Exact Consequences of Broken $O(4)$ Symmetry for Regge Trajectories. II. Integer $M \geqslant 1$*

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We obtain the complete set of constraints imposed by broken $O(4)$ symmetry on Regge-daughter sequences having integer $M \geqslant 1$. The constraints determine the pth derivative of all the daughter trajectories at zero energy in terms of a finite number of constants. The conspiring daughters of natural and unnatural parity are highly degenerate for large M; in particular, $\alpha_+(k,t)-\alpha_-(k,t)=O(t^M)$. Results are also obtained for daughter residues.

I. INTRODUCTION

IN a recent paper, the exact consequences of broken $O(4)$ symmetry were derived for Regge-daughter sequences corresponding to Toller poles with $M=0$.[1] In the present paper we apply the methods developed in I to processes involving particles with spin. Specifically, we study boson-boson scattering of a sufficiently general kind so that daughter sequences corresponding to Toller poles of arbitrary integer M can contribute. We then isolate the contribution for a definite value of M and analyze it as in I. The results of this paper and I, taken together, display the constraints imposed by broken $O(4)$ symmetry on any boson-daughter sequence. Presumably the general fermion sequence (half-integer M) can be analyzed by our methods, but we do not do so here. We refer the reader to the Introduction of I for a discussion of the background of our work, and the relation of our results to those of other people.

Our procedure is to analyze the elastic, unequal-mass t-channel process

$$S+T(s)\to T(s)+S. \tag{1}$$

Here S is a scalar meson of mass μ and $T(s)$ is a tensor meson of integer spin s, natural parity $(-1)^s$, and mass m. For convenience the t-channel scattering angle is measured between the initial-scalar and final-tensor mesons.[2] As in I, the contribution of a single t-channel Regge pole to any of the full helicity amplitudes for process (1) violates Mandelstam analyticity at $t=0$. The contradiction between the Regge representation and the Mandelstam representation is repaired by introducing daughter trajectories which are spaced at integer steps below the parent at $t=0$. A systematic reduction of the conditions imposed by analyticity leads to constraints among the daughter trajectories at $t\neq 0$. For reasons which are outlined in the Introduction of I, we term these constraints as being imposed by broken $O(4)$ symmetry.

The most delicate point of the calculation is to isolate the contribution of a Toller pole of given M to the unequal-mass process (1). For equal-mass processes of type (1), group theory asserts that only Toller poles with $M\leqslant s$ can contribute, and specifies how a given Toller pole enters.[3] As far as the matter has been studied explicitly ($s=0$ and 1), only Toller poles with $M\leqslant s$ occur in unequal-mass processes.[4] In addition, the general manner in which analyticity is restored to unequal-mass process (1) strongly supports the conjecture that Toller poles with $M\leqslant s$ can contribute, and also suggests how they enter.

To spell this out, we review the insights gained from Ref. 4 and similar papers, and how they generalize for high-spin tensor mesons in process (1). We shall study modified t-channel helicity amplitudes $F_{\lambda'\lambda}(t,u)$, which are known on general grounds to be analytic in a neighborhood of $t=0$ for all u. If neither λ' nor λ is zero, Regge poles of both natural and unnatural parity can contribute; otherwise only natural-parity Regge poles can contribute. We now consider the leading contribution to $F_{\lambda'\lambda}$ at large u, which for finite t is due to the parent t-channel Regge pole. For $M=0$, only a natural or unnatural parent is present, and its appropriately defined reduced residue must therefore be analytic at $t=0$. For $M\geqslant 1$, there are conspiring parents which are degenerate at $t=0$. In this case, the reduced parent residues need not be analytic at $t=0$ when both parents contribute, i.e., when $\lambda'\neq 0$ and $\lambda\neq 0$. They can have cancelling singularities, and in fact they *must* have cancelling singularities if there is to be a real conspiracy and not just a linear superposition of nonconspiring daughter sequences. For $s=1$, a cancellation of simple poles can occur in F_{11}, and when we restore the analyticity of F_{11} and $F_{1,-1}$ we verify explicitly in Ref. 4 that we are dealing with an $M=1$ Toller pole as it couples to these helicity amplitudes in unequal-mass vector-scalar scattering. For $s=2$, we have the new possibility of a cancellation of double poles at $t=0$ in the reduced residues in F_{22}. In this case, when we restore the analyticity we find new, more restrictive constraints on the

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[1] J. B. Bronzan, Phys. Rev. **180**, 1423 (1969). This paper, and equations contained in it, are referred to as I.

[2] With this choice one can directly use the kinematic discussion in E. Leader, Phys. Rev. **166**, 1599 (1967).

[3] D. Z. Freedman and J. M. Wang, Phys. Rev. **160**, 1560 (1967).

[4] J. B. Bronzan, Phys. Rev. **178**, 2302 (1969).

daughter trajectories. We cannot be dealing with $M=0$ or $M=1$ sequences here, since we have already analyzed such sequences and found less restrictive constraints. We conjecture that we are dealing with an $M=2$ sequence. Note that, according to our conjecture, Toller poles with $M\leqslant s$ contribute to process (1), just as for equal-mass scattering. The conclusion of this discussion is that we determine the constraints of broken $O(4)$ symmetry on a Regge-daughter sequence of quantum number M by restoring the analyticity of F_{MM} and $F_{M,-M}$ with the parent residues of natural (+) and unnatural (−) parity having the maximum allowed singularity t^{-M} at $t=0$.[5] We also determine the constraints imposed by broken $O(4)$ symmetry on the reduced residues $\gamma_{MM}^{\pm}$, but of course these results pertain to only two of the helicity amplitudes involved in the particular process (1). The constraints on the trajectories are guaranteed by factorization to be universally valid for sequences of the given M. In particular, they will apply to equal-mass scattering, where $O(4)$ is a symmetry group at $t=0$, and the symmetry is broken by $t\neq 0$.

The constraint we find on the Regge-daughter sequence is that the trajectories can be written as a power series about $t=0$ in the form

$$\alpha_{\pm}(k,t)=\alpha_0-k+\sum_{n=0}^{\infty}\frac{1}{(n+1)!}\frac{\partial^n}{\partial\alpha_0{}^n}\Bigg(\sum_{q=1}^{\infty}t^q\sum_{i=0}^{q}A_i{}^{q+}\times\frac{k!\Gamma(2\alpha_0-k+2)}{(k-i)!\Gamma(2\alpha_0-k-i+2)}\pm\sum_{q=M}^{\infty}t^q\sum_{i=0}^{q-M}A_i{}^{q-}\times\frac{k!\Gamma(2\alpha_0-k+2)\Gamma(\alpha_0-k+M+1)}{(k-i)!\Gamma(2\alpha_0-k-i+2)\Gamma(\alpha_0-k-M+1)}\Bigg)^{n+1}. \quad (2)$$

Here α_0 is the intercept of the conspiring parents at $t=0$, $k=0, 1, \cdots,$ is the daughter index, and the $A_i{}^{q\pm}$ are independent of α_0. Equation (2) is valid for $M\geqslant 1$. As in I, Eq. (2) can be expanded in t on both sides to obtain constraints on the derivatives of the trajectories. When Eq. (2) is used in this way as a generator of constraints on the derivatives, the strong restrictions imposed by broken $O(4)$ symmetry are striking. For example, for $M=1$ we find, after an appropriate redefinition of $A_i{}^{2\pm}$, that

$$\alpha_{\pm}{}^{(1)}(k)=A_0{}^{1+}+A_1{}^{1+}k(2\alpha_0-k+1)\pm A_0{}^{1-}(\alpha_0-k+1)(\alpha_0-k), \quad (3)$$

$$\begin{aligned}\alpha_{\pm}{}^{(2)}(k)=&2A_0{}^{2+}+2A_1{}^{2+}k(2\alpha_0-k+1)\\&+2A_2{}^{2+}k(k-1)(2\alpha_0-k+1)(2\alpha_0-k)\\&\pm 2A_0{}^{2-}(\alpha_0-k+1)(\alpha_0-k)\\&\pm 2A_1{}^{2-}(\alpha_0-k+1)(\alpha_0-k)k(2\alpha_0-k+1)\\&+4k(A_1{}^{1+}\mp A_0{}^{1-})[A_0{}^{1+}+A_1{}^{1+}k(2\alpha_0-k+1)\\&\pm A_0{}^{1-}(\alpha_0-k+1)(\alpha_0-k)],\end{aligned}$$

which are equivalent to the results of Ref. 4. A systematic prescription for expanding the right side of Eq. (2) in t is given in I.

According to Eq. (2), there is a high degree of degeneracy of the conspiring daughter sequences near $t=0$, and this degeneracy increases with M:

$$\alpha_+(k,t)-\alpha_-(k,t)=O(t^M). \quad (4)$$

The degeneracy is needed to assure the cancellation of the singular reduced residues, and clearly shows that M should be regarded as an index of the degree of conspiracy. It is interesting that this simple physical meaning is not evident until one breaks $O(4)$ symmetry by $t\neq 0$. However, even Eq. (4) does not display the full consequences of conspiracy. When $M=0$, $\alpha^{(1)}(0)$ and $\alpha^{(1)}(1)$ can be regarded as independent. When $M=1$, Eqs. (3) states that only three of the four slopes $\alpha_{\pm}{}^{(1)}(0)$ and $\alpha_{\pm}{}^{(1)}(1)$ are independent. They are related by the equation

$$\alpha_+{}^{(1)}(1)-\alpha_-{}^{(1)}(1)=\frac{\alpha_0-1}{\alpha_0+1}[\alpha_+{}^{(1)}(0)-\alpha_-{}^{(1)}(0)], \quad (5)$$

which is also a reflection of conspiracy.

II. ANALYTICITY CONDITIONS

A. Step I

We follow I closely in reducing the analyticity conditions, and present in detail only the new features occasioned by spin and conspiracy. The analysis presented in Ref. 4 can be used to show that t-channel helicity amplitudes $F_{\lambda'\lambda}(t,u)$ which are analytic in a neighborhood of $t=0$ for all u are related to ordinary helicity amplitudes $T_{\lambda'\lambda}(t,u)$ by

$$F_{\lambda'\lambda}(t,u)=T_{\lambda'\lambda}(t,u)/[\tfrac{1}{2}(1-z_t)]^{|\lambda'+\lambda|/2}, \quad (6)$$

where

$$\begin{aligned}z_t&=\cos\theta_t=-1-u/2p^2,\\p^2&=[t-(m+u)^2][t-(m-u)^2]/4t.\end{aligned} \quad (7)$$

It can also be shown by the methods given in Ref. 4 that the contributions of Regge poles of natural and unnatural parity to the amplitudes of interest are

$$\begin{aligned}F_{MM}(t,u)=&\gamma_{MM}{}^+(t)u^{\alpha_+(t)-M}\\&\times F(-\alpha_+(t)+M,\,-\alpha_+(t)+M,\,-2\alpha_+(t),\,-4p^2/u)\\&+\gamma_{MM}{}^-(t)u^{\alpha_-(t)-M}F(-\alpha_-(t)+M,\,-\alpha_-(t)+M,\\&-2\alpha_-(t),\,-4p^2/u),\\(-1)^MF_{M,-M}(t,u)=&[\gamma_{MM}{}^+(t)u^{\alpha_+(t)}/(4p^2)^M]\\&\times F(-\alpha_+(t)+M,\,-\alpha_+(t)-M,\,-2\alpha_+(t),\,-4p^2/u)\\&-[\gamma_{MM}{}^-(t)u^{\alpha_-(t)}/(4p^2)^M]F(-\alpha_-(t)+M,\\&-\alpha_-(t)-M,\,-2\alpha_-(t),\,-4p^2/u).\end{aligned} \quad (8)$$

Here $\gamma_{MM}{}^{\pm}(t)$ are reduced residues for Regge poles of natural and unnatural parity. When Regge poles of both parities are present and there is conspiracy, $\alpha_+(0)=\alpha_-(0)=\alpha_0$, the leading contributions to these

[5] If the residues are allowed to be less singular that t^{-M}, the corresponding Toller pole has $M'<M$. The identification of M with the degree of singularity of reduced residues has been suggested by others; for example, A. Capella, A. P. Contogouris and J. Tran Thanh Van, Orsay Report, 1968 (unpublished).

amplitudes for large u are analytic at $t=0$ if $\gamma_{MM}^{+}(t)+\gamma_{MM}^{-}(t)$ and $t^M[\gamma_{MM}^{+}(t)-\gamma_{MM}^{-}(t)]$ are analytic at $t=0$. Therefore, $\gamma_{MM}^{\pm}(t)$ may be as singular as t^{-M} at $t=0$, as stated in the Introduction. Of course, the subasymptotic terms in u are not analytic at $t=0$ because of p^2 in the arguments of the hypergeometric functions. To obtain analyticity at $t=0$ for all u, we introduce daughter trajectories $\alpha_{\pm}(k,t)$, $k=0, 1, 2, \cdots$, with $\alpha_{\pm}(k,0)=\alpha_0-k$. The daughter reduced residues must behave like t^{-k-M} at $t=0$ to restore analyticity. We sum over the contributions of the daughters and find that the amplitudes become

$$F_{MM}(t,u)=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{u^{\alpha_0-r-M}(\ln u)^s}{s!t^{r+M}}(4tp^2)^r\sum_{k=0}^{r}\Bigg([\alpha_+(k,t)-\alpha_0+k]^s R_1^{+}(k,\alpha_+(k,t))\times\frac{\Gamma(2\alpha_+(k,t)-r+k+1)}{(r-k)![\Gamma(\alpha_+(k,t)-r+k-M+1)]^2}+(\text{term with subscripts}+\to-)\Bigg),$$

$$(-1)^M F_{M,-M}(t,u)=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{u^{\alpha_0-r}(\ln u)^s}{s!t^{r}}(4tp^2)^{r-M}\sum_{k=0}^{r}\Bigg([\alpha_+(k,t)-\alpha_0+k]^s R_2(k,\alpha_+(k,t))\times\frac{\Gamma(2\alpha_+(k,t)-r+k+1)}{(r-k)!\Gamma(\alpha_+(k,t)-r+k-M+1)\Gamma(\alpha_+(k,t)-r+k+M+1)}-(\text{term with subscripts}+\to-)\Bigg), \tag{9}$$

where

$$R_1^{\pm}(k,\alpha_{\pm}(k,t))=\frac{t^{k+M}\gamma_{MM}^{\pm}(k,t)(4tp^2)^{-k}[\Gamma(\alpha_{\pm}(k,t)-M+1)]^2}{\Gamma(2\alpha_{\pm}(k,t)+1)},$$

$$R_2^{\pm}(k,\alpha_{\pm}(k,t))=R_1^{\pm}(k,\alpha_{\pm}(k,t))\frac{\Gamma(\alpha_{\pm}(k,t)+M+1)}{\Gamma(\alpha_{\pm}(k,t)-M+1)}. \tag{10}$$

As in I, we have written the R's as functions of $\alpha_{\pm}(k,t)$ rather than t, and they are analytic at $t=0$. In writing Eq. (9) we have used the expansions

$$u^{\alpha_{\pm}(k,t)}=u^{\alpha_0-k}\sum_{s=0}^{\infty}\frac{(\ln u)^s}{s!}[\alpha_{\pm}(k,t)-\alpha_0+k]^s. \tag{11}$$

In order that the amplitudes in Eq. (9) be analytic at $t=0$, we require

$$\sum_{k=0}^{r}\frac{\partial^q}{\partial t^q}\Bigg(R_1^{+}(k,\alpha_+(k,t))[\alpha_+(k,t)-\alpha_0+k]^s\times\frac{\Gamma(2\alpha_+(k,t)-r+k+1)}{(r-k)![\Gamma(\alpha_+(k,t)-r+k-M+1)]^2}+(+\to-)\Bigg)_{t=0}=0\quad(q<r+M,\quad 0\leqslant s),$$

$$\sum_{k=0}^{r}\frac{\partial^q}{\partial t^q}\Bigg(R_2^{+}(k,\alpha_+(k,t))[\alpha_+(k,t)-\alpha_0+k]^s\times\frac{\Gamma(2\alpha_+(k,t)-r+k+1)}{(r-k)!\Gamma(\alpha_+(k,t)-r+k-M+1)\Gamma(\alpha_+(k,t)-r+k+M+1)}-(+\to-)\Bigg)_{t=0}=0\quad(q<r,\quad 0\leqslant s). \tag{12}$$

Equations (12) are analogous to Eq. (I11). The rest of the analysis of step I in I can be repeated, starting with Eq. (12). The analyticity conditions emerge in the form analogous to Eq. (I24):

$$\sum_{k=0}^{r}M_{rk}(\alpha_0)\sum_{w=0}^{\infty}\sum_{u=0}^{\infty}\frac{(-1)^u}{w!u!}\frac{\partial^u}{\partial\alpha_0{}^u}[R_1^{+(w)}(k,\alpha_0-k)f_+(k,\alpha_0,q,w+u+s)+(+\to-)]=0\,(q<r+M,\ 0\leqslant s\leqslant q),$$

$$\sum_{k=0}^{r}M_{rk}(\alpha_0)\sum_{w=0}^{\infty}\sum_{u=0}^{\infty}\frac{(-1)^u}{w!u!}\frac{\partial^u}{\partial\alpha_0{}^u}[R_2^{+(w)}(k,\alpha_0-k)f_+(k,\alpha_0,q,w+u+s)-(+\to-)]=0\,(q<r,\ 0\leqslant s\leqslant q). \tag{13}$$

Here we have introduced the symbols

$$R^{\pm(w)}(k,\alpha_0-k)=\frac{\partial^w}{\partial[\alpha_\pm(k,t)]^w}R^{\pm}(k,\alpha_\pm(k,t))|_{\alpha_0-k}\,,\quad M_{rk}(\alpha_0)=\Gamma(2\alpha_0-r-k+1)/(r-k)!\,, \tag{14}$$

$$f_\pm(k,\alpha_0,q,n)=q!n!\sum''_{\substack{m_1,\cdots,m_q\\(n)}}\prod_{i=1}^{q}\frac{1}{m_i!}\left(\frac{\alpha_\pm^{(i)}(k)}{i!}\right)^{m_i},$$

and $\sum''$ means sum over all sets of q non-negative integers $\{m_i\}$ subject to the restrictions

$$\sum_{i=1}^{q}m_i=n\,,\quad \sum_{i=1}^{q}im_i=q. \tag{15}$$

$f_\pm(k,\alpha_0,q,n)/q!$ is the coefficient of t^q in the expansion of $[\alpha_\pm(k,t)-\alpha_0+k]^n$. We write $f_\pm$ as a function of α_0 rather than the trajectory derivatives in anticipation that the result of our study will be to express the derivatives $\alpha_\pm^{(i)}(k)$ in terms of α_0 and parameters, as in Eq. (3). In step II we shall write $\alpha_\pm(k,t,\alpha_0)$, recognizing that the trajectories as a whole share this property.

B. Step II

We use the inverse of $M_{rk}(\alpha_0)$,[4]

$$M^{-1}{}_{ki}(\alpha_0)=\frac{(-1)^{k-i}(2\alpha_0-k+1)}{(k-i)!\Gamma(2\alpha_0-k-i+2)}, \tag{16}$$

to invert the analyticity conditions, Eqs. (13). After manipulations similar to those in I, we obtain the analyticity conditions in a form analogous to Eq. (I32):

$$\frac{R_1^+(k,\alpha_0-k)[-z_0^+(k,t,\alpha_0)]^s}{\partial\alpha_+(k,t,\alpha_0)/\partial\alpha_0|_{\alpha_0+z_0^+(k,t,\alpha_0)}}+(+\to-)=\sum_{q=s}^{\infty}t^q\sum_{i=0}^{q-M}B_{1i}{}^{(q,s)}M^{-1}{}_{ki}(\alpha_0)\,, \tag{17a}$$

$$\frac{R_2^+(k,\alpha_0-k)[-z_0^+(k,t,\alpha_0)]^s}{\partial\alpha_+(k,t,\alpha_0)/\partial\alpha_0|_{\alpha_0+z_0^+(k,t,\alpha_0)}}-(+\to-)=\sum_{q=s}^{\infty}t^q\sum_{i=0}^{q}B_{2i}{}^{(q,s)}M^{-1}{}_{ki}(\alpha_0)\,. \tag{17b}$$

In Eqs. (17), the B's are functions of α_0 with sufficient analyticity to permit the derivation of Eq. (17).[6] The coefficient of t^q in Eq. (17a) is zero for $q<M$. The z's are solutions of the equations

$$\alpha_\pm(k,\,t,\,\alpha_0+z_0^\pm(k,t,\alpha_0))=\alpha_0-k\,,\qquad \lim_{t\to0}z_0^\pm(k,t,\alpha_0)=0. \tag{18}$$

Using Eq. (10), we can eliminate R_1 and write the analyticity conditions in the form

$$\frac{R_2^\pm(k,\alpha_0-k)[-z_0^\pm(k,t,\alpha_0)]^s}{\partial\alpha_\pm(k,t,\alpha_0)/\partial\alpha_0|_{\alpha_0+z_0^\pm(k,t,\alpha_0)}}=\tfrac12\sum_{q=s}^{\infty}t^q\sum_{i=0}^{q-M}B_{1i}{}^{(q,s)}\frac{\Gamma(\alpha_0-k+M+1)}{\Gamma(\alpha_0-k-M+1)}M^{-1}{}_{ki}(\alpha_0)\pm\tfrac12\sum_{q=s}^{\infty}t^q\sum_{i=0}^{q}B_{2i}{}^{(q,s)}M^{-1}{}_{ki}(\alpha_0)\,. \tag{19}$$

We next use Eq. (19) with $s=0$ and $s=1$ to derive equations for $z_0^\pm$. It is convenient to introduce

$$y^\pm(k,t,\alpha_0)=-\tfrac12z_0^+(k,t,\alpha_0)\mp\tfrac12z_0^-(k,t,\alpha_0)\,. \tag{20}$$

Then, Eq. (19) for $s=0$ and $s=1$ shows that

$$\begin{aligned}y^+(k,t,\alpha_0)&\left[\left(\sum_{q=0}^{\infty}t^q\sum_{i=0}^{q-M}B_{1i}{}^{(q,0)}\frac{\Gamma(\alpha_0-k+M+1)}{\Gamma(\alpha_0-k-M+1)}M^{-1}{}_{ki}(\alpha_0)\right)^2-\left(\sum_{q=0}^{\infty}t^q\sum_{i=0}^{q}B_{2i}{}^{(q,0)}M^{-1}{}_{ki}(\alpha_0)\right)^2\right]\\&=\left(\sum_{q=0}^{\infty}t^q\sum_{i=0}^{q-M}B_{1i}{}^{(q,0)}\frac{\Gamma(\alpha_0-k+M+1)}{\Gamma(\alpha_0-k-M+1)}M^{-1}{}_{ki}(\alpha_0)\right)\left(\sum_{q=1}^{\infty}t^q\sum_{i=0}^{q-M}B_{1i}{}^{(q,1)}\frac{\Gamma(\alpha_0-k+M+1)}{\Gamma(\alpha_0-k-M+1)}M^{-1}{}_{ki}(\alpha_0)\right)\\&\quad-\left(\sum_{q=0}^{\infty}t^q\sum_{i=0}^{q}B_{2i}{}^{(q,0)}M^{-1}{}_{ki}(\alpha_0)\right)\left(\sum_{q=1}^{\infty}t^q\sum_{i=0}^{q}B_{2i}{}^{(q,1)}M^{-1}{}_{ki}(\alpha_0)\right),\end{aligned}$$

[6] In I we required that the B's be entire functions of α_0. However, the discussion of Sec. III of I shows that this requirement is overly stringent. It is only necessary that there exist a point α_0^* such that the B's are analytic within a circle of radius greater than one about α_0^*. Equation (17) can be derived at α_0^* and then continued arbitrarily close to any singularity of the B's. We further stated in I that it can be shown that the parameters in the final equations, (2) and (28), can be assumed to be independent of α_0 without loss of generality.

$$y^-(k,t,\alpha_0)\left[\left(\sum_{q=0}^{\infty} t^q \sum_{i=0}^{q-M} B_{1i}{}^{(q,0)}\frac{\Gamma(\alpha_0-k+M+1)}{\Gamma(\alpha_0-k-M+1)}M^{-1}{}_{ki}(\alpha_0)\right)^2-\left(\sum_{q=0}^{\infty} t^q \sum_{i=0}^{q} B_{2i}{}^{(q,0)}M^{-1}{}_{ki}(\alpha_0)\right)^2\right]$$
$$=\left(\sum_{q=0}^{\infty} t^q \sum_{i=0}^{q-M} B_{1i}{}^{(q,0)}\frac{\Gamma(\alpha_0-k+M+1)}{\Gamma(\alpha_0-k-M+1)}M^{-1}{}_{ki}(\alpha_0)\right)\left(\sum_{q=1}^{\infty} t^q \sum_{i=0}^{q} B_{2i}{}^{(q,1)}M^{-1}{}_{ki}(\alpha_0)\right)$$
$$-\left(\sum_{q=0}^{\infty} t^q \sum_{i=0}^{q} B_{2i}{}^{(q,0)}M^{-1}{}_{ki}(\alpha_0)\right)\left(\sum_{q=1}^{\infty} t^q \sum_{i=0}^{q-M} B_{1i}{}^{(q,1)}\frac{\Gamma(\alpha_0-k+M+1)}{\Gamma(\alpha_0-k-M+1)}M^{-1}{}_{ki}(\alpha_0)\right). \quad (21)$$

If we expand $y^{\pm}$ in power series in t, we can obtain the coefficients from Eqs. (21). These coefficients have a known dependence upon k because of the lemmas

$$\frac{M^{-1}{}_{ki}(\alpha_0)}{M^{-1}{}_{k0}(\alpha_0)}=\sum_{q=0}^{i} c_{q,p}(\alpha_0)\frac{M^{-1}{}_{k,q+p}(\alpha_0)}{M^{-1}{}_{kp}(\alpha_0)}, \quad (22a)$$

$$\frac{\Gamma(\alpha_0-k+M+1)}{\Gamma(\alpha_0-k-M+1)}=\sum_{q=0}^{M} d_{q,M}(\alpha_0)\frac{M^{-1}{}_{kq}(\alpha_0)}{M^{-1}{}_{k0}(\alpha_0)}. \quad (22b)$$

Equation (22a) is derived in the Appendix of I, and Eq. (22b) is derived in the Appendix below. The c's and d's are polynomials in α_0. We use Eqs. (21) and (22) and find, for $M\geqslant 1$,

$$y^+(k,t,\alpha_0)=\sum_{q=1}^{\infty} t^q \sum_{i=0}^{q} A_i{}^{q+}(-1)^i\frac{M^{-1}{}_{ki}(\alpha_0)}{M^{-1}{}_{k0}(\alpha_0)},$$
$$y^-(k,t,\alpha_0)=\sum_{q=M}^{\infty} t^q \sum_{i=0}^{q-M} A_i{}^{q-}\frac{\Gamma(\alpha_0-k+M+1)}{\Gamma(\alpha_0-k-M+1)}\times(-1)^i\frac{M^{-1}{}_{ki}(\alpha_0)}{M^{-1}{}_{k0}(\alpha_0)}, \quad (23)$$

where the A's are functions of α_0. Hence,

$$-z_0{}^{\pm}(k,t,\alpha_0)=\sum_{q=1}^{\infty} t^q \sum_{i=0}^{q} A_i{}^{q+}(-1)^i\frac{M^{-1}{}_{ki}(\alpha_0)}{M^{-1}{}_{k0}(\alpha_0)}\pm\sum_{q=M}^{\infty} t^q \sum_{i=0}^{q-M} A_i{}^{q-}(-1)^i\frac{\Gamma(\alpha_0-k+M+1)}{\Gamma(\alpha_0-k-M+1)}\frac{M^{-1}{}_{ki}(\alpha_0)}{M^{-1}{}_{k0}(\alpha_0)}. \quad (24)$$

In this equation we can see why it is convenient to introduce the y's. Without the y's, one can easily use Eq. (22b) once too often and find the second term in Eq. (24) to be

$$\pm\sum_{q=M}^{\infty} t^q \sum_{i=0}^{q} \bar{A}_i{}^{q-}(-1)^i\frac{M_{ki}{}^{-1}(\alpha_0)}{M^{-1}{}_{ki}(\alpha_0)}.$$

However, the $\bar{A}$'s are not linearly independent. If one is careless and assumes that they are, Eq. (5) is missing, in disagreement with Ref. 4. Hence the y's help keep straight the number of independent parameters in the expressions for the z's. We also point out that Eq. (22b) assures that the expressions for $\alpha_{\pm}(k,t)$ in Eq. (2) are the same as those for $\alpha(k,t)$ in I. This is required by factorization, since in I we have actually calculated the constraints on $F_{00}(t,u)$. Of course, in F_{00} only one of the conspiring sequences contributes [natural parity for the external particles in (1)].

Equation (24) is necessary for analyticity. On the other hand, Eqs. (24) and (19) for $s=0$ imply Eq. (19) for $s>0$. We omit the proof, which consists of the use of Eqs. (20), (22), (23), and the binomial theorem to calculate $[-z_0{}^{\pm}(k,t,\alpha_0)]^s$. Hence necessary and sufficient conditions for analyticity are

$$\frac{R_2{}^{\pm}(k,\alpha_0-k)}{\partial\alpha_{\pm}(k,t,\alpha_0)/\partial\alpha_0|_{\alpha_0+z_0{}^{\pm}(k,t,\alpha_0)}}=\tfrac{1}{2}\sum_{q=M}^{\infty} t^q \sum_{i=0}^{q-M} B_{1i}{}^q\frac{\Gamma(\alpha_0-k+M+1)}{\Gamma(\alpha_0-k-M+1)}M^{-1}{}_{ki}(\alpha_0)\pm\tfrac{1}{2}\sum_{q=0}^{\infty} t^q \sum_{i=0}^{q} B_{2i}{}^q M^{-1}{}_{ki}(\alpha_0),$$
$$-z_0{}^{\pm}(k,t,\alpha_0)=\sum_{q=1}^{\infty} t^q \sum_{i=0}^{q} A_i{}^{q+}\frac{k!\Gamma(2\alpha_0-k+2)}{(k-i)!\Gamma(2\alpha_0-k-i+2)}\pm\sum_{q=M}^{\infty} t^q \sum_{i=0}^{q-M} A_i{}^{q-}\frac{k!\Gamma(2\alpha_0-k+2)\Gamma(\alpha_0-k+M+1)}{(k-i)!\Gamma(2\alpha_0-k-i+2)\Gamma(\alpha_0-k-M+1)}. \quad (25)$$

C. Step III

By analogy with I, we compute the sums

$$S_{\pm}=\sum_{n=0}^{\infty}\frac{1}{n!}\frac{\partial^n}{\partial\alpha_0{}^n}[-z_0{}^{\pm}(k,t,\alpha_0)]^n\frac{R_2{}^{\pm}(k,\alpha_0-k)}{\partial\alpha_{\pm}(k,t,\alpha_0)/\partial\alpha_0|_{\alpha_0+z_0{}^{\pm}(k,t,\alpha_0)}},$$
$$\bar{S}_{\pm}=\sum_{n=0}^{\infty}\frac{1}{n!}\frac{\partial^n}{\partial\alpha_0{}^n}[-z_0{}^{\pm}(k,t,\alpha_0)]^{n+1}\left[1+\frac{\partial z_0{}^{\pm}(k,t,\alpha_0)}{\partial\alpha_0}\right]. \quad (26)$$

Using the methods of I, we find

$$S_{\pm}=R_2^{\pm}(k,\alpha_{\pm}(k,t,\alpha_0))\,,\quad \tilde{S}_{\pm}=\alpha_{\pm}(k,t,\alpha_0)-\alpha_0+k\,. \tag{27}$$

On the other hand, we can evaluate the sums from Eqs. (25). We then obtain explicit expressions for the residues and trajectories[6]:

$$\begin{aligned}
R_2^{\pm}(k,t)=&\tfrac{1}{2}\sum_{n=0}^{\infty}\frac{1}{n!}\frac{\partial^n}{\partial\alpha_0^n}\Bigg[\Bigg(\sum_{r=M}^{\infty}t^r\sum_{j=0}^{r-M}B_{1j}{}^{r}\frac{\Gamma(\alpha_0-k+M+1)}{\Gamma(\alpha_0-k-M+1)}M^{-1}{}_{kj}(\alpha_0)\pm\sum_{r=0}^{\infty}t^r\sum_{j=0}^{r}B_{2j}{}^{r}M^{-1}{}_{kj}(\alpha_0)\Bigg)\\
&\times\Bigg(\sum_{q=1}^{\infty}t^q\sum_{i=0}^{q}A_i{}^{q+}\frac{k!\Gamma(2\alpha_0-k+2)}{(k-i)!\Gamma(2\alpha_0-k-i+2)}\pm\sum_{q=M}^{\infty}t^q\sum_{i=0}^{q-M}A_i{}^{q-}\frac{k!\Gamma(\alpha_0-k+M+1)\Gamma(2\alpha_0-k+2)}{(k-i)!\Gamma(\alpha_0-k-M+1)\Gamma(2\alpha_0-k-i+2)}\Bigg)^n\Bigg],\\
\alpha_{\pm}(k,t)=&\alpha_0-k+\sum_{n=0}^{\infty}\frac{1}{(n+1)!}\frac{\partial^n}{\partial\alpha_0^n}\Bigg(\sum_{q=1}^{\infty}t^q\sum_{i=0}^{q}A_i{}^{q+}\frac{k!\Gamma(2\alpha_0-k+2)}{(k-i)!\Gamma(2\alpha_0-k-i+2)}\\
&\pm\sum_{q=M}^{\infty}t^q\sum_{i=0}^{q-M}A_i{}^{q-}\frac{k!\Gamma(\alpha_0-k+M+1)\Gamma(2\alpha_0-k+2)}{(k-i)!\Gamma(\alpha_0-k-M+1)\Gamma(2\alpha_0-k-i+2)}\Bigg)^{n+1}.
\end{aligned} \tag{28}$$

Equations (28) are necessary for analyticity. However, a method is outlined in I—which can be generalized to the present calculation—by which one can derive Eq. (17) from Eqs. (28). Therefore, Eqs. (28) are both necessary and sufficient for analyticity, and express the full constraints imposed by broken $O(4)$ symmetry on conspiring Regge-daughter sequences which correspond, at $t=0$, to Toller poles having integer $M\geqslant 1$.

APPENDIX

Using the notation $\Gamma(x+n)/\Gamma(x)=(x)_n$, we have

$$M^{-1}{}_{kq}(\alpha_0)/M^{-1}{}_{k0}(\alpha_0)=(-1)^q(k-q+1)_q\times(2\alpha_0+2-k-q)_q\,. \tag{A1}$$

Consider the linear combination

$$I(k)=\sum_{q=0}^{M}d_{q,M}(\alpha_0)(-1)^q(k-q+1)_q\times(2\alpha_0+2-k-q)_q\,. \tag{A2}$$

I is a polynomial of degree $2M$ in k. We fix the $M+1$ coefficients $d_{q,M}(\alpha_0)$ by $d_{M,M}(\alpha_0)=1$ and the M linear inhomogeneous equations $I(\alpha_0-M+k+1)=0$, $k=0, 1, \cdots, M-1$. These equations are

$$0=\sum_{q=0}^{M}d_{q,M}(\alpha_0)(-1)^q(\alpha_0-M+k-q+2)_q\times(\alpha_0+M-k-q+1)_q\quad(0\leqslant k\leqslant M-1)\,. \tag{A3}$$

Now compute

$$I(\alpha_0+M-k)=\sum_{q=0}^{M}d_{q,M}(\alpha_0)(-1)^q(\alpha_0-M+k-q+2)_q\times(\alpha_0+M-k-q+1)_q=0\quad(0\leqslant k\leqslant M-1)\,. \tag{A4}$$

Hence $I(k)$ is the unique polynomial which behaves like k^{2M} at infinity and vanishes at the $2M$ points $k=\alpha_0-M+l+1$, $l=0, 1, \cdots, 2M-1$. Thus

$$\frac{\Gamma(\alpha_0-k+M+1)}{\Gamma(\alpha_0-k-M+1)}=\sum_{q=0}^{M}d_{q,M}(\alpha_0)\frac{M^{-1}{}_{kq}(\alpha_0)}{M^{-1}{}_{k0}(\alpha_0)}\,. \tag{A5}$$

The left side of this equation is a polynomial in α_0. The only singularities $d_{q,M}(\alpha_0)$ can have are poles cancelled by the zeros of $(2\alpha_0+2-k-q)_q$. However, these zeros move with k, and $d_{q,M}(\alpha_0)$ is independent of k. Hence $d_{q,M}(\alpha_0)$ is a polynomial in α_0.