Impact-Parameter Representations and the Coordinate-Space Description of a Scattering Amplitude

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(Received 6 January 1969; revised manuscript received 27 February 1969)

We show how impact-parameter representations with a dependence on final c.m. momentum **k**, angle θ, and impact parameter b given by $J_0(kb\sin\theta)$ follow naturally when we describe a scattering process in terms of states localized in the transverse plane in coordinate space, or more generally when we analyze it in terms of a Fourier transform with respect to the transverse momentum. We discuss this for the nonrelativistic as well as the relativistic scattering amplitude. Two classes of impact-parameter representations are considered here: one in which the impact parameter is a position coordinate, canonically conjugate to the transverse momentum; and another—a fixed-energy representation—which is obtained by reinterpreting the Fourier transform with respect to the transverse momentum, in which the impact parameter is not a position coordinate, except in the limit of infinite energy. We show that the first kind of impact-parameter representation follows from the description of a scattering process in terms of transient asymptotic states localized in the transverse coordinate space. We point out the distinctive features of each type of representation, and discuss the conditions under which each is valid.

I. INTRODUCTION

In a recent paper, an impact-parameter representation (IPR) of the scattering amplitude for finite energies was discussed for the scattering of spinless particles.[1] The generalization to arbitrary spins is discussed in a separate paper.[2] These representations involve the functions $J_m(kb\sin\theta)$ rather than the $J_m(2kb\sin\frac{1}{2}\theta)$ used more commonly till now.[3–5] In the limit of infinite energy they again lead to the eikonal approximation to the scattering amplitude.

In this paper, we discuss the relation of IPR's [with $J_0(kb\sin\theta)$, for spinless particles] with the coordinate-space representation of the scattering amplitude, and more generally, with the Fourier transform of the scattering amplitude with respect to the transverse momentum.

The representation of the scattering amplitude as a Fourier transform with respect to the transverse momentum is a natural step in the infinite-momentum frame, where the only degrees of freedom are the transverse ones. In this frame it immediately shows the relation between the IPR and the description of the scattering process in terms of states localized in the transverse coordinate space.[6,7] A similar analysis may be made for the scattering amplitude in the c.m. frame.

In Secs. II and III we discuss this for a nonrelativistic (Schrödinger) scattering amplitude and for a relativistic amplitude, respectively. In Sec. IV we show how the impact-parameter amplitudes describe a scattering process in terms of transient asymptotic states localized in the transverse plane in coordinate space.

In Sec. V we discuss a fixed-energy IPR obtained by reinterpreting the Fourier transform with respect to the transverse momentum. Finally, in Sec. VI we conclude by comparing briefly the different IPR's discussed in this paper.

II. NONRELATIVISTIC SCATTERING

Consider the scattering of two spinless particles. In the usual way we separate out the over-all motion of the c.m. of the system, and treat the relative motion as equivalent to the scattering of a single particle (with a mass equal to the reduced mass m) by a fixed potential $V(\mathbf{r})$. The relative motion is described by the Schrödinger equation[8]

$$[\nabla^2+2mE]\psi(\mathbf{r})=V(\mathbf{r})\psi(\mathbf{r}). \tag{2.1}$$

We now view the scattering process in terms of its projection on the transverse plane; we introduce the projection of the wave function on the transverse plane

* Work supported in part by the U. S. Atomic Energy Commission (Report No. NYO-2262TA-197).

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[7] For a discussion of the kinematics in the infinite-momentum frame, see H. Bacry and N. P. Chang, Ann. Phys. (N. Y.) **47**, 407 (1968).

[8] Throughout this paper the symbols $\boldsymbol{\rho}$, $\mathbf{k}_t$, and $\mathbf{b}$ are used to denote two-dimensional vectors. All other boldface symbols denote three-vectors.

by[8]

$$\psi(\mathbf{r})=\psi(\boldsymbol{\varrho},z)=\int_{-\infty}^{+\infty} d\kappa\, e^{i\kappa z}u(\boldsymbol{\varrho};\kappa)\,, \qquad (2.2)$$

where $\boldsymbol{\varrho}=(x,y)$ is the projection of the vector $\mathbf{r}=(x,y,z)$ on the transverse plane. For the potential $V(\mathbf{r})$, we similarly define

$$V(\mathbf{r})=\int_{-\infty}^{+\infty} d\kappa\, e^{i\kappa z}v(\boldsymbol{\varrho};\kappa)\,. \qquad (2.3)$$

We first consider scattering with only a direct potential (i.e., in the absence of any exchange potential). Then only one function $u(\boldsymbol{\varrho};\kappa)$ is needed to describe both forward and backward scattering. Similarly, when the two (spinless) particles are identical (so that the direct and exchange potentials are equal), the scattering is symmetric in the forward and backward directions, and only one function $u(\boldsymbol{\varrho};\kappa)$ is needed.

The equation of motion in the transverse plane is given by

$$(\nabla_t^2+k_t^2)u(\boldsymbol{\varrho},\kappa)=\int_{-\infty}^{+\infty} d\kappa' v(\boldsymbol{\varrho};\kappa-\kappa')u(\boldsymbol{\varrho},\kappa')\,, \qquad (2.4)$$

where

$$k_t^2=2mE-\kappa^2\,, \qquad (2.5)$$

and ∇_t^2 is the Laplacian in the two-dimensional transverse coordinate space.

The Green's function for (2.4) satisfies the equation

$$(\nabla_t^2+k_t^2)G(\boldsymbol{\varrho}-\boldsymbol{\varrho}';k_t)=2\pi\delta(\boldsymbol{\varrho}-\boldsymbol{\varrho}')\,. \qquad (2.6)$$

The solutions to (2.6) may be written as linear combinations of the Hankel functions $H_0^{(1)}(k_tR)$ and $H_0^{(2)}(k_tR)$, where $R=|\boldsymbol{\varrho}-\boldsymbol{\varrho}'|$. The Green's function corresponding to an outgoing-wave boundary condition is

$$G^{(+)}(\boldsymbol{\varrho}-\boldsymbol{\varrho}';k_t)=\tfrac{1}{2}i\pi H_0^{(1)}(k_tR) \qquad (2.7)$$

$$\xrightarrow[R\to\infty]{} \pi\frac{e^{i(k_tR+\pi/4)}}{(2\pi k_tR)^{1/2}}\,, \qquad (2.8)$$

which behaves like an outgoing circular wave for large distances in the transverse coordinate space.

The integral equation corresponding to (2.4), with an outgoing-wave boundary condition, is

$$u(\boldsymbol{\varrho};\kappa)=\delta(\kappa-k)+\frac{1}{2\pi}\int d^2\rho'\int_{-\infty}^{+\infty} d\kappa' G^{(+)}(\boldsymbol{\varrho}-\boldsymbol{\varrho}';k_t) \times v(\boldsymbol{\varrho}';\kappa-\kappa')u(\boldsymbol{\varrho}';\kappa')\,, \qquad (2.9)$$

where the inhomogeneous term corresponds to an incident plane wave along the z axis, $\psi(\mathbf{r})=e^{ikz}$.

Writing $\boldsymbol{\varrho}=(\rho,\phi)$ in polar coordinates, we obtain, in the limit of large distances in the transverse plane, the form[9]

$$u(\boldsymbol{\varrho};\kappa)\xrightarrow[\rho\to\infty]{}\delta(\kappa-k)+f(k_t,\phi;\kappa)\frac{e^{i(k_t\rho+\pi/4)}}{(2\pi k_t\rho)^{1/2}}\,, \qquad (2.10)$$

where

$$f(\mathbf{k}_t;\kappa)\equiv f(k_t,\phi;\kappa) =\tfrac{1}{2}\int d^2\rho' e^{-i\mathbf{k}_t\cdot\boldsymbol{\rho}'}\int d\kappa' v(\boldsymbol{\varrho}',\kappa-\kappa')u(\boldsymbol{\varrho}',\kappa')\,. \qquad (2.11)$$

In (2.11) we have defined the two-dimensional vector $\mathbf{k}_t=(k_t,\phi)$.

The coefficient $f(\mathbf{k}_t;\kappa)$ of the outgoing circular wave in (2.10) is thus just the usual scattering amplitude:

$$f(\mathbf{k})=\frac{1}{4\pi}\int d^3\mathbf{r}' e^{-i\mathbf{k}\cdot\mathbf{r}'}V(\mathbf{r}')\psi(\mathbf{r}')\,. \qquad (2.12)$$

This is as expected, since the flux at azimuth ϕ for a circular wave of momentum k_t $(=k\sin\theta)$ is proportional to the flux of an outgoing spherical wave of momentum k in the direction (θ,ϕ).

When there is also an exchange potential, the transforms (2.2) and (2.3) must be defined separately for positive and negative z. To obtain suitable analyticity properties, we define the transforms of the even and odd combinations $\psi^{(\pm)}$:

$$\psi^{(+)}(\boldsymbol{\varrho},z)=\tfrac{1}{2}[\psi(\boldsymbol{\varrho},z)+\psi(\boldsymbol{\varrho},-z)] =\int_0^{\infty} d\kappa\cos\kappa z\, u^{(+)}(\boldsymbol{\varrho};\kappa)\,, \qquad (2.13a)$$

$$\psi^{(-)}(\boldsymbol{\varrho},z)=(2i)^{-1}[\psi(\boldsymbol{\varrho},z)-\psi(\boldsymbol{\varrho},-z)] =\int_0^{\infty} d\kappa\sin\kappa z\, u^{(-)}(\boldsymbol{\varrho};\kappa)\,, \qquad (2.13b)$$

and similarly for $V^{(\pm)}(\boldsymbol{\varrho},z)$. This gives equations analogous to (2.9) and (2.11) for the even- and odd-signature wave functions $u^{(\pm)}$ and amplitudes $f^{(\pm)}$.

Defining

$$u^{(\pm)}(\boldsymbol{\varrho};\kappa)=\frac{1}{2\pi}\sum_{m=-\infty}^{+\infty} e^{im\phi}u_m^{(\pm)}(\rho;\kappa)\,, \qquad (2.14)$$

$$f^{(\pm)}(\mathbf{k}_t,\kappa)=\frac{1}{2\pi}\sum_m e^{im\phi}f_m^{(\pm)}(k_t,\kappa)\,, \qquad (2.15)$$

and noting that there is an incident wave only for $m=0$, we obtain

$$\bar{f}^{(\pm)}(k_t,k_z)\equiv\frac{1}{2\pi}\int_0^{2\pi} d\phi\, f^{(\pm)}(k_t,k_z) =\int_0^{\infty} db\, bJ_0(k_tb)A^{(\pm)}(b,k_z)\,, \qquad (2.16)$$

where

$$A^{(\pm)}(b,k_z)=\frac{1}{2\pi}\int_0^{2\pi} d\phi'\int_{-\infty}^{+\infty} d\kappa' v^{(\pm)}(b;k_z-\kappa') \times u^{(\pm)}(\mathbf{b};\kappa')\,. \qquad (2.17)$$

[9] Note that the outgoing circular wave, with unit flux, from a point source is $e^{i(k_t\rho+\pi/4)}/(2\pi k_t\rho)^{1/2}$, as $\rho\to\infty$.

Equation (2.16) gives an IPR of the nonrelativistic scattering amplitude, for fixed k_z.

III. RELATIVISTIC SCATTERING

For relativistic scattering, we again define the Fourier transform, in the transverse plane, of the even and odd signature amplitudes. Denoting

$$\begin{aligned} T^{(\pm)}(\mathbf{k}_t,k_z) &= (16p_A{}^0p_B{}^0p_C{}^0p_D{}^0)^{-1/2} \\ &\quad\times\tfrac{1}{2}[T(\mathbf{k}_t,k_z)+T(\mathbf{k}_t,-k_z)], \\ T^{(-)}(k_t,k_z) &= (16p_A{}^0p_B{}^0p_C{}^0p_D{}^0)^{-1/2} \\ &\quad\times(2\cos\theta)^{-1}[T(\mathbf{k}_t,k_z)-T(\mathbf{k}_t,-k_z)], \end{aligned} \tag{3.1}$$

for the reaction $A+B\rightarrow C+D$, where $\cos\theta=k_z/k$, $k\equiv(\mathbf{k}_t{}^2+k_z{}^2)^{1/2}$, we define

$$T^{(\pm)}(\mathbf{k}_t,k_z)=\frac{1}{2\pi}\int d^2\mathbf{k}\, e^{-i\mathbf{k}_t\cdot\mathbf{b}}A^{(\pm)}(\mathbf{b};k_z). \tag{3.2}$$

For spinless particles, $T^{(\pm)}$ must be independent of the azimuth ϕ of $\mathbf{k}_t$; therefore $A^{(\pm)}$ depends on $\mathbf{b}$ only through $|\mathbf{b}|=b$. Integrating over ϕ, we obtain the IPR:

$$T^{(\pm)}(k_t;k_z)=\int_0^\infty db\, bJ_0(bk_t)A^{(\pm)}(b,k_z). \tag{3.3}$$

Note that in (2.16) and (3.3), we have written $A^{(\pm)}(b,k_z)$ with k_z held fixed, as is natural when we regard $\mathbf{b}$ as the transverse space coordinate. This gives the inverse relations corresponding to (3.2) and (3.3) as

$$\begin{aligned} A^{(\pm)}(b,k_z) &= \frac{1}{2\pi}\int d^2k_t e^{i\mathbf{k}_t\cdot\mathbf{b}}T^{(\pm)}(k_t,k_z) &(3.4)\\ &= \int_0^\infty dk_t k_t J_0(bk_t)T^{(\pm)}(k_t,k_z). &(3.5)\end{aligned}$$

In (3.4) and (3.5), k_z is held fixed during the integration, so that an integration over all energies (or all values k of the momentum) is involved, and the amplitudes $A^{(\pm)}$ do not correspond to a fixed value of the energy.

With this form of the IPR, which is the one obtained directly on treating $\mathbf{k}_t$ and $\mathbf{b}$ as canonically conjugate variables (that is, on regarding $\mathbf{b}$ as the transverse coordinate vector), the inverse relations (3.4) and (3.5) involve the physical amplitudes at all energies. The impact-parameter amplitudes $A^{(\pm)}(b,k_z)$ are the amplitudes for the scattering of a two-particle system from an initial plane-wave state to a final state with a definite impact parameter $\mathbf{b}$ and a definite z component of the momentum. In the final state, the transverse momentum and the total momentum do not have definite values.

The validity of the fixed-k_z representation requires the integrability condition

$$\int_0^\infty dk_t k_t{}^{1/2}|T^{(\pm)}(k_t,k_z)|<\infty\,, \tag{3.6}$$

which restricts the behavior of the amplitudes $T^{(\pm)}$ as $k_t\rightarrow\infty$, for fixed k_z, or equivalently, as $s\rightarrow\infty$, $t\rightarrow -s/2$, in terms of the usual Mandelstam variables.

As an alternative to (3.2), we may define a transform for fixed θ:

$$T^{(\pm)}(\mathbf{k}_t;\theta)=\frac{1}{2\pi}\int d^2\mathbf{b}\, e^{-i\mathbf{k}_t\cdot\mathbf{b}}\bar{A}^{(\pm)}(\mathbf{b};\theta), \tag{3.7}$$

which leads to equations of the form of (3.3) to (3.5), but with θ held fixed:

$$T^{(\pm)}(k,\theta)=\int_0^\infty db\, bJ_0(bk_t)\bar{A}^{(\pm)}(b,\theta), \tag{3.8}$$

$$\bar{A}^{(\pm)}(b,\theta)=(\sin^2\theta)\int_0^\infty dk\, kJ_0(bk\sin\theta)T^{(\pm)}(k,\theta). \tag{3.9}$$

Again the inverse transforms involve only the physical amplitudes, and the integration is over all k. The integrability condition now reads

$$\int_0^\infty dk\, k^{1/2}|T^{(\pm)}(k,\theta)|<\infty\,, \tag{3.10}$$

which restricts the behavior of $T^{(\pm)}$ for $k\rightarrow\infty$ for fixed θ, or equivalently, for $s\rightarrow\infty$, $-s\leq t\leq -\frac{1}{2}s$.

In Sec. V we shall discuss a fixed-energy IPR in which the impact-parameter amplitudes $\mathcal{A}^{(\pm)}(b,k)$ correspond to a fixed value of the energy, but require a continuation of the scattering amplitude to values of $\sin\theta$ outside the physical region.

IV. IPR AND THE DESCRIPTION OF SCATTERING IN TERMS OF LOCALIZED TRANSIENT STATES

In this section, we show how the IPR is intimately related to the description of a scattering process in terms of transient asymptotic states localized in the transverse plane in coordinate space. We first consider nonrelativistic scattering.

Consider a nonrelativistic two-particle system in the c.m. frame. We define a two-particle state with a definite (two-dimensional) impact parameter $\mathbf{b}$ and a z component κ of the momentum by[10]

$$|\mathbf{b},\kappa\rangle_c=\frac{1}{2\pi}\int d^2k_t e^{-i\mathbf{k}_t\cdot\mathbf{b}}|\mathbf{k}_t,\kappa\rangle_c\,, \tag{4.1}$$

where $|\mathbf{k}_t,\kappa\rangle_c$ is a two-particle plane-wave state in the c.m. frame, with transverse momentum $\mathbf{k}_t$. We normalize the plane-wave states according to

$${}_c\langle\mathbf{k}_t,\kappa|\mathbf{k}_t{}',\kappa'\rangle_c=(2\pi)^3\delta(\mathbf{k}_t-\mathbf{k}_t{}')\delta(\kappa-\kappa')\,, \tag{4.2}$$

[10] Note that the state $|\mathbf{b},\kappa\rangle_c$ is a two-particle state in the c.m. frame, with a definite impact parameter, a definite value (namely, zero) of the total 3-momentum of the two particles, and a definite value of the z component of the relative momentum, but not a definite value of the energy.

which implies the normalization

$$_c\langle \mathbf{b},\kappa | \mathbf{b}',\kappa'\rangle_c = (2\pi)^3\delta(\mathbf{b}-\mathbf{b}')\delta(\kappa-\kappa') \qquad (4.3)$$

of the states $|\mathbf{b},\kappa\rangle_c$. The completeness of the plane-wave states, expressed by the relation

$$\int \frac{dq_z d^2q}{(2\pi)^3} |\mathbf{q},q_z\rangle_c\, {}_c\langle \mathbf{q},q_z| = 1, \qquad (4.4)$$

implies that the states $|\mathbf{b},\kappa\rangle_c$, too, form a complete set[11]:

$$\int \frac{d\kappa d^2b}{(2\pi)^3} |\mathbf{b},\kappa\rangle_c\, {}_c\langle \mathbf{b},\kappa| = 1. \qquad (4.5)$$

The significance of the description of the scattering process in terms of the impact-parameter states becomes clear when we examine these states in coordinate space.

From the classical meaning of the impact parameter, we expect an asymptotic state with a definite impact parameter to be a state that is localized in the transverse plane in coordinate space at some time in the distant past or the distant future. Such a state cannot be a stationary state, but must be described by a wave packet. For instance consider the state (4.1) at time T in the distant past. (We later let $T\to\infty$.) The coordinate space representative of the state (4.1) is given by the wave packet

$$\Phi(\boldsymbol{\rho},z;\,t,T) = \int dk_z d^2\mathbf{k}_t dE\ \theta(E) \times e^{i\mathbf{k}_t\cdot\rho+ik_z z} e^{-iE(t-T)} a(\mathbf{k}_t,k_z,E), \qquad (4.6)$$

where

$$a(\mathbf{k}_t,k_z,E) = e^{-i\mathbf{k}_t\cdot\mathbf{b}}\delta(k_z-\kappa)\delta[E-(\mathbf{k}_t^2+k_z^2)/2M], \qquad (4.7)$$

and M is the reduced mass for the two-particle system.

The wave packet (4.6) is quite different from the type of wave packets usually considered in the momentum-space description of scattering. At time $t=T\to-\infty$, (4.6) describes a state exactly localized in the transverse plane in coordinate space, with a definite z component κ of the momentum:

$$\Phi(t=T\to-\infty) = e^{i\kappa z}\delta(\boldsymbol{\rho}-\mathbf{b}). \qquad (4.8)$$

Owing to its large spread in momentum space, such a wave packet changes in character very rapidly with time.[12] Such states will be termed transient states or transients, in contradistinction to stationary states.

To examine how the wave packet (4.6) moves in the transverse plane as a function of time, as it spreads, we define the mean position (or "center") of the wave packet in the transverse plane as the expectation value of the transverse position operator $\boldsymbol{\rho}$ in the state specified by (4.6).

This expectation value is found to be just $\langle\boldsymbol{\rho}\rangle = b$, independently of the time.[13] Thus, although the wave packet (4.6) changes in character rapidly with time, its mean position continues to be at $\boldsymbol{\rho}=\mathbf{b}$. This supports the assumption that the transient state $|\mathbf{b},\kappa\rangle_c$ can be interpreted as a state with a definite impact parameter $\mathbf{b}$.

In the familiar analysis of a scattering process in terms of stationary states, an incident wave train is fed in over a long interval in the past, and the transients are eliminated by a suitable limiting process[14] or by an adiabatic switching on or off of the interaction,[15] so that only the steady-state part of the asymptotic states are retained.

Here we have a complementary description of the scattering process, where a pulse is fed in over a short time interval in the remote past. Thus we may write (4.6) as

$$\Phi(\boldsymbol{\rho},z,t,T) = \lim_{\Delta\to 0}\frac{1}{\Delta}\int_{(T-\Delta)}^{T} d\tau \int dk_z d^2k_t dE\ \theta(E) \times e^{-iE(t-\tau)} e^{i\mathbf{k}_t\cdot\rho+ik_z z} a(\mathbf{k}_t,k_z,E). \qquad (4.9)$$

This expresses the asymptotic state (in the remote past) explicitly as a short pulse.

Whereas the description of a scattering process in terms of plane-wave states is a description in terms of stationary asymptotic states, the impact-parameter amplitudes describe it in terms of transient asymptotic states.[16] Each by itself can provide a complete description of the scattering process, as the states $|\mathbf{b},\kappa\rangle_c$ and $|\mathbf{k}_t,\kappa\rangle_c$ each form a complete set. The impact-parameter representation of the usual scattering amplitude (between plane-wave states) expresses the relation between these two descriptions of a scattering process. Thus, using the completeness of the impact-parameter states $|\mathbf{b},\kappa\rangle_c$, we obtain

$$\begin{aligned}{}_c\langle \mathbf{q}_f,\kappa_f|T|\mathbf{q}_i,\kappa_i\rangle_c &= \int \frac{d^2b_i d\kappa}{(2\pi)^3}\int \frac{d^2b_f d\kappa'}{(2\pi)^3}\, {}_c\langle \mathbf{q}_f,\kappa_f|\mathbf{b}_f,\kappa'\rangle_c \times {}_c\langle \mathbf{b}_f,\kappa'|T|\mathbf{b}_i,\kappa\rangle_c\, {}_c\langle \mathbf{b}_i,\kappa|\mathbf{q}_i,\kappa_i\rangle_c \\ &= \frac{1}{(2\pi)^2}\int d^2b_i d^2b_f e^{i\mathbf{q}_i\cdot\mathbf{b}_i - i\mathbf{q}_f\cdot\mathbf{b}_f}\, {}_c\langle \mathbf{b}_f,\kappa_f|T|\mathbf{b}_i,\kappa_i\rangle_c, \end{aligned} \qquad (4.10)$$

[11] Any two-particle state in the c.m. frame may be expressed in terms of the plane-wave states $|\mathbf{k}_t,\kappa\rangle_c$, and hence in terms of the impact-parameter states $|\mathbf{b},\kappa\rangle_c$, using the inverse relation to (4.1).

[12] Note that a state with a δ-function localization in the transverse plane at some instant of time cannot describe a physical two-particle state, any more than a plane wave can. A physical two-particle state would be expressed as a superposition of these transient states (or as a superposition of stationary plane-wave states).

[13] Care is needed in evaluating the expectation value, because of the δ-function normalization of the states. The simplest device is to work with wave functions in a box in momentum space. (Note that, in momentum space, the transverse position operator is $i\partial/\partial\mathbf{k}_t$, and one evaluates the expectation value of this operator in a state (4.6) whose momentum-space representative is $e^{-i\mathbf{k}_t\cdot\mathbf{b}} \times\delta(k_z-\kappa)$, obtaining just $\mathbf{b}$.) As remarked in Ref. 13, a physical state would be a superposition of states $|\mathbf{b},\kappa\rangle$ (with values of $\mathbf{b}$ symmetrically distributed around a central value $\mathbf{b}_0$, say). Then again the expectation value of $\boldsymbol{\rho}$ may be evaluated with no difficulty, and the same result is obtained.

[14] M. Gell-Mann and M. L. Goldberger, Phys. Rev. **91**, 398 (1953).

[15] B. Lippmann and J. Schwinger, Phys. Rev. **79**, 469 (1950).

where we have used the relation

$$_c\langle \mathbf{b},\kappa | \mathbf{q},q_z\rangle_c = (2\pi)^2\delta(\kappa-q_z)e^{i\mathbf{q}\cdot\mathbf{b}}. \quad (4.11)$$

For relativistic scattering, we shall write a covariant two-particle plane-wave state in momentum space as $|\mathbf{k}_t,\kappa;\mathbf{P}\rangle$, where $\mathbf{P}$ is the total 3-momentum of the two particles. These states will be normalized covariantly according to

$$\langle \mathbf{k}_t',\kappa';\mathbf{P}'|\mathbf{k}_t,\kappa;\mathbf{P}\rangle = (2\pi)^6(4p_{\mathrm{I}}{}^0p_{\mathrm{II}}{}^0) \times\delta(\mathbf{k}_t-\mathbf{k}_t')\delta(\kappa-\kappa')\delta(\mathbf{P}-\mathbf{P}'). \quad (4.12)$$

The normalization of the states in the c.m. frame will be defined by factoring out the δ function for the total momentum:

$$\langle \mathbf{k}_t',\kappa';\mathbf{P}'|\mathbf{k}_t,\kappa;\mathbf{P}\rangle = {}_c\langle \mathbf{k}_t',\kappa'|\mathbf{k}_t,\kappa\rangle_c\,(2\pi)^3\delta(\mathbf{P}-\mathbf{P}'), \quad (4.13)$$

so that

$$_c\langle \mathbf{k}_t',\kappa'|\mathbf{k}_t,\kappa\rangle_c = (2\pi)^3(4p_{\mathrm{I}}{}^0p_{\mathrm{II}}{}^0)\delta(\mathbf{k}_t-\mathbf{k}_t')\delta(\kappa-\kappa'). \quad (4.14)$$

The impact-parameter states in the c.m. frame are now defined by

$$|\mathbf{b},\kappa\rangle_c = \frac{1}{2\pi}\int \frac{d^2k_t}{(4p_{\mathrm{I}}{}^0p_{\mathrm{II}}{}^0)^{1/2}}e^{-i\mathbf{k}_t\cdot\mathbf{b}}|\mathbf{k}_t,\kappa\rangle_c. \quad (4.15)$$

They are again normalized according to (4.3).

The completeness relation for the plane-wave states $|\mathbf{k}_t,\kappa\rangle_c$ is now given by

$$\int \frac{d\kappa d^2k_t}{(2\pi)^3(4p_{\mathrm{I}}{}^0p_{\mathrm{II}}{}^0)}|\mathbf{k}_t,\kappa\rangle_c\,{}_c\langle\mathbf{k}_t,\kappa| = 1. \quad (4.16)$$

The completeness of the states $|\mathbf{b},\kappa\rangle_c$ is again expressed by (4.5).

The coordinate space representative of the state (4.15) is given by the wave packet (4.6), with (4.7) replaced by

$$a(\mathbf{k}_t,k_z,E) = 4(p_{\mathrm{I}}{}^0p_{\mathrm{II}}{}^0)^{-1/2}e^{-i\mathbf{k}_t\cdot\mathbf{b}}\delta(k_z-\kappa) \times\delta\Big(E-\sum_{i=\mathrm{I}}^{\mathrm{II}}(\mathbf{k}_t{}^2+k_z{}^2+M_i{}^2)^{1/2}\Big), \quad (4.17)$$

where M_{I} and M_{II} are the masses of the two particles. For $t=T\to-\infty$, this wave packet is of the form

$$\Phi(\boldsymbol{\varrho},z,t=T\to-\infty) = e^{i\kappa z}\int \frac{d^2k_t e^{i\mathbf{k}_t\cdot(\rho-\mathbf{b})}}{2(\mathbf{k}^2+M_{\mathrm{I}}{}^2)^{1/4}(\mathbf{k}^2+M_{\mathrm{II}}{}^2)^{1/4}}, \quad (4.18)$$

where $\mathbf{k}^2=\mathbf{k}_t{}^2+\kappa^2$.

For equal masses, $M_{\mathrm{I}}=M_{\mathrm{II}}=M$, this is given by the simple expression

$$\Phi(t=T\to-\infty) = \frac{\pi\exp[-(\kappa^2+M^2)^{1/2}|\boldsymbol{\varrho}-\mathbf{b}|]e^{i\kappa z}}{|\boldsymbol{\varrho}-\mathbf{b}|}. \quad (4.19)$$

The wave packets (4.18) and (4.19) describe states localized in the transverse plane to within a distance of the order of $(\kappa^2+M^2)^{1/2}$. This is characteristic of relativistic scattering, where states with positive-energy components alone cannot have a δ-function localization. These wave packets again change in character rapidly with time and describe transient states.

The asymptotic states (4.18) and (4.19) are the analogs of the Newton-Wigner[17] localized single-particle states for a two-particle relativistic state localized in the transverse plane.

Again, the mean position of the wave packet Φ in the transverse plane, as defined by the expectation value of the (relativistic) relative position operator

$$\delta_R = i\left[\frac{\partial}{\partial\mathbf{k}_t}-\tfrac{1}{2}\mathbf{k}_t\left(\frac{1}{2E_{\mathrm{I}}{}^2}+\frac{1}{2E_{\mathrm{II}}{}^2}\right)\right], \quad (4.20)$$

where $E_{\mathrm{I}}=p_{\mathrm{I}}{}^0$, $E_{\mathrm{II}}=p_{\mathrm{II}}{}^0$, is seen to be just $\mathbf{b}$, independently of the time. The transient asymptotic state described by the wave packet Φ is an eigenstate of the position operator $\boldsymbol{\varrho}_R$.

The representation of the scattering amplitude for the process $A+B\to C+D$, analogous to (4.10), now has an additional factor $(16p_A{}^0p_B{}^0p_C{}^0p_D{}^0)^{1/2}$ on the right-hand side.

It is also of intetest to consider states $|b,m;\kappa\rangle_c$ with a definite value m of the angular momentum component along the z axis; we define such states by

$$|\mathbf{b},\kappa\rangle_c = \sum_{m=-\infty}^{+\infty} e^{-im(\phi+\pi/2)}|b,m;\kappa\rangle_c, \quad (4.21)$$

$$|b,m;\kappa\rangle_c = \frac{1}{2\pi}\int_0^{2\pi} d\phi\, e^{im(\phi+\pi/2)}|\mathbf{b},\kappa\rangle_c. \quad (4.22)$$

The coordinate-space representative Φ_m of the state $|b,m;\kappa\rangle_c$ is related to Φ in (4.6) in a similar way, and is obtained by replacing $a(\mathbf{k}_t,k_z,E)$ in (4.6) by

$$a_m(k_t) = (4p_{\mathrm{I}}{}^0p_{\mathrm{II}}{}^0)^{-1/2}J_m(bk_t)\delta(k_z-\kappa) \times\delta\Big(E-\sum_{i=\mathrm{I}}^{\mathrm{II}}(\mathbf{k}_t{}^2+k_z{}^2+M_i{}^2)^{1/2}\Big). \quad (4.23)$$

[16] This is a particular instance of a more general observation. A linear system can be described by examining its response either to a steady-state input or to short-lived pulses. This is familiar in the analysis of electrical networks. The two kinds of descriptions stress complementary aspects of the system; each is of particular utility in examining a different kind of property of the system. For instance, the analysis in terms of pulses is of particular interest in studying properties such as the time delay of a pulse in passing through a system. The relation of the time delay in scattering to the impact-parameter description will be discussed elsewhere.

[17] T. D. Newton and E. P. Wigner, Rev. Mod. Phys. 21, 400 (1949).

The normalization and the completeness relation for the states $|b,m;\kappa\rangle_c$ are given by

$$_c\langle b,m;\kappa|b',m';\kappa'\rangle_c = (2\pi)^2[\delta(b-b')/b]\delta(\kappa-\kappa')\delta_{mm'}: \quad (4.24)$$

$$\sum_{m=-\infty}^{+\infty}\int\frac{d\kappa db b}{(2\pi)^2}|b,m;\kappa\rangle_c\,{}_c\langle b,m;\kappa|=1. \quad (4.25)$$

Using the completeness of the states $|b,m;\kappa\rangle_c$ and the relation

$$_c\langle b,m;\kappa|\mathbf{q},q_z\rangle_c=(2\pi)^2(4p_{\mathrm{I}}{}^0p_{\mathrm{II}}{}^0)^{1/2}\times\delta(\kappa-q_z)J_m(bq_t)e^{-im\phi}, \quad (4.26)$$

we proceed as in (4.10) to obtain, for the process $A+B\to C+D$,

$$_c\langle\mathbf{q}_f,\kappa_f|T|\mathbf{q}_i,\kappa_i\rangle_c = (16p_A{}^0p_B{}^0p_C{}^0p_D{}^0)^{1/2}\sum_{m=-\infty}^{+\infty}\int_0^\infty db_ib_i\int_0^\infty db_fb_f \times J_m(b_iq_t{}^i)J_m(b_fq_t{}^f)e^{-im(\phi_i-\phi_f)} \times\langle b_fm\kappa_f|T|b_im\kappa_i\rangle, \quad (4.27)$$

where $q_t{}^i=|\mathbf{q}_i|$, etc.

When the z direction is chosen along the direction of the initial momentum, then (4.10) (or its relativistic analog) and (4.27) give

$$_c\langle\mathbf{q}_f,\kappa_f|T|\mathbf{0},\kappa_i\rangle_c = (16p_A{}^0p_B{}^0p_C{}^0p_D{}^0)^{1/2}\int\frac{d^2b}{2\pi}e^{-i\mathbf{q}_f\cdot\mathbf{b}}A(b;\kappa_f\kappa_i) \quad (4.28)$$

$$= (16p_A{}^0p_B{}^0p_C{}^0p_D{}^0)^{1/2}\int_0^\infty db\, bJ_0(bq_t{}^f)\times A(b;\kappa_f\kappa_i), \quad (4.29)$$

where

$$A(b,\kappa_f\kappa_i)=\int\frac{d^2b_i}{2\pi}\,{}_c\langle b,\kappa_f|T|b_i,\kappa_i\rangle_c \quad (4.30)$$

$$=\int_0^\infty db_ib_i\,{}_c\langle b0\kappa_f|T|b_i0\kappa_i\rangle_c. \quad (4.31)$$

Finally, we note that the IPR (3.8), with fixed θ, may be obtained by considering states $|\mathbf{b},\theta\rangle$ with coordinate space representatives of the form

$$\Phi(\boldsymbol{\rho},z,t)=\int dk_zd^2k_tdE\theta(E)e^{i\mathbf{k}_t\cdot\boldsymbol{\rho}+ik_z(z-\zeta)}e^{-iE(t-T)}\times a(\mathbf{k}_t,k_z,E), \quad (4.32)$$

where

$$a(\mathbf{k}_t,k_z,E)=e^{-i\mathbf{k}_t\cdot\mathbf{b}}\delta(k_z-k_t\cot\theta)\times\delta(E-[(\mathbf{k}_t{}^2+k_z{}^2)]/2M) \quad (4.33)$$

for nonrelativistic scattering, and

$$a(\mathbf{k}_t,k_z,E)=(4p_{\mathrm{I}}{}^0p_{\mathrm{II}}{}^0)^{-1/2}e^{-i\mathbf{k}_t\cdot\mathbf{b}}\delta(k_z-k_t\cot\theta)\times\delta(E-\sum_{\mathrm{I}}^{\mathrm{II}}(k_t{}^2+k_z{}^2+M_i{}^2)^{1/2}) \quad (4.34)$$

for relativistic scattering.

For nonrelativistic scattering, we now have

$$\Phi(z=\zeta\to-\infty,\ t=T\to-\infty)=\delta(\boldsymbol{\rho}-\mathbf{b}). \quad (4.35)$$

For relativistic scattering, in the limit $z=\zeta\to-\infty$, $t=T\to-\infty$, we obtain a wave packet localized to a distance of order $M_i{}^{-1}$.

In general, the analysis given here should be made separately for forward and backward scattering, leading to representations for the amplitudes $T^{(\pm)}$ with definite signature defined as in (3.1).

V. A FIXED-ENERGY IPR

As noted in the previous sections, the impact-parameter amplitudes $A^{(\pm)}(b;k_z)$ or $\bar{A}^{(\pm)}(b;\theta)$ defined there are not characterized by definite values of the energy.

To obtain a similar representation of the scattering amplitude with the energy held fixed, we proceed as follows.

For the scattering process characterized by momenta $p_1+p_2\to p_1'+p_2'$, we factor out the c.m. motion and write the scattering amplitude in the c.m. frame as

$$_c\langle\mathbf{k}_t',k_z'|T(E)|\mathbf{k}_t,k_z\rangle_c, \quad (5.1)$$

where $k=p_1-p_2$, $k'=p_1'-p_2'$, and E is the total energy in the c.m. frame, where $\mathbf{K}\equiv\mathbf{p}_1+\mathbf{p}_2=0$.

Choosing the z direction along the direction of the initial momentum, we write the amplitude (5.1) as $T(E,\mathbf{k}_t')$. We now define the following transforms of the even- and odd-signature amplitudes $T^{(\pm)}$, for fixed E:

$$T^{(\pm)}(E,\mathbf{k}_t')=\frac{1}{2\pi}\int d^2b\, e^{i\mathbf{k}_t'\cdot\mathbf{b}}\mathcal{A}^{(\pm)}(\mathbf{b},E) \quad (5.2)$$

$$=\int_0^\infty db\, bJ_0(bk_t')\mathcal{A}^{(\pm)}(b,E). \quad (5.3)$$

For spinless particles, $T^{(\pm)}$ and $\mathcal{A}^{(\pm)}$ depend on $\mathbf{k}_t$ and $\mathbf{b}$ only through k_t, and b, respectively.

The inverse relations read

$$\mathcal{A}^{(\pm)}(b,E)=\frac{1}{2\pi}\int d^2k_t'e^{-i\mathbf{k}_t'\cdot\mathbf{b}}T^{(\pm)}(E,k_t') \quad (5.4)$$

$$=\int dk_t'k_t'J_0(bk_t')T^{(\pm)}(E,k_t'). \quad (5.5)$$

The integration over k_t' in these inverse relations is now an integration over $\sin\theta$ from 0 to ∞, for fixed E. Thus, in order to define the representation (5.2) for fixed E, we require an analytic continuation of the amplitudes $T^{(\pm)}$ as a function of $\sin\theta$, in the unphysical region in $\sin\theta$, from 1 to ∞. We shall define this analytic continuation by writing Mandelstam representations for $T^{(\pm)}$.

We shall refer to (5.3) as the fixed-energy IPR.[18] This was the representation considered in Ref. 1.

We note that now we no longer have an interpretation of the impact parameter $\mathbf{b}$ as the spatial coordinate canonically conjugate to $\mathbf{k}_t$. This is a necessary consequence of looking for an IPR with the energy held fixed. It is only in the limit of infinite energy that the fixed-energy IPR also corresponds to a coordinate-space description of the scattering amplitude.

The validity of the representations (5.2) and (5.3) requires the integrability condition

$$\int_0^\infty d(\sin\theta)\ (\sin\theta)^{1/2}|T^{(\pm)}(k,\sin\theta)|<\infty\,. \quad (5.6)$$

In an IPR with fixed k_z or fixed θ, the impact-parameter amplitude is not simply related to the partial-wave amplitude. With the fixed-energy IPR, however, we have a simple relation between the amplitudes $\mathcal{A}^{(\pm)}(b,E)$ and the partial-wave amplitudes $a_l(E)$.

[18] We note that a fixed-energy IPR with $J_0(kb\sin\theta)$ follows from the optical (diffraction) picture of high-energy scattering. Consider the optical-sphere model, in which the high-energy scattering of the two particles is replaced by the scattering of a plane wave by an absorptive sphere. Constructing the scattered wave from Huyghens's principle, the scattering amplitude is found to be

$$f(k,\theta)=ik\int_0^R db\ bJ_0(kb\sin\theta)(1-e^{-\chi(b,k)}),$$

where

$$\chi(b,k)=\int_{-(R^2-b^2)^{1/2}}^{(R^2-b^2)^{1/2}}dz\ \gamma(b,z,k).$$

[Cf. S. Fernbach, R. Serber, and T. B. Taylor, Phys. Rev. **75**, 1352 (1949) for the corresponding expression for scattering by a homogeneous sphere. See also J. Hamilton, *Theory of Elementary Particles* (Clarendon Press, Oxford, England, 1959), pp. 13, 19.] In the above expressions, R is the radius of the sphere and γ is the (complex) absorption coefficient. For a realistic model of particle scattering, γ must vanish rapidly outside the scattering region, which has a radius of the order of a fermi. Therefore the upper limit of the integral over b may be extended to infinity, giving

$$f(k,\theta)=ik\int_0^\infty db\ bJ_0(kb\sin\theta)\,\mathcal{A}(b,k),$$

with

$$\mathcal{A}(b,k)=1-\exp[-\chi(b,k)].$$

For a recent model of high-energy scattering, using an optical picture, refer to T. T. Chou and C. N. Yang, Phys. Rev. **170**, 1591 (1968). They introduce a two-dimensional impact parameter and obtain their expressions from a representation with $J_0(2kb\sin\frac{1}{2}\theta)$. Their representation and ours coincide in the infinite-energy limit.

From (5.2) or (5.3), we immediately obtain[19]

$$a_{2n}^{(+)}(E)=\tfrac{1}{2}\int_0^\infty db\ b\mathcal{A}^{(+)}(b,E)\times\int_{-1}^{+1}d(\cos\theta)P_{2n}(\cos\theta)J_0(kb\sin\theta)$$
$$=\left(\frac{1}{2k}\right)^{1/2}\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}\int_0^\infty db\ b^{1/2}J_{2n+1/2}(kb)\times\mathcal{A}^{(+)}(b,E)\,, \quad (5.7)$$

and similarly,

$$a_{2n+1}^{(-)}(E)=\frac{\sqrt{2}\Gamma(n+\frac{1}{2})}{k^{3/2}\Gamma(n+1)}\int_0^\infty db\ b^{-1/2}J_{2n+3/2}(kb)\times\mathcal{A}^{(-)}(b,E)\,. \quad (5.8)$$

VI. CONCLUDING REMARKS

In this paper, we have shown how impact-parameter representations involving $J_0(kb\sin\theta)$ follow naturally when we consider the description of the scattering amplitude in terms of states localized in the transverse plane in coordinate space. We have examined two classes of IPR. In one, the impact parameter $\mathbf{b}$ is the transverse coordinate conjugate to the transverse momentum $\mathbf{k}_t$. We have considered two such representations, one with fixed k_z and the other with a fixed scattering angle θ. In these, the impact-parameter amplitudes involve a knowledge of the physical amplitudes at all energies. We have shown how these representations are intimately related to the description of the scattering in terms of transient asymptotic states localized in the transverse plane.

The other kind of IPR is a fixed-energy IPR, obtained by reinterpreting the Fourier transform with respect to the transverse momentum. In this representation, the impact-parameter amplitude involves a knowledge of the scattering amplitude at one energy, but for unphysical values of the sine of the scattering angle.

The existence of these different IPR's imposes different kinds of restrictions on the scattering amplitude. Thus the integrability condition (3.6) for the IPR with fixed k_z requires that the scattering amplitude fall off sufficiently rapidly at large values of the transverse

[19] In obtaining these, we have used the relations

$$\int_{-1}^{+1}d(\cos\theta)\ P_{2n}(\cos\theta)J_0(kb\sin\theta)=\left(\frac{2}{kb}\right)^{1/2}\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}J_{2n+1/2}(kb),$$

and

$$\int_{-1}^{+1}d(\cos\theta)\ P_{2n+1}(\cos\theta)\cos\theta J_0(kb\sin\theta)=\left(\frac{2}{kb}\right)^{3/2}\frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+1)}J_{2n+3/2}(kb).$$

momentum. The precipitous drop of high-energy differential cross sections at 90° suggests that this condition is well satisfied experimentally. Thus, the IPR at fixed k_z should be a useful representation for high-energy scattering.

The analogous condition for the IPR with fixed θ is that the scattering amplitude fall off sufficiently rapidly with increasing energy at a fixed angle. This condition should be well satisfied not too close to the forward direction, outside the diffraction peak in elastic scattering.

Finally, the validity of the fixed-energy IPR requires that the amplitude decrease sufficiently rapidly for large values of $\sin\theta$ for fixed energy. This is a condition on the behavior of the scattering amplitude in the unphysical region; its interpretation depends on the analytic properties of the scattering amplitude.

The IPR's considered here provide us with different ways of expressing the scattering amplitude for energies and angles outside the domain of validity of the eikonal approximation.

An important question is the analyticity properties of each of the IPR's. This has been investigated for the fixed-energy IPR[1]; in subsequent work we shall examine this in detail for the other two representations.

The results obtained here may be directly extended to the scattering of particles with arbitrary spin; this will be discussed elsewhere.

PHYSICAL REVIEW VOLUME 181, NUMBER 5 25 MAY 1969

Extended Magnetic Monopoles in Field Theory*

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(Received 3 June 1968; revised manuscript received 25 February 1969)

A field theory of electric and magnetic monopoles which are either point or extended particles is constructed, using Mandelstam's path-dependent field quantities specialized to straight-line paths. Restrictions on the paths and form factors which are needed for self-consistency have been given. It is found that the Jacobi identity is satisfied. Schiff's selection principle for quarks, originally derived in ordinary quantum mechanics, follows easily and is thus generalized to field theory.

I. INTRODUCTION

A THEORY has been proposed by Schiff[1] showing that fractionally charged quarks should be unobservable as separate entities. It makes use of Dirac's idea that the quantization of electric charge derives from the existence of a magnetic pole of strength g.[2] Schiff treated a magnetic monopole of radius R classically, and used the nonrelativistic Schrödinger equation to show that fractionally charged particles can exist if they are confined to a roughly spherical volume of radius R which contains integer total charge. The purpose of the present paper is to extend these ideas to a spin-zero field theory by showing that a self-consistent field theory of extended magnetic monopoles is possible. Several other authors have previously considered the quantum effects of magnetic monopoles.[3] We shall use a modification of the formalism of Cabibbo and Ferrari,[3] who generalized Mandelstam's quantum electrodynamics without potentials.[4] This approach uses Mandelstam's paths which extend from the charged particle to infinity, and is to be distinguished from the Dirac-Schwinger approach which uses strings of singularities. We shall introduce straight-line paths in order to extend Schiff's ideas; however, the material that follows can be generalized to arbitrary spacelike paths.

II. NOTATION

The field quantities are denoted by

$$\Phi(x,\alpha_\mu)=\varphi(x)\exp\left[ie\int_0^\infty \alpha_\mu A_\mu(x+\alpha s)ds\right]$$

for electrically charged particles and by

$$\Psi(x,\alpha_\mu)=\psi(x)\exp\left[ig\int_0^\infty \alpha_\mu B_\mu(x+\alpha s)ds\right]$$

* Research sponsored by the Air Force Office of Scientific Research, Office of Aerospace Research, U. S. Air Force, under AFOSR Contract No. AF49(638)1389.

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