Effects of the Finite Width of the ρ Meson on Some Vector-Meson-Dominance Predictions*†

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Corrections due to the finite width of the ρ meson based on a generalized effective-range formula for pion-pion scattering modify by a non-negligible amount the well-known expressions, based on vector-meson dominance, for the following branching ratios: $\Gamma(\rho^0\to e^+e^-)/\Gamma(\rho^0\to\pi^+\pi^-)$, $\Gamma(\varphi\to K\bar{K})/\Gamma(\rho^0\to\pi^+\pi^-)$, $\Gamma(\omega\to e^+e^-)/\Gamma(\rho^0\to e^+e^-)$, and $\Gamma(\varphi\to e^+e^-)/\Gamma(\rho^0\to e^+e^-)$. We also present a new current-algebra prediction for the shape and magnitude of $\sigma_{\text{tot}}(e^+e^-\to\pi^+\pi^-)$, and estimate the ρ-meson contribution to the Schwinger term and to the anomalous magnetic moment of the muon. A discussion of ρ dominance when the width is explicitly taken into account and a simple recipe for including finite-ρ-width corrections in vector-meson-dominance calculations are also given.

I. INTRODUCTION

DURING the last eight years the vector-meson-dominance (VMD) hypothesis has been used to derive predictions covering a wide range of phenomena.[1] In most of these cases the vector mesons were treated as essentially stable, or sometimes, in processes dominated by vector-meson exchange, the width was taken into account by adding a negative imaginary part, either constant or energy-dependent, to the vector-meson mass appearing in the corresponding propagator. Of course, when the dominating vector meson is very far off the mass shell, intuitively we do not expect the width to be important. For example, if we use the ρ-exchange model[2] to calculate the s-wave pion-nucleon scattering length, we would not expect the width of the ρ meson to be of any relevance. But as we approach the mass shell, we do expect the finite width to become important. Intuitively, we would expect effects of the order of the ratio of the width over the mass to appear. Therefore, if our intuition does not mislead us, we would expect, for the case of the ρ meson, finite-width effects to be of the order of $\Gamma_\rho/m_\rho\simeq 15\%$ for reasonable values of Γ_ρ. For ω and φ, this argument is somewhat less secure, because of threshold effects (for the φ) and three-particle final states (for the ω).[3]

Recently, methods to take into account the width of the ρ meson on the vector-meson-dominance prediction for $\rho\to e^+e^-$ were suggested independently by Gounaris and Sakurai[4] and by Vaughn and Wali.[5] The latter authors calculated also in their model the effect of the width of the ρ on the decay rates $\omega\to 3\pi$ and $\omega\to\pi^0\gamma$, and suggested a model to take into account the widths of ω and φ.

In the present work, we propose to pursue further the method of Ref. 4 and to calculate systematically the correction due *only* to the finite width of the ρ meson on various predictions of the VMD. In the various calculations presented in this paper, the only other unstable (under strong interactions) particles which are going to appear occasionally are ω and φ. We treat them as if they were essentially stable. We think that this is worthwhile, even if it will finally turn out that the finite-width effects of ω and φ are important. Our reasons are the following. (1) We present our theory and results in such a form that to take into account ω, φ finite-width corrections should be a completely independent problem. It will be obvious in our treatment which of our results are going to be modified because of this and which are not. (2) Even in processes where ω, φ finite-width effects may be important, it may be interesting to find how much of the finite-width corrections are due to each of the unstable particles involved. (3) Within our phase-shift approach[4] it is hard, at least for the present, to take into account finite-width effects for ω and φ. It seems that an analysis of three-particle interactions is required.

Our order of presentation is as follows. In Sec. II, starting from the assumption[4] that for a wide energy range ($s\lesssim 1$ BeV^2) the p-wave pion-pion scattering phase shift δ_1 satisfies a generalized effective-range formula of the Chew-Mandelstam type,[6] we derive an expression for the pion electromagnetic form factor. Using this form factor and the current-field identity,[7,8] we estimate the ρ-meson contribution to the Schwinger term and we derive an expression for the ρ-meson propagator which is expected to be valid for low values of s ($s\lesssim 1$ BeV^2). We then proceed to show that our assumptions

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[1] J. J. Sakurai, in *Lectures in Theoretical Physics* (University of Colorado Press, Boulder, Colo., to be published), Vol. XI.
[2] J. J. Sakurai, Ann. Phys. (N. Y.) **11**, 1 (1960).
[3] Actually in the model suggested in Ref. 5, the ω-φ finite-width effects are as large as 25%.
[4] G. J. Gounaris and J. J. Sakurai, Phys. Rev. Letters **21**, 244 (1968).
[5] M. T. Vaughn and K. C. Wali, Phys. Rev. Letters **21**, 938 (1968); Phys. Rev. **177**, 2199 (1969). See also M. T. Vaughn, M. L. Blackmon, and K. C. Wali, in *Proceedings of the Fourteenth International Conference on High-Energy Physics, Vienna, 1968* (CERN, Geneva, 1968).
[6] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).
[7] M. Gell-Mann and F. Zachariasen, Phys. Rev. **124**, 956 (1961).
[8] N. M. Kroll, T. D. Lee, and B. Zumino, Phys. Rev. **157**, 1376 (1967).

lead to complete ρ dominance of the pion electromagnetic form factor or, equivalently, of the $\rho\pi\pi$ vertex [see Eqs. (2.35) and (2.37)], and give a recipe on how ρ-width corrections may be included in calculations in a simple way. Roughly, this recipe states that the ρ meson should be treated as if it were stable and the only effects of the width is to modify the ρ propagator and the normalization of the ρ field. In connection with this we note that when $\Gamma_\rho \to 0$, both the ρ propagator and the normalization of the ρ field become the usual ones.

In Secs. III and IV, relations among the processes $\rho^0 \to e^+e^-$, $\rho^0 \to \pi^+\pi^-$, $\varphi \to K\bar{K}$, $\varphi \to e^+e^-$, $\omega \to e^+e^-$, $\omega \to \pi^0\gamma$, and $\omega \to 3\pi$ are calculated and compared with the stable-ρ-meson predictions. Finite-width effects of at most 15%, for reasonable values of Γ_ρ, are found. We wish to add here that the decay rates $\rho^0 \to e^+e^-$ and $\rho^0 \to \pi^+\pi^-$ are calculated in a completely unambiguous way and shown to agree with our recipe. In Sec. V, the form factor found in Sec. II is used to calculate the ρ contribution to the anomalous magnetic moment[9] of the muon. It is expected that our results should be better than those of other calculations, since it is known that the low-energy region, for which our pion form factor is particularly good, is most important[9] in the calculation of the ρ contribution to the anomalous magnetic moment of the muon. We also give an evaluation of the ρ contribution to the charge renormalization[8] to the order e^2. Finally, in Sec. VI we summarize our results and give our conclusions.

II. GENERAL THEORY

We assume that for a wide energy range ($s \lesssim 1$ BeV^2) the p-wave pion-pion scattering phase shift satisfies a generalized effective-range formula of the Chew-Mandelstam type[6]:

$$(k^3/\sqrt{s})\cot\delta_1 = k^2 h(s) + a + bk^2\,, \tag{2.1}$$

where

$$k = \tfrac{1}{2}(s-4m_\pi^2)^{1/2}\,, \tag{2.2}$$

$$h(s) = \frac{2}{\pi}\frac{k}{\sqrt{s}}\ln\left(\frac{\sqrt{s}+2k}{2m_\pi}\right). \tag{2.3}$$

We construct the function

$$f(s) = -ik^3/\sqrt{s} + (k^3/\sqrt{s})\cot\delta_1\,. \tag{2.4}$$

From (2.1)–(2.4), it is easy to see that $f(s)$ has the following analytical properties[10]:

(1) $f(s)$ is an analytic function without any poles.

(2) It is real in the sense of the Schwarz reflection principle. That is,

$$f(s) = f^*(s^*)\,.$$

(3) When a and b are determined by a fit to the mass and the width of the ρ meson, $f(s)$ is seen to have a zero at

$$s = s_0 = -9.4\times 10^6 m_\rho^2\,. \tag{2.5}$$

(4) $f(s)$ has only a right-hand cut, from $s=4m_\pi^2$ to $s=\infty$, right above which

$$\operatorname{Im} f(s+i\epsilon) = -k^3/\sqrt{s}\,, \tag{2.6}$$

$$\text{phase of } f(s+i\epsilon) = -\delta_1(s)\,. \tag{2.7}$$

Note that in (2.2) and (2.3) it is implied that for $0<s<4m_\pi^2$, we make the replacement

$$k \to i(m_\pi^2 - \tfrac{1}{4}s)^{1/2}\,,$$

$$\ln\left(\frac{\sqrt{s}+2k}{2m_\pi}\right) \to i\cot^{-1}\left(\frac{s}{4m_\pi^2 - s}\right)^{1/2}.$$

Using the analytical properties of $f(s)$, we can write $F_\pi(s)$ with the correct phase and singularities[11,12]

$$F_\pi(s) = f(0)/f(s)\,. \tag{2.8}$$

We now *define* the ρ mass m_ρ and the ρ width Γ_ρ by

$$\cot\delta_1|_{s=m_\rho^2} = 0\,, \tag{2.9}$$

$$d\delta_1/ds|_{s=m_\rho^2} = 1/m_\rho\Gamma_\rho\,. \tag{2.10}$$

With these definitions our form factor (2.8) can be written as follows[13]:

$$F_\pi(s) = \frac{m_\rho^2 + m_\rho\Gamma_\rho d}{m_\rho^2 - s + \Gamma_\rho(m_\rho^2/k_\rho^3)\{k^2[h(s)-h(m_\rho^2)] + k_\rho^2 h'(m_\rho^2)(m_\rho^2 - s)\} - i m_\rho\Gamma_\rho(k/k_\rho)^3(m_\rho/\sqrt{s})}\,, \tag{2.11}$$

where d is a constant that depends only on the ρ mass:

$$d = \frac{3}{\pi}\frac{m_\pi^2}{k_\rho^2}\ln\left(\frac{m_\rho + 2k_\rho}{2m_\pi}\right) + \frac{m_\rho}{2\pi k_\rho} - \frac{m_\pi^2 m_\rho}{\pi k_\rho^3}\,. \tag{2.12}$$

Near the ρ mass we can ignore the middle term in the denominator of (2.11) since it goes as $(m_\rho^2 - s)^2$, and

[9] C. Bouchiat and L. Michel, J. Phys. Radium **22**, 121 (1961); L. Durand, III, Phys. Rev. **128**, 441 (1962); **129**, 2835(E) (1963); T. Kinoshita and R. J. Oakes, Phys. Letters **25B**, 143 (1967).

[10] To be specific, we may state that in an s plane we take $\sqrt{s}$ to have a cut along the negative real axis, and $k=\frac{1}{2}(s-4m_\pi^2)^{1/2}$ to have a cut along the positive real axis.

[11] The connection between the p-wave pion-pion phase shift and the pion form factor was first discussed by P. Federbush, M. L. Goldberger, and S. B. Treiman, Phys. Rev. **112**, 642 (1958).

[12] Since our formalism is expected to be valid for $|s| \lesssim 1$ BeV^2, we have ignored the vanishing point of $f(s)$ at $s_0 = -9.4\times 10^6 m_\rho^2$. The only place where the knowledge of this point is of some use is in Sec. V, where we "estimate" the unrenormalized mass of the ρ meson. Note also a mistake in Ref. 4, where it was stated that $s_0 = -1.2\times 10^6 m_\rho^2$.

[13] Equation (2.11) is essentially equivalent to the expression for the pion form factor given by W. R. Frazer and J. R. Fulco, Phys. Rev. Letters **2**, 365 (1959); Phys. Rev. **117**, 1609 (1960). Note, however, that their v_ρ is not quite equal to our k_ρ^2. See also B. W. Lee and M. T. Vaughn, Phys. Rev. Letters **4**, 578 (1960).

(2.11) is reduced to the familiar resonance formula

$$F_\pi(s)|_{\text{near } s=m_\rho^2} = \frac{m_\rho^2[1+(\Gamma_\rho/m_\rho)d]}{m_\rho^2-s-im_\rho\Gamma_\rho(k/k_\rho)^3(m_\rho/\sqrt{s})}. \quad (2.13)$$

It is very important to note that even though our $F_\pi(s)$ is correctly normalized at $s=0$, the numerator in (2.13) is not just m_ρ^2. Numerically, we have $d=0.48$ for $m_\rho=775$ MeV. The fact that the quantity d is not zero, but has such a large value, is our most significant result. This quantity will appear again and again till the end of this paper; it is, in most cases, responsible for a major part of the finite-width corrections.

Let us now calculate the ρ propagator. This calculation will lead us to the recipe promised in the Introduction and many other useful results. The ρ propagator is given by[14]

$$\Delta_{\mu\nu}=i\int d^4x\, e^{-iq\cdot x}\langle 0|T^*\{\rho_\mu{}^3(x),\rho_\nu{}^3(0)\}|0\rangle$$
$$=\delta_{\mu\nu}R(-q^2)+q_\mu q_\nu S(-q^2)$$
$$=\int \frac{\delta_{\mu\nu}+m^{-2}q_\mu q_\nu}{q^2+m^2-i\epsilon}\sigma_\rho(m^2)d(m^2). \quad (2.14)$$

We are only interested in the $\delta_{\mu\nu}$ part of the propagator. It is only this part that would contribute for the ρ meson coupled to a conserved current:

$$R(s)=\int_{4m_\rho^2}^{\infty}\frac{\sigma_\rho(m^2)}{m^2-s-i\epsilon}dm^2. \quad (2.15)$$

At this point, in order to indicate when the various assumptions enter, it is more advantageous to calculate instead the expression

$$g(s)=\int_{4m_\rho^2}^{\infty}\frac{\rho_1{}^{(3)}(m^2)}{m^2-s-i\epsilon}dm^2, \quad (2.16)$$

where $\rho_1{}^{(3)}(m^2)$ is the spectral function which appears in Weinberg's sum rules.[15] At the end of the calculations, current-field identity[7,8] will be used to find $R(s)$. Now (2.16) just tells us that the function $g(s)$ has the following properties. (i) It is real (in the sense of the Schwarz reflection principle), analytic, without any poles, and vanishes at infinity. (ii) It has only a right-hand cut from $s=4m_\pi^2$ to $s=\infty$, just above which

$$\operatorname{Im}g(s+i\epsilon)=\pi\rho_1{}^{(3)}(s).$$

We shall construct an expression for $g(s)$, by just checking these two requirements. In order to do this, we have first to find an expression for $\rho_1{}^{(3)}(m^2)$. It is known that $\rho_1{}^{(3)}(m^2)$ is related to the colliding-beam cross section via[16,17]

$$\rho_1{}^{(3)}(m^2)=\frac{s^2}{16\pi^3\alpha^2}\sigma_{\text{tot}}(e^+e^-\to(T=1\text{ system}))|_{s=m^2}. \quad (2.17)$$

We now make the assumption that the low-energy contribution saturates the integral (2.16) for small s (say, $|s|\lesssim 1$ BeV^2). Therefore, in order to find an expression for $g(s)$ valid for small s, we can use in the integral (2.16) the expression

$$\rho_1{}^{(3)}(m^2)=\frac{s^2}{16\pi^3\alpha^2}\sigma_{\text{tot}}(e^+e^-\to\pi^+\pi^-)|_{s=m^2}, \quad (2.17')$$

or, from

$$\sigma_{\text{tot}}(e^+e^-\to\pi^+\pi^-)=[\tfrac{1}{3}\pi\alpha^2(s-4m_\pi^2)^{3/2}/s^{5/2}]|F_\pi(s)|^2, \quad (2.18)$$

$$\rho_1{}^{(3)}(s)=\frac{1}{48\pi^2}\frac{(s-4m_\pi^2)^{3/2}}{\sqrt{s}}|F_\pi(s)|^2. \quad (2.19)$$

Now from (2.19), the analytic properties of $f(s)$ given above,[18] and (2.8), we can show that[19]

$$g(s)=\frac{1}{6\pi}\frac{k_\rho^3}{\Gamma_\rho}\left(1+\frac{\Gamma_\rho}{m_\rho}d\right)F_\pi(s) \quad (2.20)$$

satisfies requirements (i) and (ii) and, consequently, is the result of the integration (2.16) when $\rho_1{}^{(3)}(m^2)$ is given by (2.19). At this point we already have a useful result. We see from (2.16) that the Schwinger term (or the vacuum expectation value of the Schwinger term if the Schwinger term is not a c number) is given by

$$C_{33}=g(0), \quad (2.21)$$

where $C_{\alpha\beta}$ is normalized so that

$$[j_0{}^\alpha(x),j_k{}^\beta(x')]_{x_0=x_0'}=if_{\alpha\beta\gamma}j_k{}^\gamma(x)\delta(\mathbf{x}-\mathbf{x}')-iC_{\alpha\beta}\partial_k\delta(\mathbf{x}-\mathbf{x}').$$

[14] By T^* we indicate the covariant part of the time-ordered product. Note also that $R(-q^2)$ and $S(-q^2)$ are related to each other by $\int[\sigma_\rho(a)/a]da=R(-q^2)+q^2S(-q^2)$.

[15] The spectral function $\rho_1{}^{(3)}(s)$ is normalized so that the ρ-meson contribution to it in the stable-particle approximation is given by $(m_\rho^2/f_\rho)^2\delta(s-m_\rho^2)$.

[16] J. J. Sakurai, in *Lectures on Currents and Mesons* (The University of Chicago Press, Chicago, to be published).

[17] A relation of this type was first written within the framework of quantum electrodynamics by T. Goto and T. Imamura, Progr. Theoret. Phys. (Kyoto) 14, 396 (1955).

[18] In particular, Eq. (2.6).

[19] We have again ignored the pole of $F_\pi(s)$ at $s=s_0=-9.4\times10^6 m_\rho^2$. Taking this pole into account, we find that the correct expression for $g(s)$, corresponding to $\rho_1{}^{(3)}(m^2)$ given in (2.19), is

$$g(s)=\frac{1}{6\pi}\frac{k_\rho^3}{\Gamma_\rho}\left(1+\frac{\Gamma_\rho d}{m_\rho}\right)\left\{F_\pi(s)-\frac{\operatorname{Res}[F_\pi(s_0)]}{s-s_0}\right\},$$

where $\operatorname{Res}[F_\pi(s_0)]$ is the residuum of the function $F_\pi(s)$ at $s=s_0$. It is easy to show that for small s, for which our formalism is valid, the second term in brackets in the equation above is completely negligible.

Therefore, the assumption that the low-energy contribution saturates the integral (2.16) for small s gives, via (2.20) and (2.21),

$$C_{33}=\frac{1}{6\pi}\frac{k_\rho^3}{\Gamma_\rho}\left(1+\frac{\Gamma_\rho}{m_\rho}d\right), \tag{2.22}$$

where d is given by (2.12).

From here on, let us adopt the gauge-field algebra,[20] in which $C_{\alpha\beta}$ is a finite constant given by

$$C_{\alpha\beta}=(m_\rho/f_\rho)^2\delta_{\alpha\beta}. \tag{2.23}$$

We thus obtain from (2.22)[21]

$$\frac{f_\rho^2}{4\pi}=\frac{3}{2}\frac{\Gamma_\rho m_\rho^2}{k_\rho^3}\left(1+\frac{\Gamma_\rho}{m_\rho}d\right)^{-1}. \tag{2.24}$$

Note that in the stable-ρ approximation, assuming that the ρ meson dominates the pion form factor, we obtain

$$f_\rho^2/4\pi=\tfrac{3}{2}(\Gamma_\rho m_\rho^2/k_\rho^3). \tag{2.24$'$}$$

We see that the two expressions, (2.24) and (2.24′), agree in the limit $\Gamma_\rho \ll m_\rho$, but for a realistic value of the ρ width they differ by about 8%. This is illustrated in Fig. 1.

Recently, Brown and Goble[22] proposed that the p-wave pion-pion phase shift be obtained by matching the effective-range formula (2.1) to the current-algebra prediction on pion-pion scattering near threshold.[23] Their procedure leads to the modified Kawarabayashi-Suzuki-Riazuddin-Fayyazuddin (KSRF) relation[24]

$$\Gamma_\rho=\frac{k_\rho^5}{3\pi c_\pi^2 m_\rho^2}\{1-k_\rho^4 h'(m_\rho^2)/3\pi c_\pi^2\}^{-1}=130\text{ MeV}, \tag{2.25}$$

if $m_\rho=775$ MeV is used. Using this value of Γ_ρ in (2.22) and (2.24), we obtain the current-algebra predictions

$$C_{33}=0.021\text{ BeV}^2 \tag{2.26}$$

and[25]

$$f_\rho^2/4\pi=2.3. \tag{2.27}$$

These predictions correspond to the triangular point in Fig. 1.

Now let us calculate the ρ propagator $R(s)$. The current-field identity[7,8] tells us that

$$(m_\rho^2/f_\rho)^2\sigma_\rho(m^2)=\rho_1^{(3)}(m^2). \tag{2.28}$$

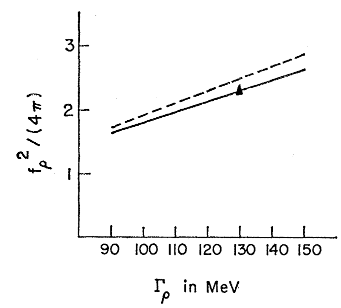

FIG. 1. The dependence of the coupling constant $f_\rho^2/4\pi$ defined by Eq. (2.23), on the ρ-meson width. The full line is based on Eq. (2.24), which takes into account the finite ρ width. The broken line corresponds to the usual VMD prediction when we treat the ρ meson as stable. The triangular point represents the current-algebra prediction.

Combining (2.15), (2.16), (2.21), (2.23), (2.24), and (2.28), we find the $\delta_{\mu\nu}$ part of the ρ propagator,

$$R(s)=(1/m_\rho^2)F_\pi(s), \tag{2.29}$$

where $F_\pi(s)$ is given in (2.11) and (2.13). We note that this result is what would have been expected by analogy with the stable ρ case. For s near m_ρ^2, we have

$$R(s)|_{\text{near } s=m_\rho^2}=\frac{1+(\Gamma_\rho/m_\rho)d}{m_\rho^2-s-im_\rho\Gamma_\rho(k/k_\rho)^3(m_\rho/\sqrt{s})}, \tag{2.30}$$

from which, using the formalism of Ref. 8,[26] we get

$$Z_1/Z=1+(\Gamma_\rho/m_\rho)d. \tag{2.31}$$

We recall that Z_1/Z gives the normalization of the renormalized ρ field. The best way to define it is by the numerator in Eq. (2.30).[26] In the hypothetical stable-ρ case, an alternative equivalent definition is

$$\langle 0|\rho_\mu(0)|\rho\rangle=\epsilon_\mu(2E_\rho)^{-1/2}(Z_1/Z)^{1/2}. \tag{2.32}$$

At this point some interesting remarks about (2.31) are in order. We note that already in writing (2.23), we have adopted the convention

$$Z_1=Z_0=(m_\rho/m_\rho^0)^2, \tag{2.33}$$

in the notation of Ref. 8, where m_ρ^0 is the unrenormalized mass of the ρ meson. This convention is the one used in gauge-field algebra,[20] and has the important advantage of establishing[27]

$$(m_\rho/f_\rho)^2=(m_\rho^0/f_\rho^0)^2,$$

that is, making $(m_\rho/f_\rho)^2$ *"renormalization-invariant."* It is important to remember[8] that once the convention (2.33) has been chosen, the normalization of the renormalized ρ field has been determined and we cannot choose it arbitrarily. In our effective-range-formula

[20] T. D. Lee, S. Weinberg, and B. Zumino, Phys. Rev. Letters **18**, 1029 (1967).

[21] Our coupling constant is normalized so that the current-field identity reads $j_\mu^\alpha(x)=(m_\rho^2/f_\rho)\rho_\mu^\alpha(x)$ $(\alpha=1,2,3)$.

[22] L. S. Brown and R. L. Goble, Phys. Rev. Letters **20**, 346 (1968); also, see Ref. 4.

[23] S. Weinberg, Phys. Rev. Letters **17**, 616 (1966). It is amusing to note that our result, depending only on the current-algebra prediction for the p-wave pion-pion scattering length, is independent of any assumptions concerning the σ terms. I wish to thank D. F. Greenberg for pointing this out to me.

[24] The pion decay constant is normalized so that $c_\pi=94$ MeV.

[25] The value of $f_\rho^2/4\pi$ that we found should be compared with the original KSRF value $f_\rho^2/(4\pi)=m_\rho^2/(8\pi c_\pi^2)=2.66$.

[26] In particular, Eq. (4.18) of Ref. 8.

[27] See Eqs. (3.10) and (3.11) of Ref. 8. Note that our f_ρ and f_ρ^0 correspond, respectively, to g_ρ and g_ρ^0 used in that reference.

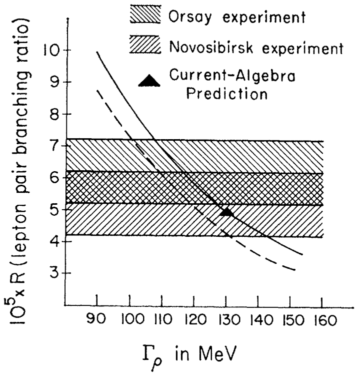

FIG. 2. The dependence of the lepton pair branching ratio on the ρ meson width. The full line is based on Eq. (3.4), which takes into account the finite ρ width. The broken line is obtained when we treat the ρ meson as stable. The experimental results of the Novosibirsk and Orsay groups are indicated. The triangular point represents the current-algebra prediction.

model, it turns out that this normalization is given by (2.31). It is also interesting that in the hypothetical case of a stable ρ meson ($\Gamma_\rho \ll m_\rho$), (2.31) gives

$$Z_1/Z|_{\text{narrow width}} = 1 ,$$

and the renormalized field has the usual normalization. That is, so long as we treat the ρ meson as a stable particle, it is perfectly consistent to have

$$Z = Z_0 = Z_1 .$$

Up to now in this section we have calculated m_ρ^2/f_ρ^2 and Z_1/Z, and found expressions for the $\pi^\pm$ form factor and the ρ-meson propagator valid for small s. Now let us focus on the question: What can all these things tell us, in a more formal way, about ρ dominance when the width is explicitly taken into account? From the current-field identity[8]

$$j_\mu{}^3(0) = (m_\rho^2/f_\rho)\rho_\mu{}^3(0) ,$$

we obtain[28]

$$\begin{aligned}\langle\pi^+(p_2)|j_\mu{}^3(0)|\pi^+(p_1)\rangle &= (m_\rho^2/f_\rho)\langle\pi^+(p_2)|\rho_\mu{}^3(0)|\pi^+(p_1)\rangle \\ &= (m_\rho^2/f_\rho)R(s)f_\rho\langle\pi^+(p_2)|\hat{J}_\mu{}^\rho(0)|\pi^+(p_1)\rangle , \end{aligned} \quad (2.34)$$

where

$$\partial_\mu \hat{J}_\mu{}^\rho(x) = 0 .$$

We now define

$$\langle\pi^+(p_2)|j_\mu{}^3(0)|\pi^+(p_1)\rangle = (p_2+p_1)_\mu F_\pi(s) \quad (2.35)$$

and

$$\langle\pi^+(p_2)|\hat{J}_\mu{}^\rho(0)|\pi^+(p_1)\rangle = (p_2+p_1)_\mu \hat{F}_\pi{}^\rho(s) , \quad (2.36)$$

where $s = -(p_2-p_1)^2$. Combining (2.29) and (2.34)–(2.36), we obtain

$$\hat{F}_\pi{}^\rho(s) = 1 . \quad (2.37)$$

[28] The current $\hat{J}_\mu{}^\rho(x)$ is the same as the one in Ref. 8.

By analogy with the corresponding treatment in the stable-ρ-meson approximation,[16] (2.37) may be considered as a definition of what we mean by ρ dominance of the $\pi^\pm$ electromagnetic form factor, in the framework of the current-field identity. Of course, it is a relation which holds only for small s. We can, therefore, make the following statement: Our effective-range formula for the pion form factor, the gauge field algebra, and the approximation to use only the two-pion intermediate contribution in the spectral function $\rho_1{}^{(3)}(m^2)$ in order to calculate the ρ propagator, lead us to complete ρ dominance in the sense of (2.37). It is also interesting to note that (2.34) and (2.37) imply that (in our theory) f_ρ plays also the role of the usual $\rho\pi\pi$ coupling constant. Therefore, our theory implies the relation

$$f_\rho = f_{\rho\pi\pi}$$

derived long ago[7,29] under the assumption of ρ dominance of the pion form factor in the stable-ρ approximation. Another interesting feature in our derivation is that the usual ρ-dominance phrase, "The ρ meson saturates the spectral function $\rho_1{}^{(3)}(s)$ for low values of s," is consistently replaced by the phrase "The low-energy two-pion contribution saturates the spectral function $\rho_1{}^{(3)}(s)$ for low values of s." In fact, this substitution is the only difference between our ρ-dominance theory which takes into account finite-ρ-width corrections, and the usual ρ-dominance theory.

Looking back now at (2.29), (2.34), and (2.37), we see that all the structure of the pion form factor comes from the ρ propagator, exactly as in the stable-ρ approximation, where VMD leads to

$$\frac{1}{m_\rho^2}F_\pi(s)|_{\rho\text{ stable}} = \frac{1}{m_\rho^2 - s} .$$

It is now natural to conjecture that this analogy is always true. That is, in any process in which ρ exchange dominates, its propagator gives all the structure so far as the ρ meson is concerned.[30] We can, therefore, give the following recipe on how to take finite-width corrections in any process dominated by the ρ meson. Treat the ρ meson as if it were stable with the following modifications *only:*

(1) Use for the $\delta_{\mu\nu}$ part of the ρ propagator

$$R(s) = (1/m_\rho^2)F_\pi(s) , \quad (2.29)$$

where $F_\pi(s)$ is given in (2.11).

(2) The renormalized ρ field is normalized as

$$\langle 0|\rho_\mu{}^\alpha(0)|\rho^\beta\rangle = (2E_\rho)^{-1/2}(Z_1/Z)^{1/2}\epsilon_\mu\delta_{\alpha\beta} , \quad (2.32)$$

[29] Y. Nambu and J. J. Sakurai, Phys. Letters 8, 79 (1962); 8, 191(E) (1962); also, Ref. 31.

[30] Actually this is the point of view underlying the work in Ref. 5. We wish to acknowledge at this point that this is what motivated us to try to prove it for the particular case of the $\pi^\pm$ form factor, in the framework of our phase-shift approach.

$$Z_1/Z=1+(\Gamma_\rho/m_\rho)\,, \tag{2.31}$$

where d is given in (2.12).

(3) The ρ field is coupled to the isovector current[1,2] with coupling constant f_ρ given by

$$f_\rho^2/4\pi=\tfrac{3}{2}(\Gamma_\rho m_\rho^2/k_\rho^3)[1+(\Gamma_\rho/m_\rho)d]^{-1}. \tag{2.24}$$

Note that in the stable-ρ-meson approximation ($\Gamma_\rho\ll m_\rho$), this recipe becomes the ordinary recipe for calculating processes assumed to be dominated by the ρ meson. We hasten to say that in Sec. III we shall calculate in a completely unambiguous way the decay rates $\rho^0\to e^+e^-$ and $\rho\to\pi\pi$ and show that we obtain the same results as the ones expected from our recipe. After that, we shall simply use it in order to calculate the decay rates of $\omega\to\pi^0\gamma$ and $\omega\to 3\pi$ using the Gell-Mann–Sharp–Wagner model,[31] and assuming a constant $f_{\pi\rho\omega}$ coupling.[32]

III. CORRECTION DUE TO FINITE WIDTH OF ρ MESON TO VMD PREDICTION FOR $\rho^0\to e^+e^-$

We first note that using (2.31) we can rewrite (2.24) as follows:

$$\Gamma_\rho=(f_\rho^2/4\pi)\tfrac{2}{3}(k_\rho^3/m_\rho^2)(Z_1/Z)\,. \tag{3.1}$$

Remembering that throughout this work we have implicitly assumed $\Gamma_\rho=\Gamma(\rho^0\to\pi^+\pi^-)$, we immediately see that (3.1) is what would have been expected from our recipe.[33]

We now focus our attention on the process $\rho^0\to e^+e^-$. We note that the most unambiguous way to define the lepton pair branching ratio,[4]

$$R=\Gamma(\rho^0\to e^+e^-)/\Gamma(\rho^0\to\pi^+\pi^-)\,, \tag{3.2}$$

is by the formula

$$\sigma_{\rm tot}(e^+e^-\to\pi^+\pi^-)\,|_{s=m_\rho^2}=3\pi(2/m_\rho)^2R\,. \tag{3.3}$$

Combining (2.11), (2.18), and (3.3), we get

$$R|_{\text{finite width}}=\frac{\alpha^2}{36}\left(\frac{m_\rho^2-4m_\pi^2}{m_\rho^2}\right)^{3/2}\left(\frac{m_\rho}{\Gamma_\rho}+d\right)^2, \tag{3.4}$$

which is to be compared with the narrow-width result

$$R|_{\text{narrow width}}=(\alpha^2/36)(m_\rho^2-4m_\pi^2/m_\rho^2)^{3/2}\times(m_\rho/\Gamma_\rho)^2. \tag{3.5}$$

Although the two expressions agree for $\Gamma_\rho\ll m_\rho$, for a realistic value of the ρ width they differ by as much as 15%. This is indicated in Fig. 2. The data of the Novosibirsk[34] and Orsay[35] colliding-beam experiments

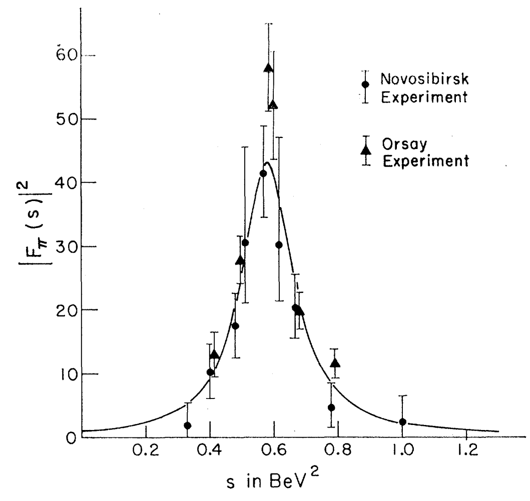

FIG. 3. Theoretically predicted $|F_\pi(s)|^2$ based on current algebra (i.e., $\Gamma_\rho=130$ MeV) and the effective-range expansion. The theoretical curve has no adjustable parameter once the ρ mass is given ($m_\rho=775$ MeV).

are also shown. For the current-algebra prediction (2.25), Eq. (3.4) gives

$$R|_{\text{current algebra}}=5.0\times10^{-5}. \tag{3.6}$$

These predictions correspond to the triangular point in Fig. 2. Also note that combining (2.24), (2.31), and (3.4), we obtain

$$\Gamma(\rho^0\to e^+e^-)=\tfrac{1}{3}\alpha^2(4\pi/f_\rho^2)m_\rho(Z_1/Z)\,. \tag{3.7}$$

This relation is exactly what would have been expected from our recipe. The current-algebra prediction for this decay rate is

$$\Gamma(\rho^0\to e^+e^-)=6.5\text{ keV}. \tag{3.8}$$

The experimental results, calculated from (3.3) and the experimental value for $\sigma_{\rm tot}(e^+e^-\to\pi^+\pi^-)$ at $s\simeq m_\rho^2$, are somewhat sensitive to the value of Γ_ρ that we use. Using for consistency with the result (3.8) $\Gamma_\rho=130$ MeV, we find

$$\Gamma(\rho\to e^+e^-)=6.7+1.4\text{ keV} \tag{3.9}$$

from the Novosibirsk data,[34] and

$$\Gamma(\rho\to e^+e^-)=8.1\pm1.3\text{ keV} \tag{3.10}$$

from the Orsay[35] data.

Using the current-algebra phase shift, we can also predict the s dependence of $|F_\pi(s)|^2$ (or, equivalently, the colliding-beam cross sections) as shown in Fig. 3. We emphasize that the theoretical curve has no adjustable parameters once the ρ mass is given. The values of the

[31] M. Gell-Mann, D. H. Sharp, and W. Wagner, Phys. Rev. Letters **8**, 261 (1962); R. F. Dashen and D. H. Sharp, Phys. Rev. **133**, B1585 (1964).
[32] S. G. Brown and G. B. West, Phys. Rev. **174**, 1777 (1968).
[33] Note that this relation is equivalent to Eq. (7.1) of Ref. 8.
[34] V. L. Auslander *et al.*, Novosibirsk Report No. 243 (unpublished).
[35] J. E. Augustin *et al.*, Phys. Rev. Letters **20**, 129 (1968); and papers presented in *Proceedings of the Fourteenth International Conference on High-Energy Physics, Vienna, 1968* (CERN, Geneva, 1968). Note that in consistency with our input $m_\rho=775$ MeV, we have used the experimental value of $\sigma_{\rm tot}(e^+e^-\to\pi^+\pi^-)$ at $s=m_\rho^2=0.6$ BeV^2, in order to calculate $R_{\rm expt}$, rather than the maximum value of $\sigma_{\rm tot}(e^+e^-\to\pi^+\pi^-)$.

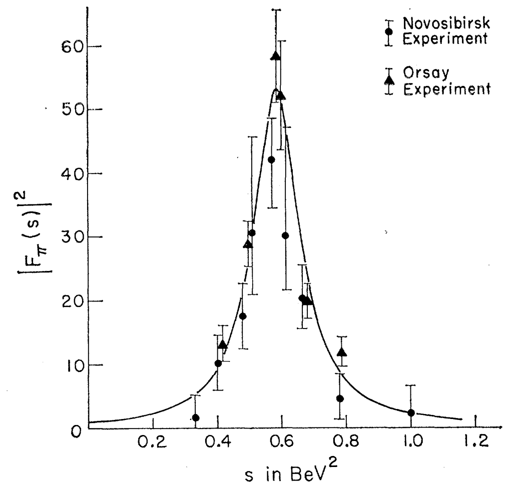

FIG. 4. Theoretically predicted $|F_\pi(s)|^2$ based on effective range [Eq. (2.11)]. The curve corresponds to $\Gamma_\rho=116$ MeV, $m_\rho=775$ MeV.

cross sections obtained by the Novosibirsk group, who have reanalyzed their data,[34] are in very good agreement with our prediction which is based on $\Gamma_\rho=130$ MeV. On the other hand, the peak cross section obtained in the Orsay experiment is considerably higher. If we put $\Gamma_\rho=116$ MeV and $m_\rho=775$ MeV in our theoretical formula (2.11), we can obtain a reasonably good "fit" of the Orsay data, without having to abandon proper normalization of the pion form factor at zero momentum transfer. This is shown in Fig. 4. The Novosibirsk group has claimed $\Gamma_\rho=105\pm20$ MeV and the Orsay group $\Gamma_\rho=112\pm12$ MeV. In connection with this, using Fig. 5, we remind ourselves that commonly quoted values of the ρ width and ρ mass may often depend on the particular manner in which the experimental data are parametrized. Curve α in Fig. 5 corresponds to our effective-range prediction for $|F_\pi(s)|^2$ given in (2.11), while curves β and γ correspond to two other forms commonly used in the literature. The same values of m_ρ and Γ_ρ have been used for all three curves. Note the large difference between curves α and γ in the low-energy region. For $0.4\lesssim s\lesssim0.6$ BeV^2, the difference between the two curves varies between 25 and 17%. The difference between curves α and β is 10% or larger for $0.4\lesssim s\lesssim0.8$ BeV^2. It is also worth mentioning that when curves α, β, and γ are plotted versus $\sqrt{s}$, the full width at half-maximum of curves α and β is the same and about[36] 123 MeV, while that of curve γ is about 130 MeV, even though in all three cases we have used $\Gamma_\rho=130$ MeV. Also note in Fig. 5 that the actual peak of $|F_\pi(s)|^2$ for curves α and β is not at $s=m_\rho^2=0.6$ BeV^2 but it is shifted towards the left by about 14 MeV. A glance at Fig. 4, which is based on $\Gamma_\rho=116$ MeV, indicates that this shift does not depend strongly on Γ_ρ. On the other hand, no such shift is observed for curve γ in Fig. 5. Going back now to Figs. 3 and 4, we observe that the value of $|F_\pi(s)|^2$ at the maximum is about 4% higher than the value at $s=m_\rho^2$. This introduces an error of about the same magnitude in the branching ratio R, if the maximum $\sigma_{\rm tot}(e^+e^-\to\pi^+\pi^-)$ is used for its evaluation in (3.4). Of course, this error could, in principle, be avoided by first fitting the data to $|F_\pi(s)|^2$ in order to find m_ρ and then use $\sigma_{\rm tot}(e^+e^-\to\pi^+\pi^-)$ at $s=m_\rho^2$. But any way this error is very small. Finally, Fig. 6 serves to give a comparison for the real and imaginary parts of our theoretical prediction for $F_\pi(s)$ with other forms commonly used in the literature. We have used for all curves $m_\rho=775$ MeV and $\Gamma_\rho=130$ MeV.

IV. EFFECT OF FINITE WIDTH OF ρ MESON ON VMD RELATIONS INVOLVING DECAY MODES OF ρ, ω, AND φ

We are working in the framework of the gauge-field algebra.[20] In this framework, the generalized first sum rule of Weinberg[37,38] applied to comparison between the third and the eighth SU_3 components of the vector currents, and the dominance of the corresponding spectral functions by ρ, ω, and φ lead to[39]

$$\left(\frac{m_\rho}{f_\rho}\right)^2=\frac{3}{4}\frac{1}{f_Y^2}[m_\varphi^2(\cos\theta_Y)^2+m_\omega^2(\sin\theta_Y)^2]. \quad (4.1)$$

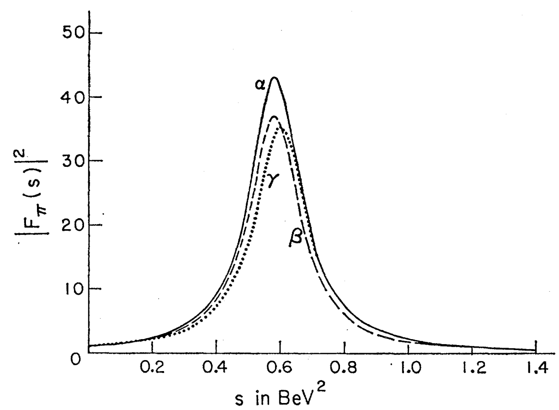

FIG. 5. Comparison of our theoretically predicted $|F_\pi(s)|^2$ with two other forms for the pion form factor commonly used in the literature. Curve α corresponds to our theoretical effective-range prediction [Eq. (2.11)]. Curve β corresponds to $F_\pi(s)=m_\rho^2/[m_\rho^2-s-im_\rho\Gamma_\rho(k/k_\rho)^3(m_\rho/\sqrt{s})\theta(s-4m_\pi^2)]$, and curve γ corresponds to $F_\pi(s)=m_\rho^2/[m_\rho^2-s-im_\rho\Gamma_\rho\theta(s-4m_\pi^2)]$. In all three curves, we have used $m_\rho=775$ MeV and $\Gamma_\rho=130$ MeV.

[36] Note that in Ref. 4, because of a mistake, it was claimed that the width at half-maximum is instead 118 MeV for our theoretically predicted form factor.

[37] S. Weinberg, Phys. Rev. Letters **18**, 507 (1967); T. Das, G. S. Guralnik, V. S. Mathur, F. E. Low, and J. E. Young, *ibid.* **18**, 759 (1967); S. L. Glashow, H. Schnitzer, and S. Weinberg, *ibid.* **19**, 139 (1967); T. Das, V. S. Mathur, and S. Okubo, *ibid.* **18**, 761 (1967); J. J. Sakurai, *ibid.* **19**, 803 (1967).

[38] R. J. Oakes and J. J. Sakurai, Phys. Rev. Letters **19**, 1266 (1967).

[39] The formalism of Ref. 8 is used.

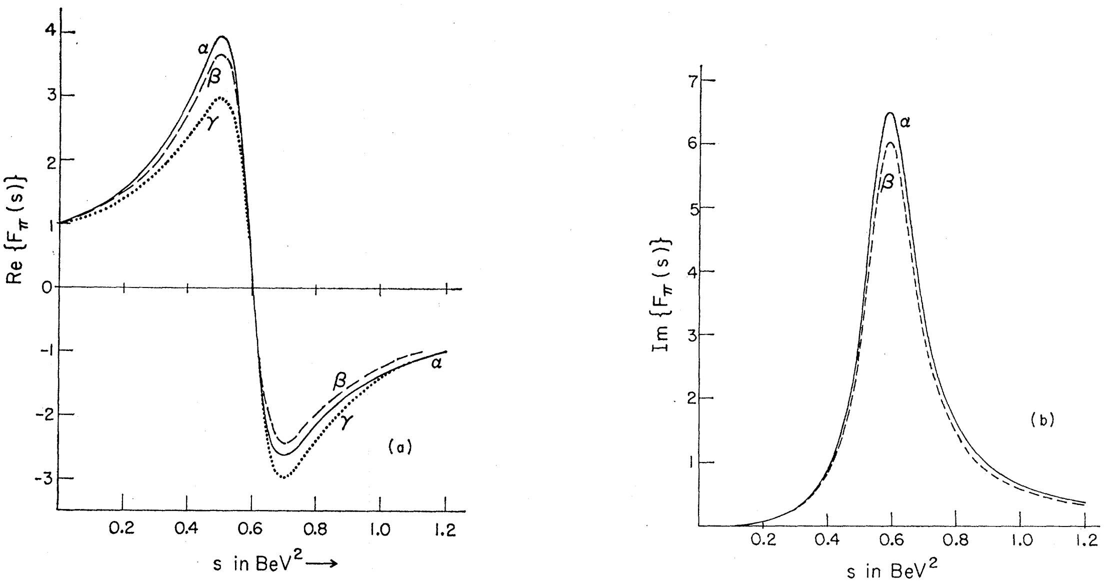

FIG. 6. The real and imaginary parts of our theoretically predicted $F_\pi(s)$ are compared with other commonly used forms. Curves α, β, and γ correspond to expressions for the form factor given in the caption of Fig. 5. In all three cases $m_\rho = 775$ MeV and $\Gamma_\rho = 130$ MeV were used.

We emphasize that this relation takes into account the finite width of the ρ meson. From (4.1) and

$$\Gamma(\rho^0 \to e^+e^-) = \tfrac{1}{3}\alpha^2(4\pi/f_\rho^2)m_\rho(Z_1/Z)\,, \qquad (3.7)$$

$$\Gamma(\omega \to e^+e^-) = \tfrac{1}{12}\alpha^2(4\pi/f_Y^2)m_\omega(\sin\theta_Y)^2\,, \qquad (4.2)$$

$$\Gamma(\varphi \to e^+e^-) = \tfrac{1}{12}\alpha^2(4\pi/f_Y^2)m_\varphi(\cos\theta_Y)^2\,, \qquad (4.3)$$

we derive in complete analogy with the stable-ρ approximation[40]

$$(Z/Z_1)\tfrac{1}{3}m_\rho\Gamma(\rho^0 \to e^+e^-) = m_\omega\Gamma(\omega \to e^+e^-) + m_\varphi\Gamma(\varphi \to e^+e^-)\,, \qquad (4.4)$$

where Z_1/Z is given in (2.31). We also recall[30,41] that the mixing angles θ_Y, θ_B, and θ are directly measurable in the lepton decays of ω and φ, since

$$\frac{\Gamma(\omega \to e^+e^-)}{\Gamma(\varphi \to e^+e^-)} = \frac{m_\omega}{m_\varphi}(\tan\theta_Y)^2 = \frac{m_\varphi}{m_\omega}(\tan\theta)^2\,. \qquad (4.5)$$

We see that in (4.4) the finite-width correction is given by the factor Z_1/Z. This is to be compared with (3.4), where the finite correction is given by $(Z_1/Z)^2$. Numerically, for $\Gamma_\rho = 130$ MeV,

$$Z_1/Z = 1.0805\,. \qquad (4.6)$$

Thus there is only an 8% finite-width correction in Eq. (4.4). Of course, by analogy with Oakes and Sakurai,[38] we can summarize (4.4) and (4.5) in a corresponding graphical representation shown in Fig. 7. We observe that the radius of the circle is reduced only by the factor

$$(1 + d\Gamma_\rho/m_\rho)^{-1/2}\,,$$

compared to the one in Ref. 38. Recent experimental Orsay results are also shown. As was observed earlier, experimentally we measure directly the leptonic branching ratio R using the total cross section $\sigma_{\rm tot}(e^+e^- \to \pi^+\pi^-)$ around the peak [see Eq. (3.3)]. Now in order to calculate the decay rate $\rho^0 \to e^+e^-$, we also need to know Γ_ρ, which, of course, depends on the experimental values of the total cross section away from the peak as well as around it. Because of increased experimental background difficulties as we go away from the peak, we believe that the experimental value for Γ_ρ may be considered as not so accurate as the experimental

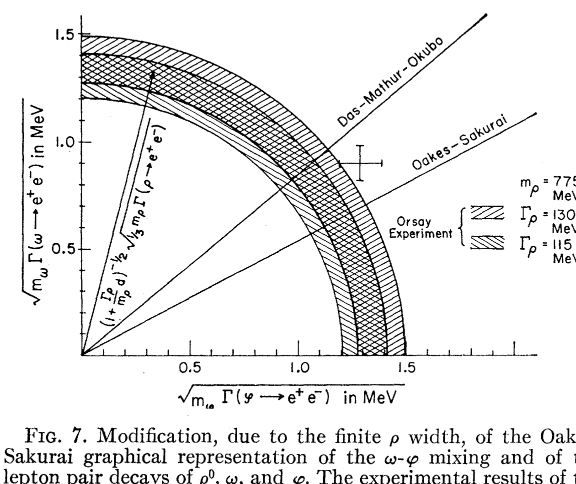

FIG. 7. Modification, due to the finite ρ width, of the Oakes-Sakurai graphical representation of the ω-φ mixing and of the lepton pair decays of ρ^0, ω, and φ. The experimental results of the Orsay group for $\Gamma_\rho = 130$ MeV and $\Gamma_\rho = 115$ MeV (see text) are also shown.

[40] T. Das, V. S. Mathur, and S. Okubo, Phys. Rev. Letters **19**, 470 (1967); also, J. J. Sakurai (Ref. 37).

[41] R. F. Dashen and D. H. Sharp (Ref. 30); see also C. A. Levinson, H. J. Lipkin, and S. Meshkov, Phys. Letters **7**, 81 (1963).

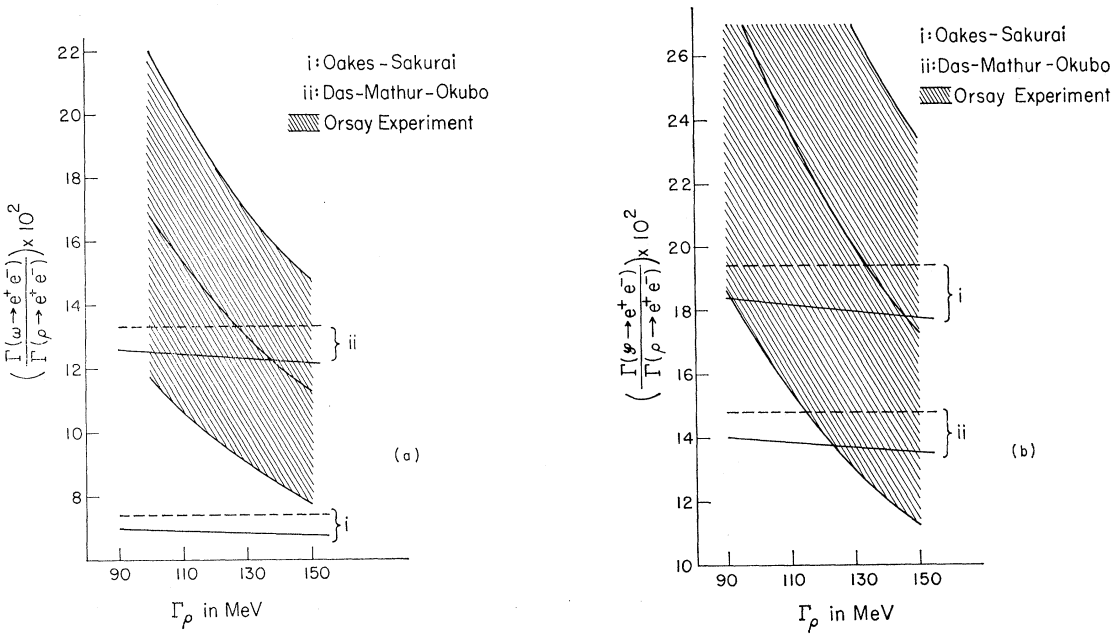

FIG. 8. The dependence of the branching ratios $\Gamma(\omega\to e^+e^-)/\Gamma(\rho^0\to e^+e^-)$ and $\Gamma(\varphi\to e^+e^-)/\Gamma(\rho^0\to e^+e^-)$ on the ρ-meson width for the Oakes-Sakurai model (curves i) and for the Das-Mathur-Okubo model (curves ii). The full lines are based on Eqs. (4.7) and (4.8) which take into account the finite ρ width. The broken lines are obtained when we treat the ρ meson as a stable particle. The experimental results of the Orsay group are indicated.

value of $\sigma_{\text{tot}}(e^+e^-\to\pi^+\pi^-)$ around the peak. Therefore, in Fig. 7 we give the experimental results for $\Gamma(\rho^0\to e^+e^-)$, for two values of Γ_ρ, namely, $\Gamma_\rho=130$ and 115 MeV. From (3.7) and (4.1)–(4.3) we can also write

$$\frac{\Gamma(\omega\to e^+e^-)}{\Gamma(\rho^0\to e^+e^-)}=\frac{m_\omega m_\rho(\sin\theta_Y)^2}{3[m_\varphi^2(\cos\theta_Y)^2+m_\omega^2(\sin\theta_Y)^2]}\left(\frac{Z}{Z_1}\right),\qquad(4.7)$$

$$\frac{\Gamma(\varphi\to e^+e^-)}{\Gamma(\rho^0\to e^+e^-)}=\frac{m_\varphi m_\rho(\cos\theta_Y)^2}{3[m_\varphi^2(\cos\theta_Y)^2+m_\omega^2(\sin\theta_Y)^2]}\left(\frac{Z}{Z_1}\right).\qquad(4.8)$$

To obtain the corresponding results in the stable-ρ-meson approximation, we have simply to put $Z=Z_1$ in (4.7) and (4.8). We see that the finite width reduces these branching ratios by the factor Z/Z_1 given in (2.31). This is illustrated in Fig. 8, where we summarize the Oakes-Sakurai[38] and Das-Mathur-Okubo[42] predictions for these branching ratios for various values of Γ_ρ. The experimental Orsay results are also shown.

An interesting effect of the width of the ρ meson is that it reduces the VMD prediction for $\Gamma(\varphi\to K\bar{K})$ by about 8%, for reasonable values of Γ_ρ. In more detail if we assume that the electromagnetic form factors of K and π are dominated by ρ, ω, and φ, we obtain via (3.7), (4.1), and the formalism[43] of Ref. 8

$$\frac{\Gamma(\varphi\to K\bar{K})}{\Gamma(\rho\to\pi\pi)}=\frac{3\{m_\varphi^2\cos^2\theta_Y+m_\omega^2\sin^2\theta_Y\}}{4m_\varphi^2}\times\left(\frac{\cos\theta_B}{\cos(\theta_Y-\theta_B)}\right)^2\left(\frac{p_{K^+K^-}{}^3+p_{K^0\bar{K}^0}{}^3}{p_{\pi\pi}{}^3}\right)\left(\frac{Z}{Z_1}\right)$$
$$=\frac{3}{4}\left(\frac{p_{K^+K^-}{}^3+p_{K^0\bar{K}^0}{}^3}{p_{\pi\pi}{}^3}\right)(\cos\theta)^2\left(\frac{Z}{Z_1}\right).\qquad(4.9)$$

The corresponding narrow-ρ-width prediction is obtained by simply putting $Z=Z_1$ in (4.9). The two formulas agree with each other in the limit $\Gamma_\rho\ll m_\rho$, but they differ by about 8% for realistic values of Γ_ρ. This is illustrated in Fig. 9, where the Oakes-Sakurai model

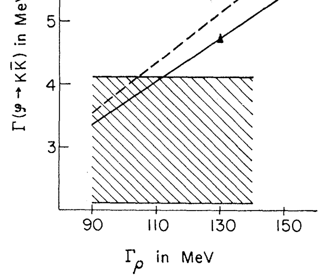

FIG. 9. Dependence of the decay rate $\Gamma(\varphi\to K\bar{K})$ on the ρ-meson width. The full line is based on Eq. (4.9), which takes into account the finite ρ width. The broken line is obtained when we treat the ρ meson as a stable particle. The currently accepted experimental results are indicated.

[42] Das, Mathur, and Okubo (Ref. 40).

[43] The relevance of (4.1) (or a relation equivalent to it obtained from the soft π, K technique) to the φ width has been discussed by V. S. Mathur, L. K. Pandit, and R. E. Marshak, Phys. Rev. Letters **16**, 947 (1966); P. P. Divakaran and L. K. Pandit, *ibid.* **19**, 535 (1967); Oakes and Sakurai (Ref. 38). Our approach is closely related to the one of the last reference.

TABLE I. The decay width $\Gamma(\omega\to 3\pi)$ in MeV for various forms of the ρ propagator ($m_\rho=775$ MeV).[a]

ρ propagator $R(s)$	$\Gamma_\rho=130$ MeV, $f_\rho^2/4\pi=2.3$, $f_{\pi\rho\omega}^2/4\pi=(0.39\pm0.07)/m_\pi^2$ $\Gamma(\omega-3\pi)$	$\Gamma_\rho=115$ MeV, $f_\rho^2/4\pi=2.05$, $f_{\pi\rho\omega}^2/4\pi=(0.34\pm0.06)/m_\pi^2$ $\Gamma(\omega-3\pi)$
$R(s)=1/(m_\rho^2-s)$	6.7±1.3	5.3±1.1
$R(s)=1/[m_\rho^2-s-im_\rho\Gamma_\rho\theta(s-4m_\pi^2)]$	6.2±1.2	5.0±0.8
$R(s)=1/[m_\rho^2-s-im_\rho\Gamma_\rho(k/k_\rho)^3(m_\rho/\sqrt{s})\theta(s-4m_\pi^2)]$	6.6±1.3	5.3±0.9
$R(s)$ given in Eqs. (2.29) and (2.11)	7.2±1.3	5.7±1.0

[a] Note that $f_{\pi\rho\omega}^2$ was calculated from the experimental decay width $\Gamma(\omega\to\pi^0\gamma)=1135\pm200$ keV, and the value of $f_\rho^2/4\pi$. The coupling constant $f_\rho^2/4\pi$ was calculated from Γ_ρ using (always) Eq. (2.24).

has been used for θ_Y and θ_B.[44] The currently accepted experimental results are also shown.[45] The triangular point indicates the current-algebra prediction ($\Gamma_\rho=130$ MeV).

Finally, our ρ-meson propagator has a rather impressive effect on the decay rate $\omega\to 3\pi$, which was calculated using the Gell-Mann–Sharp–Wagner[3] model. This is indicated[46] in Table I, where for comparison we also give the results for other commonly used forms of the ρ propagator. Also, in order to obtain some feeling for how our results depend on Γ_ρ, we give results for two values of Γ_ρ; i.e., $\Gamma_\rho=130$ MeV and $\Gamma_\rho=115$ MeV. In the calculations the coupling[47] $f_{\pi\rho\omega}$ was taken to be constant[32] as in the ordinary Gell-Mann–Sharp–Wagner model, and was fixed from the experimental value of the decay rate $\omega\to\pi^0\gamma$. For $\Gamma_\rho=130$ MeV, Eq. (2.24) gives $f_\rho^2/4\pi=2.3$, and using the commonly accepted experimental value for $\Gamma(\omega\to\pi^0\gamma)$,[45] we find[48]

$$f_{\pi\rho\omega}^2/4\pi=(0.39\pm0.08)/m_\pi^2. \tag{4.10}$$

For $\Gamma_\rho=115$ MeV, Eq. (2.24) gives $f_\rho^2/4\pi=2.05$ and correspondingly we have

$$f_{\pi\rho\omega}^2/4\pi=(0.34\pm0.06)/m_\pi^2. \tag{4.10'}$$

In order to be able to compare our ρ propagator with other forms for it, we have used in all cases listed in the Table I

$$f_{\rho\pi\pi}^2/4\pi=f_\rho^2/4\pi,$$

where $f_\rho^2/4\pi$ was calculated from (2.24) for each value of Γ_ρ.

The errors in the table are due only to errors in $f_{\pi\rho\omega}^2/4\pi$. The present experimental results[45] are

$$\Gamma(\omega\to 3\pi)_{\text{expt}}=11\pm2\text{ MeV},$$
$$\Gamma(\omega\to\pi^0\gamma)/\Gamma(\omega\to 3\pi)=0.10\pm0.03.$$

We see that our ρ propagator gives a better result by at least 10% than any other in the table. The reason is perhaps easy to see. The relevant part of the ρ propagator for the process $\omega\to 3\pi$ is for momentum transfers roughly between $s=(4m_\pi^2)\simeq0.08$ BeV^2 and $s=(m_\omega-m_\pi)^2\simeq0.42$ BeV^2. In Figs. 5 and 6, remembering that in all cases

$$R(s)=(1/m_\rho^2)F_\pi(s),$$

we can see the differences between the various propagators in this region. Of course, our propagator based on effective range is expected to be better than any other in the table, particularly for such a low-energy region. In connection with Table I, it is amusing to note that a stable-ρ propagator gives a better result than the one with "fixed width" (row 2 in Table I). It is also amusing to note that the over-all finite-width correction in $\Gamma(\omega\to 3\pi)$ will become negligible if in the row 1 in Table I Eq. (2.24′) is used, instead of (2.24), to calculate $f_\rho^2/4\pi$ from Γ_ρ. We conclude, therefore, that if we consistently neglect ρ-meson finite-width effects in the ρ propagator and in the coupling constant $f_\rho^2/4\pi$, we find the same result for $\Gamma(\omega\to 3\pi)$ as the one obtained when ρ-meson finite-width effects are taken into account in both these quantities.

We should also mention that the results of Table I are similar to those of Vaughn and Wali.[5]

V. HADRONIC CONTRIBUTION TO THE ANOMALOUS MAGNETIC MOMENT OF THE MUON

In the last year, evaluations of the leading contribution of the α^3 radiative correction[49] to $\frac{1}{2}(g-2)$ of the muon magnetic moment have been done and are shown to be not much larger than the expected order of magnitude $(\alpha/\pi)^3\sim10^{-8}$. Therefore, α^2 hadronic contributions arising from modifications of the photon propagator due to strong interactions may be at least as important. The formalism for such a calculation has been given a long time ago.[9] Our purpose here is to redo these calculations for the following reasons:

(1) We want to compare the result obtained using our pion form factor with those of other commonly used form factors, with the same ρ mass and ρ width.

[44] We have used $m_\rho=775$ MeV, $m_\omega=783$ MeV, $m_\varphi=1019$ MeV, $m_\pi=140$ MeV, $m_{K^+}=494$ MeV, $m_{K^0}=498$ MeV (Particle Data Group, Ref. 45).

[45] Particle Data Group (N. Barash-Schmidt *et al.*), Rev. Mod. Phys. **41**, 109 (1969).

[46] The kinematics for a three-pion final state is given in an immediately programmable way by M. Parkinson, Phys. Rev. **143**, 1359 (1966). I wish to thank T. H. Chang for suggesting this reference to me.

[47] The $\pi\rho\omega$ interaction was written as $\mathcal{L}=i\epsilon_{\mu\nu\lambda\sigma}f_{\pi\rho\omega}\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\varrho}_\nu\partial_\lambda\omega_\sigma$.

[48] Observe that the value of $f_{\pi\rho\omega}$ calculated from $\omega\to\pi^0\gamma$ does not depend on what form we use for the $\delta_{\mu\nu}$ part of the ρ propagator $R(s)$, so long as $R(0)=1/m_\rho^2$, as expected from gauge-field algebra.

[49] S. D. Drell, in *Proceedings of the Thirteenth Annual International Conference on High-Energy Physics, Berkeley, 1966* (University of California Press, Berkeley, 1966), p. 93.

(2) Since the last paper on the subject, the generalized Weinberg first sum rules[37,38] have been discovered, which relate f_ρ and f_Y with a relation different from the SU_3 prediction

$$f_Y=(\sqrt{\tfrac{3}{4}})f_\rho\,,$$

and new ideas about ω-φ mixing[38,42] have appeared which are consistent with these sum rules. These ideas, as well as the ρ-finite-width corrections to f_ρ, are going to modify the ω-φ contribution.

For completeness we will first review the relevant formulas, all of which have been given before.[9] The hadronic contribution to the muon magnetic moment is given by[9]

$$\tfrac{1}{2}(g-2)_h=\frac{\alpha}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds}{s^2}e^2\rho(s)G(s)\,, \tag{5.1}$$

where

$$G(s)=\int_0^1 dx\,\frac{x^2(1-x)}{x^2+(s/m_\mu^2)(1-x)}\,, \tag{5.2}$$

$$\rho(s)=\tfrac{1}{3}(2\pi)^3\sum_n\delta(p-p_n)\times\langle 0|j_\mu^{\text{em}}(0)|n\rangle\langle n|j_\mu^{\text{em}}(0)|0\rangle\,. \tag{5.3}$$

In (5.3) the sum is over all possible eigenstates of the strong interaction Hamiltonian and $j_\mu^{\text{em}}(x)$ is the hadronic part of the electromagnetic current with a factor e taken away. Just as in (2.17), $\rho(s)$ is related to the colliding-beam cross section via[16,17]

$$\rho(s)=(s^2/16\pi^3\alpha^2)\sigma_{\text{tot}}(e^+e^-\to(T=1\text{ system}))+(s^2/16\pi^3\alpha^2)\sigma_{\text{tot}}(e^+e^-\to(T=0\text{ system}))\,, \tag{5.4}$$

and therefore can, in principle, be measured. But actually all we need in order to find a reasonably accurate value for $\frac{1}{2}(g-2)_h$ is the low-energy behavior of $\rho(s)$. To see this, observe from (5.2) that for $s\gg m_\mu^2$

$$G(s)\sim 1/s\,, \tag{5.2$'$}$$

$$\tfrac{1}{2}(g-2)_h\sim\int ds\frac{\rho(s)}{s^3}\,. \tag{5.1$'$}$$

Therefore, the large-s contributions to $\rho(s)$ will be highly damped by the $1/s^3$ factor. It is therefore legitimate to take only ω, φ, and two-pion intermediate states in (5.3) and find via (2.18) that

$$e^2\rho(s)=(\alpha/12\pi)(s-4m_\pi^2)^{3/2}s^{-1/2}|F_\pi(s)|^2\theta(s-4m_\pi^2)+(\alpha\pi/f_Y^2)\{(\cos\theta_Y)^2m_\varphi^4\delta(s-m_\varphi^2)+(\sin\theta_Y)^2m_\omega^4\delta(s-m_\omega^2)\}\,. \tag{5.5}$$

Using (4.1), we write

$$\tfrac{1}{2}(g-2)_h=\tfrac{1}{2}(g-2)_\rho+\tfrac{1}{2}(g-2)_\omega+\tfrac{1}{2}(g-2)_\varphi\,, \tag{5.6}$$

where

$$\tfrac{1}{2}(g-2)_\rho=\frac{\alpha^2}{12\pi^2}\int_{4m_\pi^2}^{\infty}ds\frac{(s-4m_\pi^2)^{3/2}}{s^{5/2}}|F_\pi(s)|^2G(s)\,, \tag{5.7}$$

$$\tfrac{1}{2}(g-2)_\varphi=\frac{\alpha^2}{3\pi}\left(\frac{4\pi}{f_\rho^2}\right)\times\frac{m_\rho^2(\cos\theta_Y)^2}{m_\varphi^2(\cos\theta_Y)^2+m_\omega^2(\sin\theta_Y)^2}G(m_\varphi^2)\,, \tag{5.8}$$

$$\tfrac{1}{2}(g-2)_\omega=\frac{\alpha^2}{3\pi}\left(\frac{4\pi}{f_\rho^2}\right)\times\frac{m_\rho^2(\sin\theta_Y)^2}{m_\varphi^2(\cos\theta_Y)^2+m_\omega^2(\sin\theta_Y)^2}G(m_\omega^2)\,. \tag{5.9}$$

The coupling constant $f_\rho^2/4\pi$ is determined by Γ_ρ via Eq. (2.24). Our numerical results for $\frac{1}{2}(g-2)_\rho$ are shown in Table II, where we also give results for other commonly used expressions for the pion form factor as well as for the ρ stable approximation. We see that our form factor gives a ρ contribution which is at least $\sim$15% larger than the one obtained from any of the other expressions listed in the table. The reason is easy to see from Fig. 5, since the integral (5.7) rather strongly emphasizes the low-energy contribution. In Table III we give the total hadronic contributions $\frac{1}{2}(g-2)_h$, as well as the ones for ω and φ for $\Gamma_\rho=130$ MeV, for both the Oakes-Sakurai[37] and the Das-Mathur-Okubo[41] models. Although the hadronic effects estimated above are well below the present uncertainties in the currently accepted value[45]

$$\tfrac{1}{2}(g-2)_h^{\text{expt}}=(116\,645\pm33)\times10^{-8}\,,$$

we may expect that in the near future the accuracy will increase so that we will not only be able to check these

TABLE II. ρ-meson contribution to the magnetic moment of the muon for various forms of the pion form factor ($m_\rho=775$ MeV).[a]

	$\Gamma_\rho=130$ MeV	$\Gamma_\rho=115$ MeV
ρ meson treated as a stable particle	$4.0\times10^{-8}=3.20(\alpha/\pi)^3$	$4.5\times10^{-8}=3.60(\alpha/\pi)^3$
$F_\pi(s)=m_\rho^2/[m_\rho^2-s-im_\rho\Gamma_\rho\theta(s-4m_\pi^2)]$	$4.0\times10^{-8}=3.20(\alpha/\pi)^3$	$4.5\times10^{-8}=3.60(\alpha/\pi)^3$
$F_\pi(s)=m_\rho^2/[m_\rho^2-s-im_\rho\Gamma_\rho(k/k_\rho)^3(m_\rho/\sqrt{s})\theta(s-4m_\pi^2)]$	$4.2\times10^{-8}=3.36(\alpha/\pi)^3$	$4.7\times10^{-8}=3.76(\alpha/\pi)^3$
$F_\pi(s)$ given in Eq. (2.11)	$4.8\times10^{-8}=3.84(\alpha/\pi)^3$	$5.3\times10^{-8}=4.24(\alpha/\pi)^3$

[a] Note that the coupling constant $f_\rho^2/4\pi$, which is needed to find the results listed in the first row in this table, was calculated from Γ_ρ via Eq. (2.24).

TABLE III. Hadronic contribution to the magnetic moment of the μ^-. (In this table the ρ-meson contribution was taken from the last line in Table II; $\Gamma_\rho=130$ MeV.)

ω-φ mixing model	$\frac{1}{2}(g-2)_\omega$	$\frac{1}{2}(g-2)_\varphi$	$\frac{1}{2}(g-2)_{\text{total hadronic}}$
Oakes-Sakurai	0.3×10^{-8}	0.4×10^{-8}	$5.5\times10^{-8}=4.36(\alpha/\pi)^3$
Das-Mathur-Okubo	0.3×10^{-8}	0.5×10^{-8}	$5.6\times10^{-8}=4.45(\alpha/\pi)^3$

TABLE IV. ρ-meson contribution to the charge renormalization to the order e^2 ($m_\rho=775$ MeV, $\Gamma_\rho=130$ MeV, $f_\rho^2/4\pi=2.3$).[a]

	$(\delta e_0^2)_\rho/e^2$
Treating ρ meson as stable particle	-0.317%
$F_\pi(s)$ from effective range [Eq. (2.11)]	-0.325%

[a] Note that $f_\rho^2/4\pi$ was calculated from Γ_ρ via Eq. (2.24).

hadronic effects but also, perhaps, give some independent information about the correct ω-φ mixing and the precise form of the pion form factor.

In the remainder of this section we shall calculate two quantities of only "theoretical" interest, namely, the wave-function renormalization constant Z for the ρ field, and the hadronic contribution to the order e^2 part of the charge renormalization. The calculation is rather enlightening, and so we shall give the details.

The wave-function renormalization constant is

$$Z^{-1}=\int_{4m_\pi^2}^{\infty}\sigma_\rho(s)ds\,, \tag{5.10}$$

where $\sigma_\rho(s)$ is the expression appearing in (2.15). Using (2.15), we have

$$Z^{-1}=\frac{1}{\pi}\int_{4m_\pi^2}^{\infty}\text{Im}[R(s)]ds\,. \tag{5.11}$$

For $R(s)$, we use[50] (2.29) and (2.11) in spite of the fact that there is no damping factor in (5.11) and that expressions (2.29) and (2.11) are valid only for low s. We use these expressions as if they were valid everywhere. Remembering that $F(s)$ has a pole at $s=s_0=-9.4\times10^6m_\rho^2$, we find

$$Z^{-1}=\frac{1}{m_\rho^2}\,\text{residue}[F_\pi(s)]\Big|_{s=s_0}=15.5\,. \tag{5.12}$$

From (5.12) and our convention

$$Z=Z_0=(m_\rho/m_\rho^0)^2\,,$$

we find (with $m_\rho=775$ MeV) that

$$m_\rho^0=3\text{ BeV}\,. \tag{5.13}$$

We remind the reader that in a divergenceless theory, $Z^{-1}=\infty$. Note also that if we had not ignored the vanishing point of $f(s)$ at $s=s_0=-9.4\times10^6m_\rho^2$, the correct expression for the pion form factor would have been

$$F_\pi(s)=[(s_0-s)/s_0]f(0)/f(s)\,,$$

where $f(s)$ is given by (2.4). In this case from (2.28) and (2.19), we immediately see that

$$Z^{-1}=\infty\,, \tag{5.12'}$$

[50] It is easy to see that this result remains unaltered if, instead of using (2.29) for $R(s)$, we had used the expression which does not ignore the pole of $F_\pi(s)$ at $s_0=-9.4\times10^6m_\rho^2$ (Ref. 19).

and therefore

$$m_\rho^0=\infty\,,$$

that is, the theory becomes divergenceless.

Now let us consider the hadronic contribution to the e^2 part of the charge renormalization. It is given by[8]

$$(\delta e_0^2)_h/e^2=(\delta e_0^2)_\rho/e^2+(\delta e_0^2)_{\omega,\varphi}/e^2\,, \tag{5.14}$$

where the ρ contribution is

$$\frac{(\delta e_0^2)_\rho}{e^2}=-e^2\int_{4m_\pi^2}^{\infty}\frac{\rho_1^{(3)}(s)}{s^2}ds\,, \tag{5.15}$$

and the ω-φ contribution is

$$\frac{(\delta e_0^2)_{\omega,\varphi}}{e^2}=-\frac{e^2}{4}\int_{9m_\pi^2}^{\infty}\frac{1}{s^2}\rho^{(Y)}(s)ds\,. \tag{5.16}$$

Note that

$$\rho_1^{(3)}(s)+\tfrac{1}{4}\rho^{(Y)}(s)=\rho(s)\,, \tag{5.17}$$

and $\rho(s)$ has been given in terms of the hadronic part of the electromagnetic current[51] in (5.3). First the ρ contribution. Since high values of s in (5.15) are damped by the $1/s^2$ factor, it is perfectly legitimate to use (2.19) for $\rho_1^{(3)}(s)$. On the other hand, from (2.16) we immediately have

$$(\delta e_0^2)_\rho/e^2=-e^2g'(0)\,. \tag{5.18}$$

Using (2.20) and (5.18), it is a matter of straightforward algebra to find

$$\frac{(\delta e_0^2)_\rho}{e^2}=-\tfrac{2}{3}\alpha\left[\frac{k_\rho^3}{m_\rho^2\Gamma_\rho}+\tfrac{1}{4}h(m_\rho^2)+k_\rho^2h'(m_\rho^2)-\frac{1}{3\pi}\right]=-0.325\%\,, \tag{5.19}$$

for $\Gamma_\rho=130$ MeV. Note that if the width of the ρ meson is neglected,

$$(\delta e_0^2)_\rho/e^2=-e^2/f_\rho^2=-0.317\%\,. \tag{5.19'}$$

These results are also summarized in Table IV. The ω-φ contribution is

$$(\delta e_0^2)_{\omega,\varphi}/e^2=-e^2/4f_Y^2\,, \tag{5.20}$$

where f_Y is related to f_ρ via (4.1). The results for both the Oakes-Sakurai[37] and the Das-Mathur-Okubo[41] models are summarized in Table V.

[51] Actually, $\frac{1}{4}\rho_1^{(Y)}(s)$ is the spectral function due to the isosinglet part of the electromagnetic current.

TABLE V. Hadronic contribution to the charge renormalization to the order e^2. (The ρ-meson contribution is taken from the last line in Table IV.)

	$(\delta e_0^2)_\rho/e^2$	$(\delta e_0^2)_{\omega,\varphi}/e^2$	$(\delta e_0^2/e^2)$total hadronic
Oakes-Sakurai	-0.325%	-0.071%	-0.396%
Das-Mathur-Okubo	-0.325%	-0.079%	-0.404%

VI. SUMMARY AND CONCLUSIONS

(1) We have seen in the various calculations presented in this paper that the correction due to the finite width of the ρ meson may be as large as 15%, compared to the ordinary VMD predictions, for reasonable values of the ρ width. We have seen in more detail that such a high correction appeared in the branching ratio

$$R=\Gamma(\rho^0\to e^+e^-)/\Gamma(\rho\to\pi\pi)\,.$$

Our method gave also a correction of 10% or higher to the ρ contribution to the anomalous magnetic moment of the muon, compared to the results obtained with other commonly used forms for the $\pi^\pm$ form factor. These later results are due to the fact that our pion form factor is considerably different than the usual forms for it, particularly in the low-energy region, as was shown in Fig. 5.

(2) Smaller corrections (about 8% of the ordinary VMD predictions) were also seen in the branching ratios

$$\frac{\Gamma(\varphi\to K\bar{K})}{\Gamma(\rho\to\pi\pi)}\,,\quad\frac{\Gamma(\omega\to e^+e^-)}{\Gamma(\rho^0\to e^+e^-)}\,,\quad\frac{\Gamma(\varphi\to e^+e^-)}{\Gamma(\rho^0\to e^+e^-)}\,,$$

and in the sum rule

$$\tfrac{1}{3}m_\rho\Gamma(\rho^0\to e^+e^-)=m_\varphi\Gamma(\varphi\to e^+e^-)+m_\omega\Gamma(\omega\to e^+e^-)\,.$$

In particular, the finite ρ width tends to decrease the value of $\Gamma(\varphi\to K\bar{K})$ by the factor

$$(1+\Gamma_\rho d/m_\varphi)^{-1}=(1.0805)^{-1}$$

for $\Gamma_\rho=130$ MeV, and move it towards the right direction in order to agree with experiment.

(3) On the theoretical side, the ρ propagator was calculated by integration of the low-energy part of the spectral function $\sigma_\rho(m^2)$. To no surprise we found that the ρ propagator is responsible for all of the structure of the pion form factor for low energies, in complete analogy with the stable ρ meson, VMD expression

$$F_\pi(s)=m_\rho^2/(m_\rho^2-s)\,,$$

and consequently the $\rho\pi\pi$ coupling does not depend on s, for low s, even when the width is explicitly taken into account [see Eq. (2.37)].

(4) Finally, we have shown in our model that although the gauge-field algebra[20] with the convention

$$Z=Z_0=(m_\rho/m_\rho{}^0)^2\,,$$

immediately fixes the normalization of the renormalized vector-meson field, it is perfectly correct, in the approximation where we neglect the width of the vector meson, to take

$$Z_0=Z_1=Z\,,$$

and consequently to assume the usual normalization for the renormalized ρ field. However, this is no longer the case when the width is taken into account. In this latter case, the relation $Z_1/Z=1$ does not hold any longer; it is replaced by

$$Z_1/Z=1+(\Gamma_\rho/m_\rho)d\,,$$

provided, of course, that we stick to the convention $Z=Z_0$, which has the important advantage of making $(m_\rho/f_\rho)^2$ "renormalization-invariant."[8]

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