and

$$\int \frac{d^4q}{q^2-M_W{}^2} d^4x\, e^{-iq\cdot x}(q^2q_\mu\delta_{\lambda\tau}-q_\lambda q_\mu q_\tau) \times\langle n|T\{J_\mu{}^{K^+}(x)J_\tau{}^{\bar{K}^0}(0)\}|p\rangle = \int \frac{d^4q}{(q^2-M_W{}^2)^2}(q^2\delta_{\lambda\tau}-q_\lambda q_\tau) \times\langle n|\sum_i f^{iK^+\bar{K}^0}J_\tau{}^i(0)|p\rangle\neq 0. \quad \text{(B5)}$$

The same reasoning may be applied to the term proportional to $q_\sigma q_\tau \delta_{\mu\nu} a_{\lambda\sigma\nu}$, so we have a quadratic divergence. There is also the graph of Fig. 7(b) which we must assume is cancelled by a mass-renormalization term. The diagrams of Fig. 7(a) could lead to a violation of universality. Of course, if we imagine W_a to have strangeness zero, W_b and W_c to have strangeness one, and WWW couplings to preserve strangeness, the diagrams of Fig. 7 are forbidden.

PHYSICAL REVIEW VOLUME 181, NUMBER 5 25 MAY 1969

Solution of Nonrelativistic Partial-Wave Dispersion Relations*

PORTER W. JOHNSON
Department of Physics, Case Western Reserve University, Cleveland, Ohio 44106
(Received 23 December 1968)

The partial-wave dispersion-relation (PWDR) problem is studied in the nonrelativistic elastic case. In particular, a set of conditions on the left-hand-cut discontinuity is obtained which is sufficient to guarantee that the PWDR problem has a solution. The nature of the solutions so obtained and the possible extensions are discussed.

I. INTRODUCTION

ONE can introduce dynamical information about partial-wave amplitudes in a relatively consistent manner by requiring that the amplitude $f(k^2)$ be unitary, be analytic in the variable k^2 in the usual twice-cut plane, and have a prescribed left-hand-cut discontinuity which is regarded as input.

The problem consists of finding solutions to the singular, nonlinear integral equation (3.1) which are analytic in the desired region. The problem as stated is known not to lead to unique solutions in many cases as a result of the Castillejo-Dalitz-Dyson (CDD) ambiguity. The ancient and mysterious N/D algorithm reduces the problem to one which is less formidable in appearance. In so doing, one considerably limits the types of solutions one can obtain as a result. However, the dynamical content (e.g., the nonlinearity) of the partial-wave dispersion relation (PWDR) is carefully disguised in the N/D approach.

The rich dynamical content of the PWDR problem is indicated by the fact that although it in general has more than one solution, certain conditions must necessarily be fulfilled if it is to have any solution at all.[1] We will illustrate this here in a simple way.

In Secs. II and III we obtain and discuss conditions on ΔT which are sufficient to guarantee the existence of a certain type of solution to the PWDR problem. In Sec. IV we study the relative efficacy of these conditions by examining known solutions in simple cases. The n-pole case is discussed in Sec. V and distinct conditions are obtained there. Further discussion of the general problem is presented in Sec. VI.

II. PARTIAL-WAVE DISPERSION RELATIONS

Here we will study the construction of nonrelativistic, completely elastic partial-wave amplitudes from unitarity, analyticity in energy, and knowledge of the left-hand-cut discontinuity. That is, given the function $\Delta T(k^2)$ defined in $-\infty<k^2<-\mu^2$, we wish to learn under what conditions there exists a function $f(k^2)$ with the following properties.

(i) $f(k^2)$ is a real analytic function of k^2 in the twice-cut k^2 plane, where the cuts lie along the real axis and extend over the domains $-\infty<k^2<-\mu^2$ and $0<k^2<\infty$.

(ii) The function $f(k^2)\to 0$ as $k^2\to\infty$ within the cut plane.

(iii) The discontinuity of $f(k^2)$ across the left-hand cut $-\infty<k^2<-\mu^2$ is given by $\Delta T(k^2)$.

(iv) As one approaches the right-hand cut from above, $f(k^2+i\epsilon)$ satisfies the unitarity condition

$$\mathrm{Im} f(k^2+i\epsilon)=(k^2+i\epsilon)^{1/2}|f(k^2+i\epsilon)|^2 \qquad (2.1)$$

or

$$f(k^2+i\epsilon)=(k^2+i\epsilon)^{-1/2}e^{i\delta(k^2)}\sin\delta(k^2)\,,\quad \delta(k^2)\ \text{real}. \qquad (2.1')$$

A very simple type of necessary condition can be obtained as follows[2]:

Let $\phi(k^2)$ be any function analytic in the k^2 plane with only a right-hand cut. If ϕ is such that $k^{2+\epsilon}f\phi(k^2)\to 0$

* Supported in part by the U. S. Atomic Energy Commission under Contract No. 342-2865.

[1] A. Martin, Nuovo Cimento 38, 1326 (1965).

[2] G. Tiktopoulos (unpublished).

as $k^2 \to \infty$, then

$$\int_{-\infty}^{-\mu^2} dk^2 \phi(k^2) \Delta T(k^2) = \int_0^\infty dk^2 \, \mathrm{Im}[f\phi(k^2)],$$

where unitarity requires that the right-hand integral be finite, so that the left integral must also be finite. The interesting aspect of these conditions is that the functions ϕ are smooth (analytic) over the region of integration so that these conditions are not very dependent upon the structure of the discontinuity ΔT. We show here that a function satisfying the above properties exists if the left-hand-cut discontinuity ΔT satisfies these conditions:

(a) $\Delta T(k^2)$ is such that the function

$$T_0(k^2) = \frac{1}{\pi} \int_{-\infty}^{-\mu^2} \frac{dq^2}{q^2 - k^2} \Delta T(q^2) \tag{2.2}$$

is analytic in the k^2 plane with only a left-hand cut and approaches zero as $k^2 \to \infty$ in the cut plane.

(b) In addition, the integral

$$M = \frac{2}{\pi} \int_{-\infty}^{-\mu^2} \frac{dk^2}{\sqrt{(-k^2)}} |\Delta T(k^2)| \tag{2.3}$$

exists and is less than 1. The condition (2.3) can be viewed as a constraint of the total variation of the integral of the function ΔT over the left-hand cut. It does not place any further restrictions on the structure of ΔT.

One can examine the case in which T_0 contains pole terms by including δ-function singularities to write

$$\Delta T(k^2) = \sum_i \lambda_i \delta^{(n_i)}(k^2 + a_i^2) + \Delta T_0(k^2),$$

where $\Delta T_0(k^2)$ is the derivative of a continuous function of k^2. In this case we define

$$M = \frac{2}{\pi} \left[\sum_i \frac{\pi |\lambda_i|}{(a_i)^{2n_i+1}} \frac{\Gamma(n_i + \frac{1}{2})}{\Gamma(\frac{1}{2})} + \int_{-\infty}^{-\mu^2} \frac{dk^2}{\sqrt{(-k^2)}} |\Delta T_0(k^2)| \right] \tag{2.3'}$$

and a sufficient condition for a solution is $M < 1$.

We will show that under conditions (a) and (b) there exists a solution f such that the phase shifts defined in (2.1′) obey the constraint

$$-\tfrac{1}{2}\pi < \delta(k^2) < \tfrac{1}{2}\pi \tag{2.4}$$

for $0 \leqslant k^2 \leqslant \infty$.

It is well known that the solution to boundary-value problems of this type are plagued with the CDD ambiguity and thus one does not get unique solutions. The above restriction reduces this ambiguity, although even under such a restriction the solution would not be unique because of the latitude of choice of behavior on the unphysical sheet of the k^2 plane.

We have proved the existence of solutions under conditions (a) and (b) which satisfy the threshold condition at $l=0$ The more general threshold condition that

$$\lim_{k^2 \to 0^+} \frac{\delta(k^2)}{(k^2)^l}$$

exists will not be generally valid for our solutions. This question is discussed further at the end of Sec. V.

III. PROOF OF SUFFICIENCY OF THE CONDITION

We want to establish that $f(k^2)$ can be written in the form

$$f(k^2) = \frac{1}{\pi} \int_{-\infty}^{-\mu^2} dk'^2 \frac{1}{k'^2 - k^2} \Delta T(k'^2) + \frac{1}{\pi} \int_0^\infty \frac{dk'^2}{k'^2 - k^2} k' |f(k'^2)|^2, \tag{3.1}$$

where $\Delta T(k^2)$ is specified. We will use the N/D constructive procedure to show that such a solution exists; that is, we make the ansatz

$$f(k^2) = N(k^2)/D(k^2). \tag{3.2}$$

One can show that relation (3.1) is satisfied if N and D have the following properties:

(1) $$N(k^2) = \frac{1}{\pi} \int_{-\infty}^{-\mu^2} \frac{dq^2}{q^2 - k^2} D(q^2) \Delta T(q^2), \tag{3.3}$$

(2) $$D(k^2) = 1 - \frac{1}{\pi} \int_0^\infty \frac{dq^2}{q^2 - k^2} q N(q^2), \tag{3.4}$$

(3) N must be analytic in the left-hand-cut k^2 plane, whereas D must be analytic in the right-hand-cut k^2 plane.

(4) The function N/D must be analytic in the twice-cut plane and must vanish at ∞ like some inverse power of k^2. In particular, therefore, D cannot vanish in the cut plane.

The N/D approach should be regarded neither as panacea or anathema, but merely as a trick to construct the scattering amplitude. One can hardly expect the same trick to work in generation of all possible solutions of the problem. Conversely, one must always check to see that the amplitude so constructed does satisfy the partial-wave dispersion relation (3.1). In any case, with the restriction (2.3) we will show that one can construct an f satisfying (3.1).

We choose to uncouple Eqs. (3.3) and (3.4) by inserting (3.3) into (3.4) and inverting the order of integration[3] to obtain

$$D(k)=1+\frac{1}{\pi}\int_{\mu^2}^{\infty}dq^2D(-q^2)\Delta T(-q^2)\frac{1}{q-i\sqrt{(k^2)}}, \quad (3.5)$$

where $\text{Im}(\sqrt{k^2})\geqslant 0$ on the physical sheet.

We can write Eq. (3.5) formally as

$$D=1+KD, \quad (3.5')$$

where the kernel K operating on a function $h(k^2)$ is written

$$Kh(k^2)=\frac{2}{\pi}\int_{\mu}^{\infty}dq\frac{q}{q-i\sqrt{(k^2)}}\Delta T(-q^2)h(-q^2). \quad (3.6)$$

Suppose $|h(-q^2)|<C$ for $\mu^2<q^2<\infty$. Then for all k^2 such that $\text{Im}(\sqrt{k^2})\geqslant 0$

$$|Kh(k^2)|\leqslant\frac{2C}{\pi}\int_{\mu}^{\infty}dq\frac{q}{|q-i\sqrt{(k^2)}|}|\Delta T(-q^2)|$$

$$\leqslant\frac{2C}{\pi}\int_{\mu}^{\infty}dq|\Delta T(-q^2)|=\tfrac{1}{2}CM.$$

Under the assumption of (2.7) we have thus shown that

$$|Kh(k^2)|\leqslant\tfrac{1}{2}CM \quad (3.7)$$

for $\text{Im}(k^2)\geqslant 0$. One may thus write the Neumann series for the integral equation (3.5) and show that this series converges to a solution for $\text{Im}(k^2)\geqslant 0$. In fact, one can easily obtain the following inequality for D with $\text{Im}(k^2)\geqslant 0$:

$$|D(k)-1|\leqslant\sum_{n=1}^{\infty}(\tfrac{1}{2}M)^n=\frac{M}{2-M}. \quad (3.8)$$

Since M is strictly less than 1, then

$$0<|D(k^2)|<2 \quad (3.9)$$

for $\text{Im}(\sqrt{k^2})\geqslant 0$.

One determines $N(k^2)$ by solving (3.5) for $D(k^2)$ and then substituting the result into Eq. (3.3) for N.[4] The condition (3.8) first obtained upon D along with (2.7) on the left-hand-cut discontinuity is sufficient to establish that N is analytic in the k^2 plane with a left-hand cut. Further, for any point in the cut plane,

$$|N(k^2)|\leqslant 4[\sqrt{d(k^2)}]^{-1}M,$$

where $d(k^2)$ is the shortest distance between the point k^2 and the left-hand cut.

These results imply that the function $f(k^2)=N(k^2)/D(k^2)$ constructed here must satisfy the partial-wave dispersion relation (3.1). Note that $N(k^2)$ is purely real for $k^2\geqslant 0$. Since D never vanishes on the left-hand cut, the phase shift $\delta(k^2)$ satisfies the inequality

$$|\tan\delta(k^2)|<\infty,$$

so that the condition (2.8) is satisfied for our phase shift.

IV. EXAMPLES

We have shown that certain conditions upon the left-hand-cut discontinuity are sufficient to guarantee the existence of a solution to the PWDR problem. Here we will apply these conditions to simple cases.

(a) $T_0(k^2)=(\lambda M/4k^2)\ln(1+4k^2/\mu^2)$.

Martin[1] has shown that $\lambda M/\mu<2.5$ is necessary for any solution to this problem to exist. Our condition implies that $\lambda M/\mu<0.5$ is sufficient to guarantee a solution without CDD poles.

(b) Consider the following amplitude:

$$f(k^2)=T_0(k^2)\Big/\left[1-\frac{1}{\pi}\int_0^{\infty}\frac{k'dk'^2}{k'^2-k^2}T_0(k'^2)\right]$$

$$=\frac{\lambda M}{4k^2}\ln\left(1+\frac{4k^2}{\mu^2}\right)\Big/$$

$$\left[1-\frac{\lambda M}{2\sqrt{(-k^2)}}\ln\left(1+\frac{2\sqrt{(-k^2)}}{\mu}\right)\right], \quad (4.1)$$

where $T_0(k^2)$ is the same as in (a).[1] The amplitude is unitary and analytic in the cut plane if $\lambda M/\mu<1$. The constraint for a sufficient condition is that

$$\lambda M\int_{\mu/2}^{\infty}\frac{dp}{p^2}\left[1-\frac{\lambda M}{2p}\ln\left(1+\frac{2p}{\mu}\right)\right]^{-1}\leqslant 1. \quad (4.2)$$

One can bound this integral to show that (4.2) is satisfied if $\lambda M/\mu<\frac{1}{6}$, whereas (4.2) cannot be satisfied if $\lambda M/\mu=0.25$.

(c) We consider s-wave scattering from the potential $V(r)=ge^{-r}$ for which

$$T_0(k^2)=\frac{1}{2}\sum_{r=1}^{\infty}\frac{(-g)^r}{r!(r-1)!}\frac{1}{k^2+\frac{1}{4}r^2}$$

It has been proved[5] that the corresponding partial-wave dispersion relation has a solution if $-g<1.4457$. Our condition on T_0 is met if $I_0(2\sqrt{|-g|})<1.5$ or $|g|<0.40$.

These examples give some indication of the "breadth of sufficiency" of the above conditions.

V. N-POLE CASE; THRESHOLD

Here we will consider the case in which

$$T_0(k^2)=g\sum_{i=1}^{N}\frac{\lambda_i}{k^2+a_i^2}, \quad (5.1)$$

[3] This step is justified *a posteriori*.

[4] It is also possible to use relation (5.8) to obtain N from D.

[5] R. Blankenbecler, M. L. Goldberger, N. N. Khuri, and S. B. Treiman, Ann. Phys. (N. Y.) 10, 62 (1959).

where λ_i and a_i are real, and we write for definiteness $0<a_1<a_2<\cdots<a_N$. We can write $f=N/D$ as above and obtain

$$N(k^2)=g\sum_{i=1}^{N}\frac{\lambda_i}{k^2+a_i^2}D(-a_i^2), \tag{5.2a}$$

$$D(k)=1-g\sum_{i=1}^{N}\frac{\lambda_i}{a_i-ik}D(-a_i^2). \tag{5.2b}$$

One can also write N and D in the form[6]

$$N(k^2)=A(k^2)/C_+,$$
$$D(k)=B(k)/C_+, \tag{5.3}$$

where

$$C_+=\det\{C_{ij}\}=\det\{\delta_{ij}+g\lambda_i/(a_i+a_j)\},$$

$$A(k^2)=\left|\begin{array}{c|c} 0 & g\lambda_i/(a_i^2+k^2) \\ \hline 1 & \\ \vdots & C_{ij} \\ 1 & \end{array}\right|,$$

and[7]

$$B(k)=\left|\begin{array}{c|c} 1 & -g\lambda_i/(a_i-ik) \\ \hline 1 & \\ \vdots & C_{ij} \\ 1 & \end{array}\right|.$$

We show that the following conditions on T_0, or, equivalently, upon g and λ_i, are sufficient to guarantee a solution of the PWDR problem (we require all λ_i to be of the same sign):

$$g\in D,$$

where the domain D has the following properties:

(1) D is connected;
(2) D contains the point $g=0$;
(3) for every point g in D,

$$C_+(g)=\det\{\delta_{ij}+g[\lambda_i/(a_i+a_j)]\} \tag{5.4}$$

is positive; and

(4) for every point in g in D,

$$C_-(g)=\det\{\delta_{ij}-g[\lambda_i/(a_i+a_j)]\} \tag{5.5}$$

is also greater than zero.

One can rewrite C_+ by writing it as a polynomial in g:

$$C_+=1+\sum_{p=1}^{N}g^p\Big(\sum_{i_1<i_2<\cdots<i_p}\Big)\Delta(a_{i_1}\cdots a_{i_p}),$$

where

$$\Delta(c_1\cdots c_p)=\det\left(\frac{\lambda_i}{c_i+c_j}\right)=\prod_{i=1}^{p}\frac{\lambda_i}{2c_i}\prod_{i<j}\left(\frac{c_i-c_j}{c_i+c_j}\right)^2.$$

We are guaranteed of a solution to the problem if D has no zeros in the upper half k plane. Since in Eq. (5.2b) we have required $D(k=\infty)=1$ if D does not vanish in the upper half k plane, then $D(k)$ will be positive everywhere along $(\mathrm{Re}k=0, \mathrm{Im}k\geqslant 0)$. The function $f(p)=D(ip)$ cannot vanish unless p is real if $D(-a_i^2)$ are all positive. (Either f or $-f$ is a Herglotz function, depending upon the sign of the λ_i.) Thus as g is increased, all the zeros of D remain on the imaginary k axis, at least until one of them enters the upper half k plane.[8] Thus the first zero to enter the region $\mathrm{Im}k>0$ will do so either at $k=0$ or $k=\infty$.

Now if $C_+(g)$ vanishes at some value g_0, it is not possible to require that $D(k^2=\infty, g_0)=1$; in other words, a zero of D may enter the upper half k plane at $g=g_0$. Similarly, $A(k=0)=C_-(g)$ and, if $C_-(g)$ vanishes, a zero of D occurs at $k=0$ and may enter the forbidden region. Thus the requirement $g\in D$ is *sufficient* to guarantee that a solution to the PWDR problem exists. Note that if $\lambda_i>0$, $C_+(g)$ never vanishes, whereas if $\lambda_i<0$, $C_-(g)$ never vanishes.

The sufficient condition obtained in Sec. III ($M<1$) requires that

$$g\sum_{i=1}^{N}\frac{|\lambda_i|}{2a_i}<\tfrac{1}{4} \tag{5.6}$$

One can verify independently that C_+ and C_- are positive under this restriction. In fact, one can verify that if the left-hand side of (5.6) is less than 1, C_+ and C_- are still positive. (See the Appendix.) In the examples given in Sec. IV the more general condition $M<4$ is still sufficient to obtain a solution, as one would expect it to be, by analogy with the n-pole problem with residues of the same sign.

If the numbers λ_i are not all positive, then zeros of D can enter the upper half k plane at any point on the real k plane; when they are on the real axis they are exactly cancelled by a zero of N, since for real, positive k

$$D(-k^*)=D(k)^*, \tag{5.7}$$

$$D(k)-D(-k)=2ikN(k^2). \tag{5.8}$$

One cannot expect a zero of N to follow the zeros of D as they enter the upper k plane, although such cancellation is not ruled out.

We now consider the threshold conditions on the amplitude:

If we assume for simplicity that $C_-(g)\neq 0$, the threshold condition is simply that $N(k^2)\sim k^{2l}$ as $k\to 0$.

[6] E. M. Nyman, Nuovo Cimento **37**, 492 (1965).

[7] Note that one can show that

$$B(k=0)=\det\{\delta_{ij}-g[\lambda_i/(a_i+a_j)]\}.$$

[8] The zeros of $D(k,g)$ move continuously in k on the Riemann sphere as g varies.

This is equivalent to

$$\left.\frac{\partial^p N(k^2)}{\partial(k^2)^p}\right|_{k^2=0}=0 \quad (p=0,\cdots,l-1)$$

or

$$T_p=\begin{vmatrix} 0 & g\lambda_i/a_i^{2p} \\ 1 & \\ \vdots & C_{ij} \\ 1 & \end{vmatrix}=0 \quad (p=0,\cdots,l-1).$$

One can easily check to see whether the desired threshold conditions are satisfied without needing a complete solution to the problem. The threshold condition is manifestly nonlinear in g and thus will be satisfied only at special values of g.

VI. DISCUSSION

We have obtained sufficient conditions for a solution of the $l=0$ PWDR problem to exist for a wide class of left-hand-cut discontinuities in Sec. III, and we have generalized these conditions in Sec. V, where the left-hand cut is replaced by a series of poles of residues of fixed sign.

Our solutions do not have CDD poles in the latter case since D remains finite off the imaginary k axis. One can, of course, construct solutions containing CDD poles by explicitly inserting them in D when the N/D separation is made.

In nonpathological cases where $\Delta T(k^2)$ is of fixed sign, it should be possible to approximate $T_0(k^2)$ by poles, so that the results of Sec. V should be of more general validity. More properly, one should view this procedure as the determinental approximation of the integral equation (3.5) for D. Under fairly general restrictions that equation is essentially Fredholm for k^2 real; $-\infty<k^2<-\mu^2$, so that one can generate a solution there by standard techniques. Then one can explicitly perform the Hilbert transform to obtain $D(k)$ everywhere for $\mathrm{Im}k>0$.

Now Eq. (5.2b) is to be considered as a determinental approximation to Eq. (3.5), and in fact the quantity $C_+(g)$ is the determinental approximation[9] to the Fredholm determinant of the kernel K in (3.5). Similarly, $C_-(g)$ is the determinental approximation of the Fredholm determinant of the kernel $-K$.

With nonpathological conditions on ΔT with $\Delta T>0$, one would expect the roots of $D(k,g)$ to move continuously in k as g is varied, so that conditions (1)–(4) should be sufficient to guarantee a suitable solution of the integral equation (5.5) by the Fredholm alternative and to guarantee that the solution of the N/D problem satisfy the desired conditions. [Of course, we interpret $C_\pm(g)$ as the exact Fredholm determinants of the kernels $\pm K$.]

The relativistic, inelastic PWDR problems can be studied from this same approach. Details will be given elsewhere.

[9] W. V. Lovitt, *Linear Integral Equations* (Dover Publications, Inc., New York, 1950), pp. 24ff.

ACKNOWLEDGMENTS

I would like to thank Professor S. B. Treiman of Princeton University, Professor P. B. Kantor of Case Western Reserve University, and Professor R. Blankenbecler of the University of California, Santa Barbara for stimulating and illuminating conversations upon this subject.

APPENDIX

We show that $C_\pm(g)$ do not vanish if

$$g\sum_{i=1}^{N}\frac{|\lambda_i|}{2a_i}<1. \qquad (A1)$$

Define a matrix

$$M_{ij}=g[(\lambda_i\lambda_j)^{1/2}/(a_i+a_j)]. \qquad (A2)$$

Now the matrix M is positive definite since

$$\det M_{ij}=g^N\prod_{i=1}^{N}\frac{|\lambda_i|}{2a_i}\prod_{i<j}\left(\frac{a_i-a_j}{a_i+a_j}\right)^2\geqslant 0 \qquad (A3)$$

and all the subdeterminants are of the same form and thus also positive. Thus the eigenvalues of M are all positive.

But $\mathrm{Tr}M$ is equal to the sum of the eigenvalues of M, so every eigenvalue μ satisfies the constraint

$$0<\mu<g\sum_{i=1}^{N}\frac{|\lambda_i|}{2a_i}<1, \qquad (A4)$$

where μ satisfies the equation

$$\det\{g[|\lambda_i\lambda_j|^{1/2}/(a_i+a_j)]-\delta_{ij}\mu\}=0. \qquad (A5)$$

If either $C_+(g)$ or $C_-(g)$ were to vanish, the matrix M would have an eigenvalue equal to $+1$. Since all the eigenvalues of M are less than 1, the numbers $C_\pm(g)$ cannot vanish. Hence the result is proved.