Complex-l-Plane Singularities in the Veneziano Formula*

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The continued partial-wave projection of the Veneziano formula is performed and the complex-l-plane singularities are investigated. It is explicitly shown that in the $\pi\pi$ amplitudes given by the Veneziano-Lovelace model there are an infinite series of Regge poles with parallel trajectories spaced by one unit and an essential singularity as $\mathrm{Re}l \to -\infty$. For the even-signature amplitude, besides the singularities mentioned above, additive fixed poles are shown to be present at nonsense wrong-signature points. The classification of the Regge-pole family in terms of Lorentz poles and the positivity condition for the Regge-pole residues are also discussed.

A FORMULA for a scattering amplitude that obeys the requirements of Regge asymptotics and crossing symmetry in all channels in the case of linearly rising trajectories has been proposed by Veneziano.[1] We study here the continued partial-wave projection of the Veneziano formula.

Although in this paper we limit ourselves to the Veneziano-Lovelace[2] model for $\pi\pi$ scattering, our method can be quite generally extended to study any scattering amplitude of the Veneziano type.

Our main results are the following:

(i) The only singularities in the complex l plane for the odd-signature amplitude are an infinite series of Regge poles with parallel trajectories spaced by one unit and an essential singularity as $\mathrm{Re}l \to -\infty$. Because of the presence of this essential singularity, the infinite series of Regge poles does not converge.

The explicit form of the residue functions shows that they do satisfy the usual analyticity, reality, and threshold properties. Furthermore, they decrease exponentially for large positive $\mathrm{Re}s$, which is consistent with the experimental rapid decrease of the elastic widths, but they diverge exponentially as $\mathrm{Re}s \to -\infty$.

For the even-signature amplitude, besides the singularities mentioned above, additive fixed poles which come from the $A(t,u)$ term [see Eq. (2)] are present at nonsense wrong-signature points.

(ii) At $s=0$, the Regge pole family is analyzed in terms of Lorentz poles.[3] An infinite number of Lorentz poles spaced by one unit are shown to be present.

(iii) The partial-wave amplitude for physical l has the correct analytic properties in s. The positivity condition of the Regge residues at physical points on the trajectories places strong restrictions on the intercepts of the trajectories. In the $\pi\pi$ case, the most stringent condition that we found for the intercept of the degenerate ρ-f^0 trajectory is $\alpha(0)\geq\frac{1}{2}$.[4] It is interesting to remark that the lowest value of the ρ-f^0 intercept consistent with the positivity condition is just the value required[2] in order to satisfy the Adler self-consistency condition.

We define the amplitude[2]

$$A(s,t)=-\gamma\frac{\Gamma(1-\alpha(s))\Gamma(1-\alpha(t))}{\Gamma(1-\alpha(s)-\alpha(t))}. \tag{1}$$

The $\pi\pi$ amplitudes for the three isostates in the s channel are given by

$$\begin{aligned} A^0&=\tfrac{3}{2}[A(s,t)+A(s,u)]-\tfrac{1}{2}A(t,u),\\ A^1&=A(s,t)-A(s,u),\\ A^2&=A(t,u). \end{aligned} \tag{2}$$

γ in Eq. (1) is a constant, and $\alpha(s)$ are the degenerate ρ-f^0 trajectories assumed to take the linear form

$$\alpha(s)=as+b. \tag{3}$$

We wrote the Veneziano-Lovelace formula (1) with only one term, since this has been shown to give a good theoretical description of the available experimental data.[2,5]

First we study the two combinations $[A(s,t)\pm A(s,u)]$. The $\pi\pi$ amplitudes given by Eq. (2) satisfy fixed-s dispersion relations. Therefore, the partial-wave amplitudes of $[A(s,t)\pm A(s,u)]$ continued through the Froissart-Gribov definition for even and odd signatures are given by the single expression

$$a(l,s)=\gamma\frac{\alpha(s)}{aq^2}\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha(s)+1)}{\Gamma(\alpha(s)+1)n!}Q_l\left(1+\frac{n+1-b}{2aq^2}\right). \tag{4}$$

Clearly, the expression (4) defines a holomorphic function of l whenever it converges, i.e., for $\mathrm{Re}l>\mathrm{Re}\alpha(s)$. In order to investigate the l-plane singularities, we have to perform the sum in (4). To this end, we use the fol-

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[1] G. Veneziano, Nuovo Cimento **57A**, 190 (1968).

[2] C. Lovelace, Phys. Letters **28B**, 265 (1968); see also J. Shapiro and J. Yellin (Ref. 10).

[3] M. Toller, Nuovo Cimento **54A**, 295 (1968). A complete list of references can be found in this paper.

[4] This condition is exact in the limit of zero pion mass. For physical pion mass, see Eq. (17).

[5] M. Ademollo, G. Veneziano, and S. Weinberg, Phys. Rev. Letters **22**, 83 (1969).

lowing integral representation[6]:

$$Q_l(z)=\frac{1}{2^{l+1}\Gamma(l+1)}\int_{-1}^{1}dt(1-t^2)^l\int_0^\infty dx\, x^l e^{-(z+t)x},$$

$$\mathrm{Re}l>-1\,,\quad \mathrm{Re}z>1\,. \tag{5}$$

We carry out our summation in the region

$$\mathrm{Re}l>\mathrm{Re}\alpha(s)\,,\quad \mathrm{Re}\left(1+\frac{(1\text{-}b)}{2aq^2}\right)>1\,. \tag{6}$$

Later, we shall analytically continue the final answer in l and s.

Using the representation (5), we get

$$a(l,s)=\gamma\frac{\alpha(s)}{aq^2}\frac{1}{2^{l+1}\Gamma(l+1)}\int_{-1}^{1}dt(1-t^2)^l\int_0^\infty dx\, x^l e^{-(1+t)x}$$
$$\times e^{-[(1-b)/2aq2]x}(1-e^{-x/2aq2})^{-\alpha(s)-1}\,. \tag{7}$$

Here we define a function $f(y)$ by

$$f(y)=e^{-(1-b)y}[(1-e^{-y})/y]^{-\alpha(s)-1}\,. \tag{8}$$

Using Taylor's theorem, we can do the following expansion:

$$f(y)=\sum_{k=0}^{N}\frac{f^{(k)}(0)}{k!}y^k+R_{N+1}(y)\,, \tag{9}$$

with

$$R_{N+1}(y)=\frac{1}{N!}\int_0^y(y-t)^N f^{(N+1)}(t)dt\,, \tag{10}$$

where the non-negative integer N can be chosen arbitrarily large. Note, however, that $R_{N+1}(y)$ does not vanish as $N\to\infty$ unless $0\leq y<2\pi$. The function $f^{(k)}(0)$ is the kth derivative of $f(y)$ at $y=0$ and is a polynomial in $\alpha(s)$ and b.

Using the expansion (9), we can perform the integration in (7). The final answer is

$$a(l,s)=\sum_{k=0}^{N}2\gamma\alpha(s)\frac{f^{(k)}(0)}{k!}(4aq^2)^{\alpha(s)-k}$$
$$\times\frac{\Gamma(\alpha(s)-k+1)\Gamma(l-\alpha(s)+k)}{\Gamma(l+\alpha(s)-k+2)}+\gamma\alpha(s)\frac{\sqrt{\pi}}{2^{2l+1}\Gamma(l+\frac{3}{2})}$$
$$\times(2aq^2)^{\alpha(s)}\int_0^\infty dx\, x^{-\alpha(s)-2}e^{-x}M_{0,l+\frac{1}{2}}(2x)R_{N+1}\left(\frac{x}{2aq^2}\right), \tag{11}$$

where $M_{0,l+\frac{1}{2}}(2x)$ is the Whittaker function.[6] The function $[M_{0,l+\frac{1}{2}}(2x)]/[\Gamma(l+\frac{3}{2})]$ is regular in l. Therefore, the only singularities in l in the last term of (11) are due to the divergence of the integral at the lower limit of integration, where the behavior of the integrand is $O(x^{l+N-\alpha(s)})$. Thus, it is easily seen that the last term in (11) is regular for $\mathrm{Re}l>\mathrm{Re}\alpha(s)-N-1$.

Now, from the expression (11), the analytic structure in l is apparent. By choosing N arbitrarily large, we can conclude that in any finite region of the complex l plane the only singularities are moving poles with parallel trajectories spaced by one unit. There is also an essential singularity as $\mathrm{Re}l\to-\infty$, as can be seen from the nonvanishing of R_{N+1} when N goes to infinity. This example strongly suggests that it is impossible to construct an amplitude with the correct analytic properties in s and t starting from an infinite sum of Regge poles.[7] The structure of our continued partial-wave amplitude is such that the Mandelstam-Sommerfeld-Watson transformation can be performed, leading to an asymptotically smooth Regge behavior for any t except on the real axis. We stress once again that, although the background integral can be pushed back to the left as far as we want, the infinite series of Regge poles does not converge.

From the expression (11), it is easy to calculate the residue $\beta_m(s)$ of the Regge poles at $l=\alpha(s)-m$ ($m=0$, 1, $2,\cdots$), which are given by

$$\beta_m(s)=(4aq^2)^{\alpha(s)-m}\sum_{n=0}^{m}2\gamma\alpha(s)(4aq^2)^{m-n}\frac{f^{(n)}(0)}{n!}$$
$$\times\frac{\Gamma(\alpha(s)-n+1)}{\Gamma(2\alpha(s)-m-n+2)}\frac{(-1)^{m-n}}{(m-n)!}\,. \tag{12}$$

This expression of the residue functions does satisfy the usual analyticity, reality, and threshold properties.

For large $|s|$ with $|\arg[\alpha(s)]|<\pi$, the residue functions $\beta_m(s)$ behave like $[\alpha(s)]^{m-1}\exp[-(2\ln 2-1)\alpha(s)]$ for $m\geq1$ and like $\exp[-(2\ln 2-1)\alpha(s)]$ for $m=0$. This exponential decrease for large positive $\mathrm{Re}s$ is satisfactory, being consistent with the experimental rapid decrease of the elastic widths, while the exponential increase as $\mathrm{Re}s\to-\infty$ is not suggested by the phenomenological behavior of the residue functions and also does not allow one to write a partial-wave dispersion relation.[8] It is also easily seen that the residue functions have the symmetry properties implied by the Mandelstam symmetry at half-integers: When a trajectory goes through a half-integer l_0, either its residue vanishes or a compensating trajectory goes through $-l_0-1$ in such a way that no spurious singularities are introduced.

Let us now look at the Lorentz-pole content of the Veneziano-Lovelace representation at $s=0$. The residues of the members of a Regge-pole family generated

[6] Bateman Manuscript Project, *Higher Transcendental Functions*, edited by A. Erdélyi (McGraw-Hill Book Co., New York, 1953).

[7] See, e.g., N. N. Khuri, Phys. Rev. **176**, 2026 (1968).

[8] This point has been studied independently by R. Carlitz (private communication).

by a Lorentz pole are related by[9]

$$\frac{\beta_{2n}(0)}{\beta_0(0)}=\frac{(2n)!}{2^{2n}(n!)^2}\frac{\Gamma(\alpha+1-n)}{\Gamma(\alpha+1)}\frac{\Gamma(\alpha+\frac{3}{2})}{\Gamma(\alpha+\frac{3}{2}-n)}, \quad (13)$$

where $\beta_{2n}(0)$ is the residue of the pole at $l=\alpha-2n$.

This expression, together with (11), shows that unless very complicated cancellations do occur the representation studied contains an infinite series of Lorentz poles at $\alpha(0)-m$, $m=0, 1,\cdots$. That such cancellations do not occur has been explicitly checked for the first few values of m, and it appears extremely unlikely that they occur for large m.

For physical values of l, the amplitude can be written as

$$a_l(s)=-\gamma\frac{(aq^2)^l}{l!}(-1)^l\left(\frac{d^l}{d\epsilon^l}K(l,s;\epsilon)\right)_{\epsilon=-b+1}, \quad (14)$$

with

$$K(l,s;\epsilon)=\Gamma(1-\alpha(s))\int_{-1}^{1}dt(1-t^2)^l \times\frac{\Gamma(\epsilon+2aq^2(t+1))}{\Gamma(-\alpha(s)+\epsilon+2aq^2(t+1))}. \quad (15)$$

This expression is easily obtained from (7) and shows that the partial-wave amplitudes are analytic functions of s with a left-hand cut starting at $q^2=-\frac{1}{4}m_\rho{}^2$ and a series of poles on the positive s axis, as expected.

Now, the residues of the poles at the non-negative integers must be positive numbers, and this places strong restrictions on the intercept of $\alpha(s)$. Let us see how these conditions look. The positivity condition for the residues on the second trajectory $[\alpha(s)-1]$ turns out to be

$$3b+4am_\pi{}^2-1\geq 0, \quad (16)$$

i.e., $b\geq\frac{1}{3}$ for massless pions. For the third trajectory, the positivity condition is energy-dependent and the strongest result is obtained from the residue of the first recurrence on the trajectory. It is

$$(2b-1)[7b-2+5(4am_\pi{}^2)]+(4am_\pi{}^2)^2\geq 0, \quad (17)$$

i.e., $b\geq\frac{1}{2}$ for massless pions.

Similar conditions can be derived from the study of the lower-lying trajectories. It has been explicitly checked[10] that, under the condition $b\geq\frac{1}{2}$ for zero-mass pions, no negative widths appear, at least up to $l=25$.

As has already been remarked, it is an interesting feature of the representation that the lowest value of $\alpha(0)$ consistent with the positivity condition is just the one required by the Adler self-consistency condition and agrees rather well with the experimental results.[2]

Now we briefly discuss the contribution of the $A(t,u)$ term. The same method used above gives the following expression for the continued partial-wave amplitude of $A(t,u)$:

$$a^{tu}(l,s)=-\gamma\frac{\alpha(s)-\delta}{aq^2}\frac{\sqrt{\pi}}{2^{l+1}\Gamma(l+\frac{3}{2})}\int_0^\infty dx\, x^l g(x^2), \quad (18)$$

where $\delta=3b+4am_\pi{}^2-1$ and $g(x^2)$ is defined by

$$g(x^2)=(2x)^{-l-1}M_{0,l+\frac{1}{2}}(2x)(e^{x/4aq^2}+e^{-x/4aq^2})^{-\alpha(s)+\delta-1}. \quad (19)$$

From the study of Eqs. (18) and (19), it is easily seen that fixed poles are present at nonsense wrong-signature points and the residue of the fixed pole at $l=-2k-1$ is given by

$$-\gamma\frac{\alpha(s)-\delta}{aq^2}\frac{(\sqrt{\pi})2^{2k}}{\Gamma(-2k+\frac{1}{2})}\frac{[g^{(k)}(0)]_{l=-2k-1}}{k!}. \quad (20)$$

These fixed poles are additive in the full partial-wave amplitude. It is well known that additive fixed poles do not invalidate the usual dip mechanism at all. In fact, in the Veneziano-Lovelace amplitude, one can still expect dips at nonsense wrong-signature points.

Note added in manuscript. After submitting the manuscript, one of us (S.M.) was informed that D. Sivers and J. Yellin have also investigated the *J*-plane structure of the Veneziano formula (private communication from J. Yellin).

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[9] P. Di. Vecchia and F. Drago, Phys. Rev. **178**, 2329 (1969).

[10] J. E. Mandula (private communication). After this work was completed, we received an unpublished report by J. Shapiro and J. Yellin, where the positivity condition was also studied with results similar to ours.