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Investigation of the Accuracy of the Tomonaga Intermediate-Coupling Approximation*

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The Tomonaga intermediate-coupling approximation and a two-wave-function generalization of it are applied to the system consisting of charged scalar mesons interacting with a static-source nucleon. The generalization allows mesons in the nucleon ground state to occupy either of two states in momentum space. Expressions are obtained for the ground-state wave functions and energy, and for the differential cross section for the elastic scattering of positive mesons by protons. The cross section associated with the one-wave-function approximation differs significantly from the two-wave-function cross section. There is good agreement between the one- and two-wave-function results for static properties of the nucleon.

INTRODUCTION

A MAJOR obstacle to the field-theoretic understanding of the interaction of mesons with a static nucleon is the difficulty of solving the equations of field theory when the meson-nucleon coupling constant is too large to allow perturbation-theoretic treatments of these equations but too small to allow the use of strong-coupling approximations. The Tomonaga intermediate-coupling approximation for the interaction of mesons with a static-source nucleon constitutes an attempt to surmount this obstacle.[1] This approximation effects a considerable simplification of the Hamiltonian of the meson-nucleon system by assuming that the mesons surrounding the physical nucleon can be described with a single-wave function chosen so as to minimize the energy of the nucleon state. It is possible to solve the field-theoretic equations for the Tomonaga approximate physical nucleon state and to compute the differential cross section for meson-nucleon scattering.[2] Indeed, Friedman, Lee, and Christian have found that the Tomonaga approximation for the interaction of pseudoscalar mesons with nucleons predicts the 3-3 phase shift for pion-nucleon scattering to within 10%, up to 200 MeV.[3]

A theoretical investigation of the accuracy with which the Tomonaga method approximates exact static-source theory has not previously been performed. In this paper we shall construct a two-wave-function generalization of the Tomonaga approximation. The accuracy of the Tomonaga intermediate-coupling approximation will then be tested by comparing the results obtained from the one- and two-wave-function forms of this approximation applied to the system of charged, scalar mesons interacting with a static-source nucleon. The charged scalar system is used since it is sufficiently simple to permit easy application of the Tomonaga approximation but retains many of the characteristics of more complicated and realistic systems. In particular, an exact treatment of the charged scalar system requires the use of an infinite number of meson wave functions.

After describing the Tomonaga approximation and its generalization, we shall obtain expressions for the ground-state wave functions for the one- and two-wave-function cases. We shall then use these wave functions to obtain certain characteristics of the physical nucleon state and the differential cross section for the scattering of positive mesons by protons. It will be seen that the cross sections resulting from different Tomonaga approximations disagree by orders of magnitude, indicating that the Tomonaga approximation is unreliable.

I. PHYSICAL PROTON

We consider a charged scalar field with the Hamiltonian

$$H=\int d^3k\ \omega_k(a_{\mathbf{k}}^\dagger a_{\mathbf{k}}+b_{\mathbf{k}}^\dagger b_{\mathbf{k}})+g\int d^3k\ \frac{\rho(\mathbf{k})}{(2\omega_k)^{1/2}}[\tau^+(a_{\mathbf{k}}+b_{\mathbf{k}}^\dagger)+\tau^-(a_{\mathbf{k}}^\dagger+b_{\mathbf{k}})],$$

where $a_{\mathbf{k}}^\dagger$ and $b_{\mathbf{k}}^\dagger$ are, respectively, creation operators for positive and negative mesons, the meson mass is unity, and

$$\omega_k=(k^2+1)^{1/2}.$$

The source function $\rho(\mathbf{k})$ is defined by

$$\rho(\mathbf{k})=(2\pi)^{-3/2}\int d^3x\ \rho(\mathbf{x})e^{i\mathbf{k}\cdot\mathbf{x}},$$

where $\rho(\mathbf{x})$ has been arbitrarily chosen to be the normalized function

$$\rho(\mathbf{x})=(M^2/4\pi x)e^{-Mx}.$$

The nucleon mass M is taken to be seven times the meson mass. The effect of varying the source function has not been investigated.

We now consider two arbitrary complete sets of orthonormal functions, $\{F_{s+}(k)\}$ and $\{F_{s-}(k)\}$, and express the meson destruction operators as series expansions

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[1] S. Tomonaga, Progr. Theoret. Phys. (Kyoto) 2, 6 (1947).

[2] T. D. Lee and R. Christian, Phys. Rev. 94, 1760 (1954).

[3] M. H. Friedman, T. D. Lee, and R. Christian, Phys. Rev. 100, 1494 (1955).

with these functions[4]:

$$a_{\mathbf{k}}=\sum_{slm} k^{-1}F_{s+}(k)Y_{lm}(\hat{k})a_{slm}\,,$$

$$b_{\mathbf{k}}=\sum_{slm} k^{-1}F_{s-}(k)Y_{lm}(\hat{k})b_{slm}\,.$$

Y_{lm} is the lm spherical harmonic. The operator expansion coefficients a_{slm} and b_{slm} satisfy

$$[a_{slm},a_{s'l'm'}{}^{\dagger}]=[b_{slm},b_{s'l'm'}{}^{\dagger}]=\delta_{s's}\delta_{l'l}\delta_{m'm}$$

and, respectively, destroy mesons with the wave functions $F_{s+}(k)Y_{lm}(\hat{k})/k$ and $F_{s-}(k)Y_{lm}(\hat{k})/k$. In terms of a_{slm} and b_{slm}, the Hamiltonian is

$$H=\sum_{s'slm}(W_{s's+}a_{s'lm}{}^{\dagger}a_{slm}+W_{s's-}b_{s'lm}{}^{\dagger}b_{slm})+\sum_{s}(f_{s+}\tau^{+}a_{slm}+f_{s-}\tau^{-}b_{slm}+\text{H.c.})\,,$$

where $l=m=0$ in the second summation,

$$W_{s's\pm}=\int_0^\infty dk\;\omega_k F_{s\pm}{}^*(k)F_{s\pm}(k)\,,$$

$$f_{s\pm}=(4\pi)^{1/2}g\int_0^\infty dk\;ku(k)F_{s\pm}(k)\,,$$

and

$$u(k)=\rho(k)/(2\omega_k)^{1/2}\,.$$

Since only s-wave mesons interact with the source, the ground state of the system involves only s-wave mesons, and only s-wave mesons are scattered by the physical proton. We shall drop terms from the Hamiltonian involving non-s-wave mesons and suppress the subscripts l and m. This gives the effective Hamiltonian

$$H=\sum_{s=1}^{\infty}\sum_{s'=1}^{\infty}(W_{s's+}a_{s'}{}^{\dagger}a_s+W_{s's-}b_{s'}{}^{\dagger}b_s)+\sum_{s=1}^{\infty}(f_{s+}\tau^{+}a_s+f_{s-}\tau^{-}b_s+\text{H.c.})\,.\quad(1)$$

When using the ordinary Tomonaga method, one assumes that all the mesons which significantly affect the ground state of the system have the same wave function $F/(4\pi)^{1/2}k$. The approximation is then made that only mesons with this wave function are present in the ground state, and the function F is chosen so as to minimize the ground-state energy. The sets $\{F_{s\pm}\}$ may be assumed to contain F. Then only those terms in the Hamiltonian (1) which are associated with F correspond to mesons present in the ground state and have nonvanishing ground-state expectation values. Thus, the ground-state energy and the approximate ground-state eigenvector can be obtained by solving the effective Hamiltonian

$$H_{1\text{wf}}=W(a^{\dagger}a+b^{\dagger}b)+f[(\tau^{+}a+\tau^{-}b)+\text{H.c.}]$$

where

$$W=\int_0^\infty dk\;\omega_k F^*(k)F(k)\,,$$

$$f=(4\pi)^{1/2}g\int_0^\infty dk\;kF(k)u(k)\,,$$

and

$$\int_0^\infty dk\;F^*(k)F(k)=1\,.$$

The operators $a^{\dagger}$ and $b^{\dagger}$, respectively, create positive and negative mesons with the wave function $F/(4\pi)^{1/2}k$. The ground state of $H_{1\text{wf}}$ may be computed numerically with arbitrary accuracy to obtain the results ordinarily associated with the Tomonaga approximation. In the following discussion we shall refer to this as the 1wf approximation.

The Tomonaga approximation can now be generalized to permit the use of approximate ground states involving any finite number of wave functions. An n-wave-function approximation is formed by assuming that only mesons associated with n members of the sets $\{F_{s\pm}\}$ are present in the ground state. The effective Hamiltonian is obtained by dropping all terms in (1) not involving mesons with the given n wave functions. The n wave functions are then chosen to minimize the ground-state energy of the approximate Hamiltonian subject to the constraint that the wave functions for positive and negative mesons form separately orthogonal sets.

We shall consider two examples of the generalized Tomonaga approximation:

(a) the two-charge wave-function approximation ($2q$) in which there are two nonvanishing wave functions, one for each charge of meson. The effective Hamiltonian is

$$H_{2q}=W_{+}a^{\dagger}a+W_{-}b^{\dagger}b+(f_{+}\tau^{+}a+f_{-}\tau^{-}b+\text{H.c.})\,,$$

where

$$\int_0^\infty dk\;F_{\pm}{}^*(k)F_{\pm}(k)=1\,,$$

$$W_{\pm}=\int_0^\infty dk\;\omega_k F_{\pm}{}^*(k)F_{\pm}(k)\,,$$

and

$$f_{\pm}=(4\pi)^{1/2}g\int_0^\infty dk\;ku(k)F_{\pm}(k)\,.$$

The operator $a^{\dagger}$ creates positive mesons with the wave function $F_{+}(k)/(4\pi)^{1/2}k$, while $b^{\dagger}$ creates negative mesons with the wave function $F_{-}(k)/(4\pi)^{1/2}k$.

(b) the two-momentum-space wave-function approximation ($2p$) in which there are two orthogonal wave functions independent of meson charge. The effective

[4] E. M. Henley and W. Thirring, in *Elementary Quantum Field Theory* (McGraw-Hill Book Co., New York, 1962), Chap. 17, p. 186.

TABLE I. Static characteristics of the proton.[a]

	1wf	2q	2p	
		$g^2=0.01$		
N_1^+			0	$x_1=$ 0.0007973
N_2^+	0.0039	0.0039	0.0039	$x_2=$300.632
N_{12}^+			0	$\lambda_0=$ 0.0067
M_1^+			0	$\lambda_+=$ 0.0030
M_2^+	0.0624	0.0624	0.0624	$\lambda_-=$ 3.4004
$-E$	0.01337	0.01337	0.01337	
		$g^2=1.581$		
N_1^+			0.00269	$x_1=$ 0.108
N_2^+	0.3039	0.3166	0.3211	$x_2=$ 2.372
N_{12}^+			0.02165	$\lambda_0=$ 0.6404
M_1^+			0.04045	$\lambda_+=$ 0.3031
M_2^+	0.4450	0.4507	0.4493	$\lambda_-=$ 3.4502
$-E$	1.753	1.766	1.787	
		$g^2=2.8$		
N_1^+			0.0243	$x_1=$ 2.06
N_2^+	0.4280	0.4355	0.4331	$x_2=$ 11.0
N_{12}^+			0.0952	$\lambda_0=$ 0.8644
M_1^+			0.1229	$\lambda_+=$ 0.4185
M_2^+	0.4808	0.4840	0.4689	$\lambda_-=$ 3.279
$-E$	2.893	2.893	2.938	
		$g^2=4.0$		
N_1^+			0.00507	$x_1=$ 0.0775
N_2^+	0.5130	0.5274	0.5359	$x_2=$ 1.740
N_{12}^+			0.0337	$\lambda_0=$ 0.9852
M_1^+			0.0506	$\lambda_+=$ 0.4848
M_2^+	0.4999	0.5007	0.4963	$\lambda_-=$ 3.060
$-E$	3.875	3.905	3.962	

[a] N and M parameters for the 1wf and 2q approximations are tabulated as N_2^+ and M_2^+. Column headings indicate the approximation to which the data apply. Natural units are used with the meson mass equal to unity.

Hamiltonian is

$$H_{2p}=\sum_{s=1}^{2}\sum_{s'=1}^{2} W_{s's}(a_{s'}^\dagger a_s+b_{s'}^\dagger b_s)+\sum_{s=1}^{2}[f_s(\tau^+a_s+\tau^-b_s)+\text{H.c.}],$$

where

$$\int_0^\infty dk\, F_{s'}^*(k)F_s(k)=\delta_{s's},$$

$$W_{s's}=\int_0^\infty dk\, \omega_k F_{s'}^*(k)F_s(k),$$

and

$$f_s=(4\pi)^{1/2}g\int_0^\infty dk\, ku(k)F_s(k).$$

The operators $a_s^\dagger$ and $b_s^\dagger$, respectively, create positive and negative mesons with the wave function $F_s(k)/(4\pi)^{1/2}k$. The ground states of both of the effective Hamiltonians may be computed numerically with arbitrary accuracy.

The wave functions for the three approximations are chosen to satisfy

$$\frac{\delta}{\delta F}\langle p|H_{1\text{wf}}|p\rangle=0,\quad \frac{\delta}{\delta F_\pm}\langle p|H_{2q}|p\rangle=0, \tag{2}$$

and

$$\frac{\delta}{\delta F_s}\langle p|H_{2p}|p\rangle=0,\quad s=1\text{ or }2$$

for the 1wf, 2q, and 2p approximations, respectively. $|p\rangle$ represents the physical proton state. Analysis of Eq. (2) leads to an expression for F of the form

$$F(k)=-(4\pi)^{1/2}gAku(k)/(\omega_k+\lambda_0)$$

for the 1wf approximation, where A is a normalization constant and λ_0 is a parameter chosen to minimize the energy of the ground state. For the 2q approximation we find that

$$F_\pm(k)=-(4\pi)^{1/2}gA^\pm ku(k)/(\omega_k+\lambda_\pm),$$

where $A^\pm$ are normalization constants, and the parameters $\lambda_\pm$ are chosen to minimize the ground-state energy. For the 2p approximation

$$F_1(k)=(4\pi)^{1/2}gku(k)\frac{\alpha(\omega_k+\beta)}{x_1\omega_k^2+x_2\omega_k+1},$$
$$F_2(k)=(4\pi)^{1/2}gku(k)\frac{\delta\omega_k+\gamma}{x_1\omega_k^2+x_2\omega_k+1}, \tag{3}$$

where α, β, and γ are selected such that F_1 and F_2 are orthonormal. The constant δ may be equated to zero by choosing an appropriate basis in the space defined by F_1 and F_2, while x_1 and x_2 are chosen to minimize the ground-state energy.

The functions F, $F_\pm$, F_1, and F_2 all have the same functional form, that of Eq. (3). The only independent parameters appearing in these functions are λ_0, $\lambda_\pm$, x_1, and x_2. These parameters are analogous to undetermined Lagrange multipliers and will be referred to as such in the following discussion.

The energy-minimizing values of the Lagrange multipliers and ground-state energy have been computed numerically using a CDC-1604 computer. This was done by choosing the Lagrange multipliers for a given approximation so as to minimize E in

$$H_{\text{eff}}|p\rangle=E|p\rangle$$

for states $|p\rangle$ of unit positive charge. H_{eff} is the effective Hamiltonian. When the Lagrange multipliers have their energy-minimizing values, E is the approximate ground-state energy, and $|p\rangle$ is the approximate physical proton state. The following constants, which are required in the computation of meson-proton scattering cross sections, were also computed:

$$N^+=\langle p|a^\dagger a|p\rangle,\quad M^+=\langle p|a\tau^+|p\rangle,$$

for the 1wf and 2q approximations, and

$$N_s^+=\langle p|a_s^\dagger a_s|p\rangle,\quad N_{12}^+=\langle p|a_1^\dagger a_2|p\rangle,$$
$$M_s^+=\langle p|a_s\tau^+|p\rangle,$$

for the $2p$ approximation. The Lagrange multipliers, N and M parameters, and ground-state energy for the three approximations are displayed as functions of g^2 in Table I.

II. SCATTERING OF MESONS BY NUCLEONS

We now consider the scattering of positive mesons by protons in the various Tomonaga approximations. The scattering state $|\psi\rangle$, representing a proton and an incoming positive meson of energy ω_0 and momentum k_0, is calculated using the variational principle

$$\delta\langle\psi|H-E-\omega_0|\psi\rangle=0, \quad (4)$$

where H is the exact charged scalar Hamiltonian, and E is the energy of the proton state.[5] The trial state is taken to be

$$|\psi\rangle=\int d^3k\ \chi(\mathbf{k})a_{\mathbf{k}}^{\dagger}|p\rangle, \quad (5)$$

with χ a function to be determined.[6]

Substitution of (5) into (4) yields the following integral equation for χ:

$$(\omega_k-\omega_0)\chi(\mathbf{k})=\int d^3k'\ K(k,k')\chi(\mathbf{k}'). \quad (6)$$

The kernel is

$$K(k,k')=-\tfrac{1}{2}\{g\langle p|[a_{\mathbf{k}'}u(k)+a_{\mathbf{k}}u(k')]\tau^+|p\rangle - \langle p|a_{\mathbf{k}}^{\dagger}a_{\mathbf{k}'}(\omega_{k'}-\omega_0)|p\rangle-\langle p|a_{\mathbf{k}'}^{\dagger}a_{\mathbf{k}}(\omega_k-\omega_0)|p\rangle\}.$$

For the 1wf case, this is

$$K(k,k')=-[(\omega_{k'}-\omega_0)f(k)f(k')N^+ + gu(k')f(k)M^+],$$

where

$$f(k)=F(k)/(4\pi)^{1/2}k.$$

TABLE II. Meson-nucleon elastic scattering cross sections[a].

	ω	1wf	$2q$	$2p$	Pert. theor.
Columns below approximation names show $k^2(d\sigma/d\Omega)\times10^5$					
$g^2=0.01$	1.2	2.807	2.839	2.914	2.94
	2.0	5.506	5.670	5.690	5.91
	2.6	4.940	5.181	5.145	5.45
	3.0	4.279	4.557	4.500	4.85
Columns below approximation names show $k^2(d\sigma/d\Omega)\times10^2$					
$g^2=1.581$	1.2	3.021	4.594	9.065	
	2.0	2.992	7.025	4.886	
	2.6	1.359	4.060	2.237	
	3.0	7.553	2.527	1.331	
$g^2=2.8$	1.2	3.937	6.199	72.35	
	2.0	3.664	9.009	41.28	
	2.6	1.588	4.924	26.86	
	3.0	0.8621	2.939	19.75	
$g^2=4.0$	1.2	4.629	7.353	20.86	
	2.0	4.280	10.63	7.968	
	2.6	1.832	5.720	3.382	
	3.0	0.9868	3.371	1.948	

[a] Column headings indicate the energy and approximation to which the data apply. Natural units are used with the meson mass equal to unity.

The 1wf kernel is the sum of separable terms. The solution to (6) for this case is

$$\chi_{1\mathrm{wf}}(\mathbf{k})=\delta(\mathbf{k}-\mathbf{k}_0)-\frac{1}{D_{1\mathrm{wf}}}\frac{gu(k_0)}{\omega_k-\omega_0-i\epsilon}M^+f(k),$$

where

$$D_{1\mathrm{wf}}=1+N^+ + \int d^3k\ g\frac{u(k)}{\omega_k-\omega_0-i\epsilon}f(k)M^+.$$

It follows that the differential cross section for the 1wf case is

$$\left.\frac{d\sigma}{d\Omega}\right|_{1\mathrm{wf}}=(4\pi^2\omega_0)^2[gu(k_0)f(k_0)M^+]^2\bigg/\left\{[4\pi^2g\omega_0k_0u(k_0)f(k_0)M^+]^2+\left[(1+N^+)+P\int d^3k\ g\frac{u(k)}{\omega_k-\omega_0}M^+f(k)\right]^2\right\}.$$

The kernels for the $2q$ and $2p$ cases are also the sum of separable terms and the solution of (6) for these cases leads to the following expressions for the differential cross section:

$$\left.\frac{d\sigma}{d\Omega}\right|_{2q}=[4\pi^2\omega_0u^2(k_0)Rg^2]^2\bigg/\left\{[4\pi^2\omega_0k_0u^2(k_0)Rg^2]^2+(\omega_0+\lambda_+)^2\left(1-Rg^2(\omega_0+\lambda_+)P\int d^3k\ \frac{u^2(k)}{(\omega_k+\lambda_+)^2(\omega_k-\omega_0)}\right)^2\right\}$$

with $R=(M^+)^2/N^+$ for the $2q$ case, and

$$\left.\frac{d\sigma}{d\Omega}\right|_{2p}=\frac{(4\pi^2\omega_0)^2\{gu(k_0)[(M_1^+ + M_1^+N_2^+ - N_{12}^+M_2^+)f_1(k_0)+(M_2^+ + M_2^+N_1^+ - N_{12}^+M_1^+)f_2(k_0)]\}^2}{(\mathrm{Im}D_{2p})^2+(\mathrm{Re}D_{2p})^2},$$

where

$$D_{2p}=(1+N_1^+)(1+N_2^+)-(N_{12}^+)^2+\int d^3k\ g\frac{u(k)}{\omega_k-\omega_0-i\epsilon}\times[f_1(k)(M_1^+ + M_1^+N_2^+ - N_{12}^+M_2^+)+f_2(k)(M_2^+ + M_2^+N_1^+ - N_{12}^+M_1^+)]$$

for the $2p$ case.

[5] W. Kohn, Phys. Rev. 84, 495 (1951).

[6] We do not know what error is involved in the one-meson approximation used in the state (4).

The cross sections for the three approximations have been evaluated numerically using the CDC-1604. The results are shown in Table II along with the differential cross sections obtained from first-order perturbation theory. As can be seen from the table, for g^2 values equal to or greater than 1.581 there is sharp disagreement among the cross sections obtained using the three types of Tomonaga approximation. The 1wf and $2q$ cross sections differ by as much as a factor of 2.7, while the 1wf and $2p$ cross sections disagree by more than a factor of 20 for certain values of g^2 and ω_0. On the other hand, at $g^2=0.01$ there is good agreement among the cross sections obtained from the three types of Tomonaga approximation and perturbation theory. This agreement is to be expected inasmuch as the perturbation-theoretic proton state approaches the 1wf proton state as g^2 approaches zero.

The cross sections obtained from the 1wf and $2q$ approximations do not agree with the cross section obtained from strong-coupling theory at large values of g^2.[7] The behavior of the cross sections therefore differs from that of the ground-state energy and N and M parameters obtained from the 1wf and $2q$ approximations. The latter quantities approach agreement with strong-coupling theory as g^2 becomes large.[8] Computation of the strong-coupling limits of the $2p$ approximation was not feasible with the computer facilities available.

III. INTERPRETATION OF RESULTS

The results displayed in Tables I and II show that the values of the ground-state energy and N and M parameters obtained from the various Tomonaga approximations are in close agreement but that there is considerable disagreement among the cross sections associated with different approximations. Since the wave functions associated with all three approximations have the same functional form, that of Eq. (3), it follows that the similarities and differences in the results obtained from the approximations considered here are attributable to the values of the Lagrange multipliers associated with each approximation. Perfect agreement among the cross sections for the scattering of positive mesons by protons requires

$$x_1=0 \quad \text{and} \quad \lambda_0=\lambda_+=x_2^{-1}. \tag{7}$$

However, as can be seen from Table I, the computed values of the Lagrange multipliers depart from this condition by orders of magnitude. The cross sections, which depend explicitly on the distribution in momentum space of the mesons surrounding the physical proton, reflect the departure of the Lagrange multipliers from conditions (7). There is no way of knowing which, if any, of the three approximations considered here has yielded the correct Lagrange multipliers. We conclude that Tomonaga-type approximations using few wave functions do not yield reliable scattering cross sections outside the region of validity of first-order perturbation theory.

The ground-state energy and N and M parameters depend only on the integrated distribution of bound mesons in momentum space. As can be seen from Table I, the values of these quantities obtained from the various approximations agree to within 3%. This indicates that the ground-state energy and N and M parameters are probably accurately computed using any of the approximations we have considered. However, these quantities are not observable in the laboratory, so the utility of the approximations is severely restricted.

It is, of course, conceivable that accurate calculations of dynamical quantities could be made with Tomonaga-type approximations if sufficiently many wave functions were included in the Hamiltonian. However, the complexity of the numerical computations involved in determining the N and M parameters, ground-state energy, and Lagrange multipliers increases so rapidly with the number of wave functions used that it is scarcely feasible to attempt to solve higher-order Tomonaga approximations with computing equipment presently available.

Note added in proof. The following papers also deal with the accuracy of the intermediate coupling approximation: R. Stroffolini, Phys. Rev. **104**, 1146 (1956); R. J. Drachman, *ibid.* **109**, 996 (1958); **125**, 1758 (1962).

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[7] See Ref. 2, Eqs. (23) and (24), and footnote 9.
[8] See Ref. 2, Figs. 1 and 2, and Table I.