{}^{(0)}(t) + 2\int_0^\infty d\omega\, \rho(\omega) g^2(\omega) \times \int_0^t dt' e^{-i\omega(t-t')} R_3(t) R_-(t'). \tag{3.9}$$

Assuming, as before, that the bandwidth of $\rho(\omega)g^2(\omega)$ is comparable to ϵ, we make the replacement

$$R_-(t') = R_-(t) e^{-i\epsilon(t'-t)} \tag{3.10}$$

in the integral; hence for $t \gg \epsilon^{-1}$,

$$\dot{R}_-(t) = -i\epsilon R_-(t) + (\gamma - 2i\Omega) R_3(t) R_-(t) + 2i\textstyle\sum_k g_k R_3(t) a_k{}^{(0)}(t), \tag{3.11}$$

where γ and Ω are obtained by replacing ϵ_{qc} by ϵ in expressions (2.17b) and (2.18b), respectively. Similar considerations applied to (3.6) lead to

$$\dot{R}_3(t) = -\gamma R_+(t) R_-(t) - i\textstyle\sum_k g_k [R_+(t) a_k{}^{(0)}(t) - \text{H.c.}]. \tag{3.12}$$

For the single-atom case, Eqs. (3.11) and (3.12) become

$$\dot{\sigma}(t) = -[i(\epsilon - \Omega) + \tfrac{1}{2}\gamma]\,\sigma(t) + i\textstyle\sum_k g_k [2Q(t) - 1]\, a_k{}^{(0)}(t), \tag{3.13}$$

$$\dot{Q}(t) = -\gamma Q(t) - i\textstyle\sum_k g_k [\sigma^\dagger(t) a_k{}^{(0)}(t) - \text{H.c.}]. \tag{3.14}$$

Since (3.11) and (3.12) are nonlinear if $N > 1$, they appear to offer little hope for exact solutions, even for the average values $\langle R_-(t)\rangle$ and $\langle R_3\rangle$ in the simple case where no radiation is present initially. In the following, we will show that an appropriate set of transition operators satisfy linear equations for all values of N, and all initial radiation states. If no photons are present initially, then the corresponding equations for the reduced density operator components can be solved by a straightforward iteration procedure. It will also be possible to obtain (in simple cases) a solution for the correlation function $\langle R_+(t) R_-(t')\rangle$ where $t \geq t' > 0$, without resorting to the fluctuation-regression theorem.

Following Dicke,[9] we define

$$R_1 = \tfrac{1}{2}(R_- + R_+), \quad R_2 = \tfrac{1}{2} i (R_- - R_+); \tag{3.15}$$

then R_1, R_2, R_3, and $R^2 = R_1{}^2 + R_2{}^2 + R_3{}^2$ satisfy the relations

$$[R_i, R_j] = i\epsilon_{ijk}, \tag{3.16a}$$

$$[R^2, R_j] = 0 . \tag{3.16b}$$

If

$$R^2 |r, m\rangle = r(r+1)|r, m\rangle, \quad R_3 |r, m\rangle = m |r, m\rangle, \tag{3.17}$$

where r and m are integers or half-integers, satisfying $0 \leq r \leq \frac{1}{2}N$ and $-r \leq m \leq r$; then

$$R_- |r, m\rangle = [(r+m)(r-m+1)]^{1/2} |r, m-1\rangle \tag{3.18a}$$
$$= C_{r,m} |r, m-1\rangle . \tag{3.18b}$$

The transition operators are defined as[10]

$$\mathcal{P}_{r,m;r,m'}(t) = e^{iHt}\,|r,m\rangle\langle r,m'|\,e^{-iHt}, \quad (3.19)$$

and the equations of motion are

$$\begin{aligned}\dot{\mathcal{P}}_{r,m;r,m'} &= i(m-m')\epsilon\,\mathcal{P}_{r,m;r,m'} \\ &+ i\sum_k g_k\{a_k^{\dagger}[C_{r,m}\,\mathcal{P}_{r,m-1;r,m'} \\ &- C_{r,m'+1}\,\mathcal{P}_{r,m;r,m'+1}] \\ &+ [C_{r,m+1}\,\mathcal{P}_{r,m+1;r,m'} \\ &- C_{r,m'}\,\mathcal{P}_{r,m;r,m'-1}]\,a_k\}, \end{aligned} \quad (3.20)$$

where all operators are to be evaluated at time $t \gg \epsilon^{-1}$. Since the same subscript r is carried along in each term of (3.20), we will simplify the notation by dropping r from all subsequent expressions. Substituting (3.8) into (3.20) and using arguments similar to those that led to (3.11) and (3.12), we obtain the *linear* set of equations

$$\begin{aligned}\dot{\mathcal{P}}_{m,m'}(t) &= -\nu_{m,m'}\,\mathcal{P}_{m,m'}(t) \\ &+ \gamma C_{m+1}C_{m'+1}\,\mathcal{P}_{m+1,m'+1}(t) \\ &+ i\sum_k g_k\{a_k^{(0)\dagger}(t)[C_m\,\mathcal{P}_{m-1,m'}(t) \\ &- C_{m'+1}\,\mathcal{P}_{m,m'+1}(t)] + [C_{m+1}\,\mathcal{P}_{m+1,m'}(t) \\ &- C_{m'}\,\mathcal{P}_{m,m'-1}(t)]\,a_k^{(0)}(t)\}, \end{aligned} \quad (3.21)$$

where $a_k^{(0)}(t) = a_k(0)\exp(-i\omega_k t)$, and

$$\begin{aligned}\nu_{m,m'} &\equiv \nu_{r,m;r,m'} \\ &\equiv \tfrac{1}{2}(C_m^{\,2} + C_{m'}^{\,2})\gamma \\ &+ i(C_m^{\,2} - C_{m'}^{\,2})\Omega - i(m-m')\epsilon. \end{aligned} \quad (3.22)$$

For the single atom case, where $\mathcal{P}_{-\frac{1}{2},\frac{1}{2}} = \sigma$ and $\mathcal{P}_{\frac{1}{2},\frac{1}{2}} = Q$, expressions (3.21) reduce to (3.13) and (3.14).

If general solutions to Eqs. (3.21) can be found, then the system operators would follow from identities such as

$$R_-(t) = \sum_r \sum_{m=-r}^{r} C_m\,\mathcal{P}_{m-1,m}(t), \quad (3.23a)$$

$$R_+(t) = R_-^{\dagger}(t), \quad (3.23b)$$

$$R_3(t) = \sum_r \sum_{m=-r}^{r} m\,\mathcal{P}_{m,m}(t); \quad (3.24)$$

however, it is usually sufficient to solve for the average values of such operators. With this in mind, we consider the reduced density matrix

$$\rho^{(S)}_{m',m}(t) = \langle \mathcal{P}_{m,m'}(t)\rangle. \quad (3.25)$$

If no photons are present initially; i.e.,

$$\rho(0) = \rho^{(S)}(0)\,|0\rangle\langle 0|, \quad (3.26)$$

where $|0\rangle$ is the vacuum state, then Eqs. (3.21) and (3.25) lead to the result

$$\begin{aligned}\dot{\rho}^{(S)}_{m',m}(t) &= -\nu_{m,m'}\,\rho^{(S)}_{m',m}(t) \\ &+ \gamma C_{m+1}C_{m'+1}\rho^{(S)}_{m'+1,m+1}(t), \end{aligned} \quad (3.27)$$

with the solutions

$$\begin{aligned}\rho^{(S)}_{m',m}(t) &= \exp(-\nu_{m,m'}t)[\rho_{m',m}(0) \\ &+ \gamma C_{m+1}C_{m'+1}\int_0^t dt' \\ &\times \exp(\nu_{m,m'}t')\rho^{(S)}_{m'+1,m+1}(t')]. \end{aligned} \quad (3.28)$$

When either m or m' equals r, the solution reduces to

$$\rho_{m',m}(t) = \rho_{m',m}(0)\exp(-\nu_{m,m'}t).$$

From Eqs. (3.6), (3.7), and (3.12), the average radiation rate is

$$\begin{aligned}W(t) &= \frac{d}{dt}\sum_k \langle a_k^{\dagger}(t)a_k(t)\rangle \\ &= \gamma\langle R_+(t)R_-(t)\rangle; \end{aligned} \quad (3.29)$$

and according to (3.23) and (3.25), this can be written as

$$W(t) = \gamma \sum_{r,m} C_m^{\,2}\rho^{(S)}_{m,m}(t). \quad (3.30)$$

According to postulate (iii) in the introduction, one can obtain correlation functions such as $\langle R_+(t)R_-(t')\rangle$ from a knowledge of $\langle R_+(t)\rangle$ and $\rho_{m,m}(t')$. To see this, we note that according to (3.23),

$$\begin{aligned}\langle R_+(t)\rangle &= \sum_{r,m} C_m\rho^{(S)}_{m-1,m}(t) \\ &= \sum_{r,m} F_m(t)\rho^{(S)}_{m-1,m}(0) \end{aligned} \quad (3.31a)$$

$$= \sum_{r,m} F_m(t) \langle \mathcal{O}_{m,m-1}(0) \rangle, \tag{3.31b}$$

where coefficients $F_m(t) \equiv F_{r,m}(t)$ are obtained from the solutions of Eqs. (3.27). The correlation function can be expressed as

$$\langle R_+(t) R_-(t') \rangle = \mathrm{Tr}\{\rho(t') e^{iH(t-t')} \times R_+(0) e^{-iH(t-t')} R_-(0)\}, \tag{3.32}$$

where $\rho(t') = e^{-iHt'} \rho(0) e^{iHt'}$,

and $t \geq t'$. If

$$\rho(t') \simeq \rho^{(S)}(t') |0\rangle\langle 0| \tag{3.33}$$

is a valid approximation, then

$$\langle R_+(t) R_-(t') \rangle \simeq \mathrm{Tr}\{\rho^{(S)}(t') \times \langle 0| e^{iH(t-t')} R_+(0) e^{-iH(t-t')} |0\rangle R_-(0)\} . \tag{3.34}$$

Expanding $R_+(0)$ according to (3.23), and noting that the system operators[11]

$$\langle 0| e^{iH(t-t')} \mathcal{O}_{m,m-1}(0) e^{-iH(t-t')} |0\rangle$$

obey Eqs. (3.27), we can rewrite (3.34) as

$$\sum_{r,m} F_m(t-t') \mathrm{Tr}\{\rho^{(S)}(t') |0\rangle \times \langle 0| \mathcal{O}_{m,m-1}(0) R_-(0)\} . \tag{3.35}$$

Using (3.33) again, we finally obtain

$$\langle R_+(t) R_-(t') \rangle \simeq \sum_{r,m} F_m(t-t') \langle \mathcal{O}_{m,m-1}(t') \times R_-(t') \rangle \tag{3.36a}$$

$$= \sum_{r,m} F_m(t-t') C_m \rho^{(S)}_{m,m}(t') , \tag{3.36b}$$

which is a statement of the fluctuation-regression theorem[6-8] relating $\langle R_+(t) R_-(t') \rangle$ to $\langle R_+(t) \rangle$.

In general, this result should be exact only if $t' = 0$. It would therefore be instructive to compute the exact (or nearly exact) value of $\langle R_+(t) \times R_-(t') \rangle$ that can be obtained from Eqs. (3.21), provided that the number of atoms in the system is small. Specifically, we note that if (3.21) is multiplied on the right by $|0\rangle$, then the last set of terms disappears, and the solutions can be written as

$$\mathcal{O}_{m,m'}(t)|0\rangle = \exp(-\nu_{m,m'} t)\{\mathcal{O}_{m,m'}(0) + \gamma C_{m+1} C_{m'+1} \int_0^t dt' \times \exp(\nu_{m,m'} t') \mathcal{O}_{m+1,m'+1}(t') + i \sum_k g_k a_k^\dagger(0) \int_0^t dt' \exp(\nu_{m,m'} t') \times \exp(i\omega_k t')[C_m \mathcal{O}_{m-1,m'}(t') - C_{m'+1} \mathcal{O}_{m,m'+1}(t')]\} |0\rangle . \tag{3.37}$$

Since $C_{-r} = C_{r+1} = 0$, the solution for $\mathcal{O}_{-r,r}(t)|0\rangle$ is

$$\exp(-\nu_{-r,r} t) \mathcal{O}_{-r,r}(0)|0\rangle .$$

This can be substituted back into (3.37) to obtain $\mathcal{O}_{-r+1,r}(t)|0\rangle$, which will, in turn, enable one to calculate $\mathcal{O}_{-r+2,r}(t)|0\rangle$, and so on. Proceeding in this way, one can evaluate the required operators in the sequence $(-r, r)$; $(-r+1, r)$; ...; $(r-1, r)$; $(-r, r-1)$; $(-r+1, r-1)$; ...; $(r-2, r-1)$; ...; $(-r, -r+1)$. The $(m-1, m)$ operators may then be combined according to (3.23) to find $R_-(t)|0\rangle$.

As an illustration of the results (3.27) to (3.37), we consider the two-atom case, for which r can be 0 or 1. Equations (3.18), (3.22), and (3.28) yield the solutions

$$\rho^{(S)}_{0,0}(t) = \rho^{(S)}_{0,0}(0), \quad \text{for } r = 0, \tag{3.38}$$

and, for $r = 1$, (and ignoring Ω)

$$\rho^{(S)}_{1,1}(t) = e^{-2\gamma t} \rho^{(S)}_{1,1}(0) , \tag{3.39a}$$

$$\rho^{(S)}_{0,0}(t) = e^{-2\gamma t} \times [\rho^{(S)}_{0,0}(0) + 2\gamma t \rho^{(S)}_{1,1}(0)] , \tag{3.39b}$$

$$\rho^{(S)}_{0,1}(t) = e^{(i\epsilon - 2\gamma)t} \rho^{(S)}_{0,1}(0) , \tag{3.39c}$$

$$\rho^{(S)}_{-1,0}(t) = e^{(i\epsilon - \gamma)t} \times [\rho^{(S)}_{-1,0}(0) + 2(1 - e^{-\gamma t}) \rho^{(S)}_{0,1}(0)] . \tag{3.39d}$$

Equation (3.30) then gives

$$W(t) = 2\gamma[(1 + 2\gamma t) \rho^{(S)}_{1,1}(0) + \rho^{(S)}_{0,0}] e^{-2\gamma t} . \tag{3.40}$$

Expression (3.31b) takes the form

$$\langle R_+(t) \rangle = 2^{1/2} e^{(i\epsilon - \gamma)t} [(2 - e^{-\gamma t}) \times \langle \mathcal{O}_{1,0}(0) \rangle + \langle \mathcal{O}_{0,-1}(0) \rangle] ; \tag{3.41}$$

hence according to (3.36), (3.39a), and (3.39b),

$$\langle R_+(t)R_-(t')\rangle = 2e^{(i\epsilon-\gamma)t}e^{-(i\epsilon+\gamma)t'} \times\{\rho_{0,0}^{(S)}(0)+[2(1+\gamma t')-e^{-\gamma(t-t')}]\rho_{1,1}^{(S)}(0)\}. \quad (3.42)$$

[Note that this reduces to $W(t)/\gamma$ if $t'=t$.] The exact calculation, based on (3.37), turns out to give the same result as (3.36), at least for this example. The operators are

$$\mathcal{O}_{-1,1}(t)|0\rangle = e^{-(\gamma+2i\epsilon)t}\mathcal{O}_{-1,1}(0)|0\rangle, \quad (3.43a)$$

$$\mathcal{O}_{0,1}(t)|0\rangle = e^{-(\gamma+i\epsilon)t}\left(e^{-\gamma t}\mathcal{O}_{0,1}(0)|0\rangle + 2^{1/2}i\sum_k g_k \frac{e^{-i(\epsilon-\omega_k)t}-e^{-\gamma t}}{\gamma+i(\omega_k-\epsilon)} \times a_k^{\dagger}(0)\mathcal{O}_{-1,1}(0)|0\rangle\right), \quad (3.43b)$$

$$\mathcal{O}_{-1,0}(t)|0\rangle = e^{-(\gamma+i\epsilon)t}\Big\{\mathcal{O}_{-1,0}(0)|0\rangle + 2(1-e^{-\gamma t}) \times \mathcal{O}_{0,1}(0)|0\rangle + 2^{1/2}i\sum_k g_k[\gamma+i(\omega_k-\epsilon)]^{-1} \times\Big[2e^{-\gamma t}-1-\exp[i(\omega_k-\epsilon)t] - i\gamma\frac{e^{i(\omega_k-\epsilon)t}-1}{\omega_k-\epsilon}\Big]a_k^{\dagger}(0)\,\mathcal{O}_{-1,1}(0)|0\rangle\Big\}. \quad (3.43c)$$

Using (3.23), along with the identity

$$\int_{-\infty}^{\infty}\frac{dx}{x^2}\,\frac{(e^{-ixt}-1)(e^{ixt'}-1)}{x^2+\gamma^2} = (\pi/\gamma^3)[2\gamma t' - e^{-\gamma(t-t')} + e^{-\gamma t} + e^{-\gamma t'} - 1], \quad (t\geq t') \quad (3.44)$$

and noting that $\gamma = 2\pi\rho(\epsilon)g^2(\epsilon)$, one can recover expression (3.42) by a lengthy but straightforward computation.

IV. CONCLUSION

Transition operator techniques have enabled one to derive damped equations of motion for atomic variables by a procedure that is not only simpler, but more nearly exact than those normally used in such problems. The calculations were not limited to second order in g_k, nor was it necessary to factorize the density operator at times $t>0$. It was also unnecessary to specify the initial state of the damping radiation; however, when this is specified, the transition operator equations lead immediately to those for the reduced density matrix.

The agreement between Eqs. (2.22)–(24) and the results of Scully and Lamb, and between the two methods of evaluating $\langle R_+(t)R_-(t')\rangle$ in Sec. 3, indicate that the factorization of $\rho(t)$ into system and reservoir portions is a better approximation than it might appear. In this context, it would be of interest to compare the transition operator and density operator techniques for more complicated problems; e.g., a problem where the initial radiation is neither in the vacuum state nor in thermal equilibrium.

---

[1] M. Lax, Phys. Rev. 145, 110 (1966).

[2] M. O. Scully and W. E. Lamb, Jr., Phys. Rev. 159, 208 (1967).

[3] I. R. Senitzky, Phys. Rev. 119, 670 (1960); 131, 2827 (1963).

[4] W. H. Louisell, Radiation and Noise in Quantum Electronics (McGraw-Hill Book Company, Inc., New York, 1964), Secs. 7.1–7.3.

[5] M. Lax, Phys. Rev. 157, 213 (1967).

[6] M. Lax and W. H. Louisell, IEEE J. Quantum Electron. QE-3, 47 (1967).

[7] M. Lax, Phys. Rev. 129, 2342 (1963).

[8] M. O. Scully and W. E. Lamb, Jr., Phys. Rev. 166, 246 (1968).

[9] R. H. Dicke, Phys. Rev. 93, 99 (1954).

[10] The techniques presented here can be applied equally well to the more general operator $|r,m\rangle\langle r',m'|$, where $r\neq r'$.

[11] These operators correspond to the $\tilde{\sigma}_{m-1,m}(t)$ in Lax's notation (Ref. 7).