Iterations of Regge Cuts

P. V. LANDSHOFF AND J. C. POLKINGHORNE
Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge, England

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The Sudakov technique is used to calculate contributions corresponding to iterations of two-Reggeon cuts. A form is found which agrees with earlier conjectures and which consistently takes into account the requirements of unitarity. It is shown that the expression obtained can be used as the basis for a reformulated multi-Reggeon bootstrap. It is claimed that this version correctly takes into account known cancellation effects. However, it requires more input information than previous versions.

I. INTRODUCTION

It is now recognized that Regge poles by themselves are not sufficient to take account of the high-energy behavior of scattering amplitudes. The effects of cuts must also be included.

The simplest Feynman graph that produces a Regge cut is the double-cross graph discussed by Mandelstam.[1] This is drawn in Fig. 1, where the bubbles represent complete scattering amplitudes. If the asymptotic behavior of the bubbles is represented by Regge-pole contribution, the graph generates a two-Reggeon cut.

But the graph of Fig. 1 does not by itself give an adequate model of the two-Reggeon cut, since it produces a branch point that is more singular than is allowed by unitarity.[2] It seems[3] that to get a realistic model one must sum over iterations of the graph. This sum is represented in Fig. 2, where the first term corresponds to Fig. 1. The simplest iteration is shown in Fig. 3, corresponding to the second term in Fig. 2. The subject of this paper is the calculation of the contribution that the iterations make to the two-Reggeon cut.

The contribution from the nth iteration has been conjectured[4] to take the form (signature factors being omitted for simplicity)

$$\frac{1}{4\pi i}\int dl\, s^l \int d^2\kappa_1 d^2\kappa_2 \cdots d^2\kappa_{n-1} \times \frac{\hat{X}XX\cdots X\hat{X}}{\prod_{i=1}^{n-1}\left[l-\alpha[(\kappa_i+\tfrac{1}{2}q)^2]-\alpha[(\kappa_i-\tfrac{1}{2}q)^2]+1\right]}. \quad (1.1)$$

Here s is the asymptotic variable $s=(p+p')^2$, the momentum transfer is $q^2=t$, and the κ_i are $(n-1)$ two-dimensional momenta running round the loops associated with pairs of Reggeons. The functions $\hat{X}$ and X may be regarded, respectively, as particle-Reggeon and Reggeon-Reggeon amplitudes, corresponding to the blobs in Fig. 2. The form (1.1) agrees with the rules of Gribov's Reggeon calculus,[5] though the iterations considered by Gribov always involve intermediate states with two elementary particles which trivially give product forms for repeated Regge singularities. It is contrary to the spirit of the hybrid "Reggeized Feynman integral" model to include diagrams with these intermediate states since their presence would lead to an essential singularity which is not present in a properly Reggeized theory.[4] Instead, we couple Reggeons only through third spectral function diagrams of the cross type. This paper therefore constitutes an extension of the Gribov calculus to diagrams with four-Reggeon vertices, having as its immediate object the verification of (1.1). We find that it is indeed correct, though with qualifications which we detail below.

Besides its phenomenological importance, the iteration of the two-Reggeon cut plays an important role in our understanding of the Gribov-Pomeranchuk essential singularity at $l=-1$. It has been shown[4] that the cut obtained by summing (1.1) over n has the correct

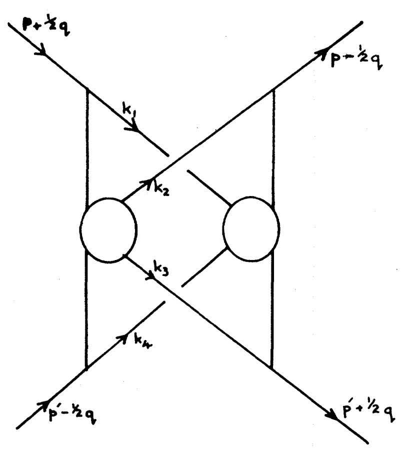

FIG. 1. Simplest Feynman diagram that produces a Regge cut. The bubbles represent complete scattering amplitudes.

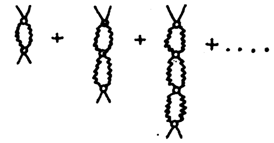

FIG. 2. Sum of iterations of the graph of Fig. 1. The first term corresponds to Fig. 1.

[1] S. Mandelstam, Nuovo Cimento **30**, 1148 (1963). J. C. Polkinghorne, J. Math. Phys. **4**, 1396 (1963).
[2] J. Bronzan and C. Jones, Phys. Rev. **160**, 1494 (1967).
[3] J. C. Polkinghorne, Nucl. Phys. **B6**, 441 (1968).
[4] D. I. Olive and J. C. Polkinghorne, Phys. Rev. **171**, 1475 (1968).
[5] V. N. Gribov, Zh. Eksperim. i Teor. Fiz. **53**, 654 (1967) [English transl.: Soviet Phys.—JETP **26**, 414 (1968)].

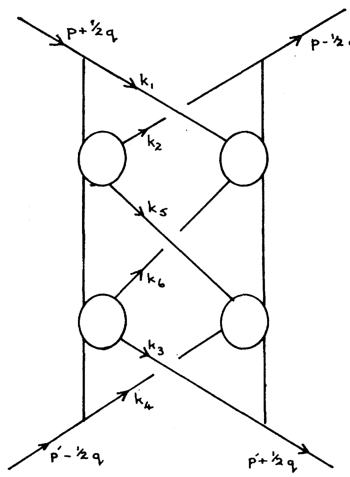

FIG. 3. Simplest iteration, corresponding to the second term in Fig. 2.

structure to eliminate the essential singularity. A particular feature of (1.1) that is necessary for this elimination is that, when the Reggeons are on the mass shell and so, in the present model, represent physical zero-spin particles, the functions $\hat{X}$ and X should coincide with each other and with the coupling of the physical particles to the $l=-1$ pole in the single-cross graph. A check on this is made here.

A stronger result also holds: When two of its four Reggeons become physical zero-spin particles, the function X becomes equal to $\hat{X}$. To be more accurate, X may be defined in such a way that this is true. The lack of a complete definition of X arises because, in addition to contributing to the second term of Fig. 2, the graph of Fig. 3 also contributes to the first term. The choice of which part of the contribution shall be assigned to each of the two terms involves a certain arbitrariness.

This arbitrariness may need consideration in another very interesting application of the sum of the series in Fig. 2. This is the suggestion of Chew and Pignotti, and of Halliday,[6] that infinite sums of this type, as well as containing the two-Reggeon cut, shall also contain a Regge pole, and so lead to the generation of a kind of bootstrap. However, the calculation of these authors differs from that discussed here in that they sum the Amati–Fubini–Stanghellini (AFS) cuts[7] which appear in multiparticle unitarity integrals. We shall take the opportunity to discuss the multi-Reggeon bootstrap program in relation to Mandelstam cuts.

The arrangement of the present paper is as follows. First, in Sec. II, we repeat Gribov's analysis[5] of the graph of Fig. 1. This provides a useful preliminary to the analysis of the more complicated graphs, and also yields for us the definition of the functions $\hat{X}$. In Sec. III, we then apply the method to the graph of Fig. 3. Properties of the function X that we obtain here we analyze in Sec. IV, and it is compared with the function $\hat{X}$. Then, in Sec. V, we sketch an analysis of the general term in the series of Fig. 2. Finally, in Sec. VI we discuss the multi-Reggeon bootstrap program formulated in terms of Mandelstam cuts.

We are conscious that in common with previous applications of Gribov's method our analysis is somewhat lacking in mathematical rigor, even though the results are perhaps compelling. A particular difficulty that we encounter is the subject of the Appendix.

II. DOUBLE-CROSS GRAPH

We label the momenta as in Fig. 1, and for algebraic simplicity assign equal mass m to each particle. This implies that

$$\begin{aligned} p\cdot q=p'\cdot q=0,\\ p^2=p'^2=m^2+\tfrac{1}{4}q^2\equiv\tau. \end{aligned} \tag{2.1}$$

The set of internal momenta will be described by variables similar to, though not quite identical with, the Sudakov variables used by Gribov.[5] We write

$$k_i=x_ip+y_ip'+\kappa_i, \tag{2.2}$$

where

$$\kappa_i\cdot p=\kappa_i\cdot p'=0.$$

The two-dimensional vectors κ_i are spacelike: $\kappa_i^2\leqslant 0$. Energy-momentum conservation requires that

$$\sum x_i=1,\quad \sum y_i=1,\quad \sum \kappa_i=0. \tag{2.3}$$

In terms of the new variables, the squared momenta in the four lines of the upper cross are given by

$$\begin{gathered} x_1y_1s+(x_1-y_1)^2\tau+\kappa_1^2,\\ (x_1-1)y_1s+(x_1-y_1-1)^2\tau+(\kappa_1-\tfrac{1}{2}q)^2,\\ x_2y_2s+(x_2-y_2)^2\tau+\kappa_2^2,\\ (x_2-1)y_2s+(x_2-y_2-1)^2\tau+(\kappa_2+\tfrac{1}{2}q)^2, \end{gathered} \tag{2.4}$$

where $s=(p+p')^2$ is the asymptotic variable. Similar expressions hold for the lower cross. All these squared momenta are external masses for the Reggeon bubbles and, following Gribov, we suppose that these amplitudes decrease rapidly when these masses become large, so that as $s\to\infty$ the dominant contribution to the integral comes from those values of the integration variables that make them finite. This is an abstraction from known properties of ladder diagrams which generate[8] Regge poles in perturbation theory. Similar considerations lead us to restrict the region of integration so that the squared masses of the Reggeons are also finite.

[6] G. F. Chew and A. Pignotti, Phys. Rev. **176**, 2112 (1968); I. G. Halliday, Nuovo Cimento **60A**, 177 (1969).

[7] D. Amati, S. Fubini, and A. Stanghellini, Nuovo Cimento **26**, 896 (1962); but see S. Mandelstam, *ibid.* **30**, 1126 (1963); J. C. Polkinghorne, Phys. Letters **4**, 24 (1963).

[8] R. J. Eden, P. V. Landshoff, D. I. Olive, and J. C. Polkinghorne, in *The Analytic S-Matrix* (Cambridge University Press, Cambridge, 1966), Sec. 3.6.

These are given by

$$(x_1+x_2-1)(y_1+y_2)s+(x_1+x_2-y_1-y_2-1)^2\tau+(\kappa_1+\kappa_2\pm\tfrac{1}{2}q)^2. \quad (2.5)$$

We shall further suppose that our regions of integration are restricted to finite values of the two-dimensional momenta κ_i. Although we could proceed some way with our argument without recourse to this assumption we should be forced, within our present understanding, to adopt it before the end of the paper. A similar observation has been made by Winbow[9] in the course of his applications of the Gribov method. As we discuss in the Appendix, it is possible to regard the assumption as a further piece of abstraction from a Feynman-integral model of Regge poles.

The condition that each of the expressions in (2.4) be finite then yields

$$x_1,\ x_2 \text{ finite}, \quad y_1,\ y_2=O(s^{-1}). \quad (2.6)$$

[The proof of this is facilitated by the simple observation that not only must each of the first two expressions in (2.4) be finite, but also their difference.] Similar considerations applied to the lower cross give

$$x_3,\ x_4=O(s^{-1}), \quad y_3,\ y_4 \text{ finite}. \quad (2.7)$$

The results (2.6) and (2.7) automatically result in the finiteness of the Reggeon masses (2.5) and also mean that these may be approximated by $(\kappa_1+\kappa_2\pm\frac{1}{2}q)^2$. Further, the leading terms in the energy variables associated with the Reggeon amplitudes,

$$(x_2+x_3)(y_2+y_3)s+(x_2+x_3-y_2-y_3)^2\tau+(\kappa_2+\kappa_3)^2,$$
$$(x_1+x_4)(y_1+y_4)s+(x_1+x_4-y_1-y_4)^2\tau+(\kappa_1+\kappa_4)^2,$$

are simply x_2y_3s and x_1y_4s. Finally, to order s^{-1}, y_1, y_2, x_3, x_4 disappear from the conditions (2.3).

Thus, the result of the calculation may be written in the form

$$\frac{1}{2}\int d^2\kappa[\hat{X}(\kappa)]^2 s^{\alpha[(\kappa+\frac{1}{2}q)^2]+\alpha[(\kappa-\frac{1}{2}q)^2]-1}, \quad (2.8)$$

where, apart from constant factors,

$$\hat{X}(\kappa)=\frac{1}{2}\int\frac{d^2\kappa_1d^2\kappa_2\,\delta^{(2)}(\kappa_1+\kappa_2-\kappa)dx_1dx_2d\bar{y}_1d\bar{y}_2\,\delta(x_1+x_2-1)}{[x_1\bar{y}_1+x_1^2\tau+\kappa_1^2-m^2][-x_2\bar{y}_1+x_2^2\tau+(\kappa_1-\frac{1}{2}q)^2-m^2]}\times\frac{x_1^{\alpha[(\kappa+\frac{1}{2}q)^2]}x_2^{\alpha[(\kappa-\frac{1}{2}q)^2]}g_1g_2}{[x_2\bar{y}_2+x_2^2\tau+\kappa_2^2-m^2][-x_1\bar{y}_2+x_1^2\tau+(\kappa_2+\frac{1}{2}q)^2-m^2]}. \quad (2.9)$$

Here we have used the fact that

$$d^4k_i\to\tfrac{1}{2}s\,dx_idy_id^2\kappa_i$$

as $s\to\infty$, and have made a further change of variable

$$y_1s=\bar{y}_1,\quad y_2s=\bar{y}_2. \quad (2.10)$$

The functions g_1 and g_2 are the Reggeon coupling functions, which depend on the Reggeon masses and the masses of the lines in the cross. We have used (2.6) to neglect some of the terms in the expressions (2.4).

Some properties of the function $\hat{X}$ are explored in Sec. IV. Here we remark that the result (2.8) agrees with (1.1) for the case $n=2$. We have not explicitly inserted signature factors. They can be introduced as discussed by Gribov, and lead to the conclusion that the cut occurs in the amplitude of signature equal to the product of the signatures of the participating Reggeons.[10] Also, a consideration of the way the poles in the y_1 and y_2 integrations are disposed with respect to the real axis by the (implicit) $i\epsilon$ prescription shows that in fact (2.9) gives nonzero values only for x_1 and x_2 lying between 0 and 1. This has been discussed by Gribov.[5]

[9] G. A. Winbow, Phys. Rev. **177**, 2533 (1969).

[10] See also J. C. Polkinghorne, Nuovo Cimento **56A**, 755 (1968).

III. TRIPLE-CROSS GRAPH

We now consider the graph of Fig. 3, where the momenta are labelled as indicated, and we use again the parametrization (2.2) for the momenta k_i.

The discussion of the momenta in the end crosses exactly parallels that of Sec. II, and leads to the same conclusions (2.6) and (2.7). For the middle cross, the masses squared in its four lines are given by

$$\begin{aligned}&x_5y_5s+(x_5-y_5)^2\tau+\kappa_5^2,\\&x_6y_6s+(x_6-y_6)^2\tau+\kappa_6^2,\\&-(x_3+x_4+x_6)(y_1+y_2+y_5)s+(x_3+x_4+x_6+y_1+y_2+y_5)^2\tau+(\kappa_1+\kappa_2+\kappa_5-\tfrac{1}{2}q)^2,\\&-(x_3+x_4+x_5)(y_1+y_2+y_6)s+(x_3+x_4+x_5+y_1+y_2+y_6)^2\tau+(\kappa_1+\kappa_2+\kappa_6-\tfrac{1}{2}q)^2,\end{aligned} \quad (3.1)$$

where we have made use of the energy-momentum conservation conditions analogous to (2.3). The condition that these expressions be finite has certain solutions for which x_5 and x_6 are large [i.e., $O(s^\gamma)$, $\gamma>0$], provided that y_5 and y_6 correspond suitably. However, from (2.6), (2.7), and (2.3) we already know that their sums (x_5+x_6) and (y_5+y_6) cannot be large. Further, a combination of the expressions (3.1) leads to the condition that

$$(x_5+x_6)(y_5+y_6)s+(x_5+x_6)(y_1+y_2)s +(x_3+x_4)(y_5+y_6)s-2(x_5+x_6)(y_5+y_6)\tau \quad (3.2)$$

be finite [we have omitted some terms that are of manifestly of lower order because of (2.6) and (2.7)]. This leads to three types of solution to the finiteness requirements.

In the first we have

$$x_5+x_6=O(s^{-1}),\quad y_5+y_6 \text{ finite.} \quad (3.3)$$

The resulting contribution may be evaluated by making a change of variables:

$$x_5=\lambda s^{-1}\bar{x}_5,\quad x_6=\lambda s^{-1}\bar{x}_6,\quad \bar{x}_5+\bar{x}_6=1.$$

If one works through this, one finds that the answer must be interpreted as contributing to the first term of Fig. 2 rather than the second, in that the dominant contribution corresponds to the energy variables $(k_3+k_6)^2$, $(k_4+k_5)^2$ associated with the lower pair of bubbles being finite. This represents a model-dependent addition to the lower $\hat{X}$ function, corresponding to replacing the lower cross of Fig. 1 by the subdiagram below, and including the middle cross of Fig. 3. Since the latter subdiagram has a third Mandelstam spectral function, a contribution of this form must be expected.[1,11]

The second class of solutions is

$$x_5+x_6 \text{ finite},\quad y_5+y_6=O(s^{-1}). \quad (3.4)$$

This again results in a contribution to the first term of Fig. 2 rather than to the second, and yields a correction to the upper $\hat{X}$ function.

Interpolating between the solutions (3.3) and (3.4) is the third possibility

$$x_5+x_6=O(s^{-\gamma}),\quad y_5+y_6=O(s^{\gamma-1}),\quad 0<\gamma<1. \quad (3.5)$$

This leads us to make the change of integration variables

$$x_5=s^{-\gamma}\bar{x}_5,\quad y_5=s^{\gamma-1}\bar{y}_5,\quad x_6=s^{-\gamma}\bar{x}_6,\quad y_6=s^{\gamma-1}\bar{y}_6. \quad (3.6)$$

In order not to increase the number of integration variables we impose the constraint

$$\bar{x}_5+\bar{x}_6=1. \quad (3.7)$$

This procedure is manifestly asymmetric between x and y. We shall discuss this in the next section and show that, correctly interpreted, our answers will not suffer by it. It can only be done at all with γ real if $x_5+x_6>0$, so we first break the integration region into two parts, $x_5+x_6\gtrless 0$, and in the latter make the change of variables $x\to -x$, $y\to -y$. The Jacobian is

$$dx_5dx_6dy_5dy_6=s^{-2}\ln s\,\delta(\bar{x}_5+\bar{x}_6-1)d\bar{x}_5d\bar{x}_6d\bar{y}_5d\bar{y}_6. \quad (3.8)$$

We also make the change of integration variables given in (2.10), together with a corresponding one involving x_3 and x_4.

The basic assumption underlying the Gribov method is that one can take the limit $s\to\infty$ inside the integrations, provided the resulting integral is convergent. Thus, for example, when we have made the changes of integration variable the last expression in (3.1) is equal to

$$-\bar{x}_5\bar{y}_6+(\kappa_1+\kappa_2+\kappa_6-\tfrac{1}{2}q)^2 \quad (3.9)$$

plus functions of the $\bar{x}$ and $\bar{y}$ that are multiplied by negative powers of s. These we neglect, in comparison with (3.9). With this procedure we obtain the result

$$\frac{1}{2}\int d^2\kappa d^2\kappa'\,\hat{X}(\kappa)X(\kappa,\kappa')\hat{X}(\kappa')\phi(\kappa,\kappa',s), \quad (3.10)$$

where $\kappa'=\kappa_1+\kappa_2$, $\kappa=\kappa_3+\kappa_4$, $\hat{X}$ is the function defined in (2.9),

$$X(\kappa,\kappa') =\frac{1}{2}\int\frac{d^2\kappa_5d^2\kappa_6\,\delta^{(2)}(\kappa+\kappa'+\kappa_5+\kappa_6)d\bar{x}_5d\bar{x}_6d\bar{y}_5d\bar{y}_6}{[\bar{x}_5\bar{y}_5+\kappa_5^2-m^2][-\bar{x}_6\bar{y}_5+(\kappa'+\kappa_5-\frac{1}{2}q)^2-m^2]} \times\frac{\delta(\bar{x}_5+\bar{x}_6-1)g_1g_2g_1'g_2'\bar{x}_5^{\alpha[(\kappa+\frac{1}{2}q)^2]}\bar{x}_6^{\alpha[(\kappa-\frac{1}{2}q)^2]}}{[\bar{x}_6\bar{y}_6+\kappa_6^2-m^2][-\bar{x}_5\bar{y}_6+(\kappa'+\kappa_6-\frac{1}{2}q)^2-m^2]} \times\bar{y}_5^{\alpha[(\kappa'-\frac{1}{2}q)^2]}\bar{y}_6^{\alpha[(\kappa'+\frac{1}{2}q)^2]} \quad (3.11)$$

and

$$\phi(\kappa,\kappa',s)=\tfrac{1}{2}(1+\Xi)s^{-1}\ln s\int_0^1 d\gamma\, s^{(1-\gamma)a+\gamma b} =\tfrac{1}{2}(1+\Xi)\frac{s^{a-1}-s^{b-1}}{a-b}, \quad (3.12a)$$

with

$$a=\alpha[(\kappa'+\tfrac{1}{2}q)^2]+\alpha[(\kappa'-\tfrac{1}{2}q)^2],\quad b=\alpha[(\kappa+\tfrac{1}{2}q)^2]+\alpha[(\kappa-\tfrac{1}{2}q)^2]. \quad (3.12b)$$

In (3.12), Ξ is the product of the signatures of the four Reggeons, so that $\frac{1}{2}(1+\Xi)$ is equal to 0 or 1. This factor arises from the breaking up of the integration region into the two parts $x_5+x_6\gtrless 0$ before the transformation of variables (3.6) and (3.7). Thus, as might be expected, there is no contribution when the signatures of the two cuts to either side of the central cross are opposite; the factor $(1+\Xi)$ forbids the mixing of signatures. When $\frac{1}{2}(1+\Xi)=1$ we see on inserting (3.12) into (3.10) that we have recovered (1.1) for the case $n=3$.

[11] H. J. Rothe, Phys. Rev. **159**, 1471 (1967).

Properties of the function X are examined in the next section.

IV. PROPERTIES OF $\hat{X}$ AND X

In this section we examine and compare the functions $\hat{X}$ and X. Consider $\hat{X}$, given by (2.9). We first note that our result has a slightly different appearance from that of Gribov, because of our slightly different definition of the parameters x and y. It can be reduced to the Gribov form by the transformation

$$\bar{y}\to\bar{y}-\tau x. \tag{4.1}$$

This removes the terms $x_1{}^2\tau$ and $x_2{}^2\tau$ from the first and third factors in the denominator, and in the fourth and second factors it replaces them by $x_1\tau$ and $x_2\tau$, respectively (we have used the δ function in the numerator). We show that these terms may also be omitted, provided the terms $(\kappa_1-\frac{1}{2}q)^2$ and $(\kappa_2+\frac{1}{2}q)^2$ are, respectively, replaced by $(\kappa_1-p-\frac{1}{2}q)^2$ and $(\kappa_2-p+\frac{1}{2}q)^2$. We do this in order to be able to compare $\hat{X}$ with the expression (3.11) for X, which does not have the terms involving τ.

Initially, we treat the coupling functions g_1 and g_2 as constant. Introduce "Feynman" parameters λ_1, λ_2, λ_3, λ_4 by using the identity

$$\frac{1}{x_1\bar{y}_1+\kappa_1{}^2-m^2}=i\int_0^\infty d\lambda_1\exp[i\lambda_1(x_1\bar{y}_1+\kappa_1{}^2-m^2)] \tag{4.2}$$

and three similar relations. The convergence of the integral (4.2) is ensured by Feynman's prescription of adding a small negative imaginary part to m^2. The κ_1 and κ_2 integrations may now be performed, with the result, apart from constant factors,

$$\begin{aligned}\int d\lambda\int dx_1dx_2d\bar{y}_1d\bar{y}_2\,\delta(x_1+x_2-1)&\\ \times C^{-1}x_1{}^{\alpha[(\kappa+\frac{1}{2}q)^2]}x_2{}^{\alpha[(\kappa-\frac{1}{2}q)^2]}&\\ \times\exp\{i[\lambda_1x_1\bar{y}_1+\lambda_2x_2(\tau-\bar{y}_1)&\\ +\lambda_3x_2\bar{y}_2+\lambda_4x_1(\tau-\bar{y}_2)+C^{-1}D]\}.&\end{aligned} \tag{4.3}$$

Here $C=\lambda_1+\lambda_2+\lambda_3+\lambda_4$ and D is the D function[12] for the upper single cross in Fig. 1, but with p replaced by zero. The $\bar{y}$ integrations in (4.3) give δ functions, and if these are combined with the δ function already appearing in (4.3), this yields

$$x_1=\lambda_2/(\lambda_1+\lambda_2)=\lambda_3/(\lambda_3+\lambda_4),$$
$$x_2=\lambda_1/(\lambda_1+\lambda_2)=\lambda_4/(\lambda_3+\lambda_4).$$

Substitution of these values into the τ-dependent terms of the exponent in (4.3) gives a contribution identical with that which would result from omitting the τ terms and restoring the p dependence in D. As is explained elsewhere,[13] the effect of the variation of the coupling functions g_1, g_2 (which of course is crucial to the work of Sec. II) may be taken into account by making some weak assumptions about their analytic properties.

This result allows us to deduce immediately that, when X in (3.11) is continued in K' to the point where

$$\alpha[(\kappa'-\tfrac{1}{2}q)^2]=\alpha[(\kappa'+\tfrac{1}{2}q)^2]=0,$$

that is,

$$(\kappa'-\tfrac{1}{2}q)^2=(\kappa'+\tfrac{1}{2}q)^2=m^2,$$

then X coincides with $\hat{X}$.

However, a similar result does not follow if instead we put the exponents of $\bar{x}_5$ and $\bar{x}_6$ in (3.11) equal to zero. This difference can immediately be traced back to the unsymmetrical change of variables in (3.6) and (3.7); we could equally well have replaced (3.7) by the constraint

$$\bar{y}_5+\bar{y}_6=1, \tag{4.4}$$

which would simply have the result that it replaces $\delta(\bar{x}_5+\bar{x}_6-1)$ in (3.11) by $\delta(\bar{y}_5+\bar{y}_6-1)$, giving a new function $X'(K,K')$.

To analyze this difference, make yet another change of variable in (3.11):

$$\bar{y}_5\to\lambda\bar{y}_5,\quad \bar{y}_6\to\lambda\bar{y}_6,\quad \bar{y}_5+\bar{y}_6=1. \tag{4.5}$$

This yields

$$X(\kappa,\kappa')=\frac{1}{2}\int\frac{d^2\kappa_5d^2\kappa_6\,\delta^{(2)}(\kappa+\kappa'+\kappa_5+\kappa_6)d\bar{x}_5d\bar{x}_6d\bar{y}_5d\bar{y}_6d\lambda}{(\lambda\bar{x}_5\bar{y}_5+\kappa_5{}^2-m^2)[-\lambda\bar{x}_6\bar{y}_5+(\kappa'+\kappa_5-\frac{1}{2}q)^2-m^2]}\times\frac{\delta(\bar{x}_5+\bar{x}_6-1)\delta(\bar{y}_5+\bar{y}_6-1)g_1g_2g_1'g_2'\lambda^{a+1}\bar{x}_5{}^a\bar{x}_6{}^a\bar{y}_5{}^a\bar{y}_6{}^a}{(\lambda\bar{x}_6\bar{y}_6+\kappa_6{}^2-m^2)[-\lambda\bar{x}_5\bar{y}_6+(\kappa'+\kappa_6-\frac{1}{2}q)^2-m^2]}, \tag{4.6}$$

where a is defined in (3.12b). On the other hand, X' is given by the same expression, but with λ^{a+1} replaced by λ^{b+1}. Thus, the difference between X and X' contains the factor $\lambda^{a+1}-\lambda^{b+1}$, which tends to zero when $a\to b$. So the difference between the integrand in (3.10) and the similar one involving X' does not contain a double pole in the l plane when $a\to b$, but rather the *sum* of separate poles. That is, the difference between using X and X' in (3.10) is that it leads to different separations of the total contribution from the graph of Fig. 3 into parts identified with the first term in Fig. 2 and with the second term.

[12] Reference 8, Sec. 1.5.

[13] I. T. Drummond, P. V. Landshoff, and W. J. Zakrzewski, Phys. Letters **28B**, 676 (1969).

There are many other functions equivalent to X in the same sense as is X'. Instead of (3.6) and (3.7), use the variables defined by

$$x_5=\lambda^{\mu}s^{-\gamma}\bar{x}_5\,,\quad y_5=\lambda^{1-\mu}s^{\gamma-1}\bar{y}_5\,,$$
$$x_6=\lambda^{\mu}s^{-\gamma}\bar{x}_6\,,\quad y_6=\lambda^{1-\mu}s^{\gamma-1}\bar{y}_6\,,$$
$$\bar{x}_5+\bar{x}_6=1\,,\quad \bar{y}_5+\bar{y}_6=1\,,\quad 0\leqslant\mu\leqslant 1\,.$$

This results in an integral like (4.6), but with λ^{a+1} replaced by

$$\lambda^{\mu b+(1-\mu)a+1}.$$

This term is independent of μ when $a=b$, so all these possibilities lead to an identical *double*-pole structure.

Finally, we show that when the masses of all four Reggeons are such that $\alpha=0$, both X and $\hat{X}$ are equal to the residue of the $l=-1$ pole in the simple single-cross graph. For under these conditions it is easy also to do the x integrations in (4.3), with a result that may be expressed in the form

$$\int d\lambda\; C^{-1}e^{iC^{-1}D}\delta(\lambda_3\lambda_1-\lambda_2\lambda_4)$$
$$=\int ds_1 d\lambda\; C^{-2}e^{iC^{-1}D}=\int ds_1\, F(s_1,q^2)\,. \qquad (4.7)$$

Here F is the Feynman integral for the single-cross graph, with energy variable s_1 and momentum transfer $q^2=t$. Now, since, as $s_1\to\infty$, $F\to 0$ faster[14] than s_1^{-1}, the Froissart-Gribov integral for the positive-signature Regge amplitude corresponding to F converges right down to $l=-1$. The residue of the pole at $l=-1$ is thus obtained by taking the residue of $Q_l(z)$ inside the integral, with the result that it is just

$$2i\int_{s_1>0} ds_1\,\mathrm{Im}F(s_1,t)\,. \qquad (4.8)$$

Expressing this as an integral of F over a contour around the right-hand cut, and opening the contour, we obtain (4.4).

V. FURTHER ITERATIONS

We have not made a complete investigation of the general iteration, but give here an outline of how we expect it to work out.

Label momenta k_i on the lines of the crosses as before, with k_{2r-1} and k_{2r} ($r=1, 2, \cdots, n$) referring to the rth cross. Introduce parameters x_i and y_i, and momenta κ_i, as before. Then the masses on the lines in the rth cross are

$$k_{2r-1}^2=x_{2r-1}y_{2r-1}s+(x_{2r-1}-y_{2r-1})^2\tau+\kappa_{2r-1}^2\,,$$
$$k_{2r}^2=x_{2r}y_{2r}s+(x_{2r}-y_{2r})^2\tau+\kappa_{2r}^2\,,$$
$$(p-k_1-k_2-\cdots-k_{2r-1})^2$$
$$=-(x_{2r}+\cdots+x_{2n})(y_1+\cdots+y_{2r-1})s$$
$$+(x_{2r}+\cdots x_{2r}+y_1+\cdots y_{2r-1})^2\tau$$
$$+(\kappa_1+\cdots\kappa_{2r-1}-\tfrac{1}{2}q)^2\,,$$
$$(p-k_1-\cdots-k_{2r-2}-k_{2r})^2$$
$$=-(x_{2r-1}+x_{2r+1}+\cdots+x_{2n})(y_1+\cdots+y_{2r-2}+y_{2r})s$$
$$+(x_{2r-1}+x_{2r+1}+\cdots+x_{2n}+y_1+\cdots+y_{2r+2}+y_{2r})^2\tau$$
$$+(\kappa_1+\cdots+\kappa_{2r-2}+\kappa_{2r}+\tfrac{1}{2}q)^2 \qquad (5.1)$$

and the masses on the Reggeons connecting this cross to the $(r+1)$th are

$$(p-k_1-\cdots-k_{2r})^2$$
$$=-(x_{2r+1}+\cdots+x_{2n})(y_1+\cdots+y_{2r})s$$
$$+(x_{2r+1}+\cdots+x_{2n}+y_1+\cdots+y_{2r})^2\tau$$
$$+(\kappa_1+\cdots+\kappa_{2r}\pm\tfrac{1}{2}q)^2\,, \qquad (5.2)$$

with corresponding energy variables

$$(k_{2r}+k_{2r+1})^2=(x_{2r}+x_{2r+1})(y_{2r}+y_{2r+1})s$$
$$+(x_{2r}+x_{2r+1}-y_{2r}-y_{2r+1})^2\tau$$
$$+(\kappa_{2r}+\kappa_{2r+1})^2\,, \qquad (5.3)$$
$$(k_{2r-1}+k_{2r+2})^2=(x_{2r-1}+x_{2r+2})(y_{2r-1}+y_{2r+2})s$$
$$+(x_{2r-1}+x_{2r+2}-y_{2r-1}-y_{2r+2})^2\tau$$
$$+(\kappa_{2r-1}+\kappa_{2r+2})^2.$$

We expect that the relevant solution to the condition that the masses be finite is

$$\kappa_i \text{ finite}\,,$$
$$x_{2r-1},\, x_{2r}=O(s^{-\gamma_r})\,, \qquad (5.4)$$
$$y_{2r-1},\, y_{2r}=O(s^{\gamma_{r-1}})\,,$$

where

$$O=\gamma_1\leqslant\gamma_2\leqslant\cdots\leqslant\gamma_n=1\,. \qquad (5.5)$$

Then the first terms in the last two expressions in (5.1) may, respectively, be approximated by

$$-x_{2r}y_{2r-1}s\,,\quad -x_{2r-1}y_{2r}s\,,$$

the first term in (5.2) may be omitted, and the first terms in the expressions in (5.3) become, respectively,

$$x_{2r}y_{2r+1}s\,,\quad x_{2r-1}y_{2r+2}s\,.$$

Making these approximations, and the changes of variable

$$x_{2r-1}=s^{-\gamma_r}\bar{x}_{2r-1}\,,\quad y_{2r-1}=s^{\gamma_{r-1}}\bar{y}_{2r-1}\,,$$
$$x_{2r}=s^{-\gamma_r}\bar{x}_{2r}\,,\quad y_{2r}=s^{\gamma_{r-1}}\bar{y}_{2r}\,,$$

with the constraints

$$\bar{x}_{2r-1}+\bar{x}_{2r}=1\,,$$

[14] Reference 8, Sec. 3.4.

and integrating the γ_r over the region (5.5), we obtain an expression that corresponds exactly to (1.1).

VI. MULTI-REGGEON BOOTSTRAP

The infinite sum of iterated cut contributions of the form (1.1) represents the effects of taking into account the requirements of unitarity in the nonasymptotic channel. This can be seen in several ways. The presence of Regge poles at sense right-signature integers means that (1.1) has normal threshold singularities corresponding to the two-particle intermediate states given by the particles lying on the trajectories. These singularities were discussed in Ref. 4, where it is also shown that one gets a discontinuity formula of the correct unitarity form only by taking the infinite sum. (This is exactly analogous to the well-known fact that if one considers normal thresholds associated with Feynman integrals corresponding to two line bubbles, it is only the infinite set of such interated bubbles which gives the correct unitarity form.) Furthermore, it is only the infinite set which eliminates the Gribov-Pomeranchuk essential singularity[4] and softens the nature of the cut singularity to make it noninfinite and hence consistent with unitarity.[3]

If this infinite sum is to be performed in other than a formal sense, it will have to be achieved by writing an integral equation constructed in an obvious way analogous to the Bethe-Salpeter equation. A solution of this equation will involve a denominator function which can vanish for certain values of l. Thus, the solution of the equation will not only give the unitarized cut but it will contain within it also the possibility of Regge poles, and hence the possibility of a bootstrap procedure in which these poles are identified with the poles originally used to generate the cuts. If we make the strong and presumably unrealistic assumption that the X functions can be approximated by a separable form, then the integral equation becomes an algebraic equation summing a geometric series and the possibility of poles from a vanishing denominator is immediately explicit.

We have not written down the equations expressing these ideas because they are almost identical in form with the equations of the multi-Reggeon bootstrap given by Chew and Pignotti and by Halliday.[6] There are, however, two important differences of interpretation between their equations and those which result from the discussion of this section:

(a) The cuts discussed by Chew and Pignotti and by Halliday are Amati–Fubini–Stanghellini (AFS) cuts in multiparticle unitarity integrals. It is known in models[7] and also from general considerations relating to the properties of Regge residues[11] that these cuts are cancelled in the complete amplitude. Thus the presence of the poles associated with these cuts is dubious. The cuts which we discuss here are Mandelstam cuts[1] which are actually present in the amplitude and so the poles associated with them will be truly present also.

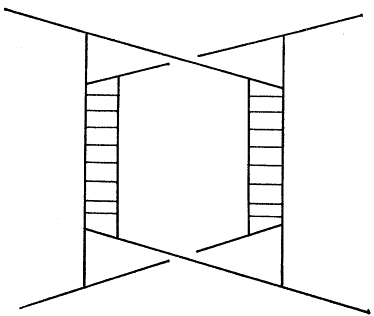

FIG. 4. Set of ladder insertions that would be made in Fig. 1.

(b) In the AFS cut case, the kernel of the integral equation can be expressed in terms of particle–two-Reggeon coupling functions which can themselves be redetermined from a similar calculation applied to the $2 \rightarrow 3$ production process amplitude.[6] Thus a relatively simple closed bootstrap is possible. In our case the kernel of the integral equation is expressed in terms of the four-Reggeon coupling functions. These depend upon third-spectral-function properties, and some of the subtleties in their calculation were exhibited in Sec. III, where it was shown that they receive additional terms from higher-order iterations. It is clearly not going to be easy to find a way of calculating these functions from simple constructs. They will have to appear as input information and the bootstrap then used to determine the trajectory functions self-consistently. This is a less impressive program but we believe it is the correct one because of the AFS cancellation phenomenon.

APPENDIX

In the text we have assumed that the dominant contribution to the integrals arise from finite values of the two-dimensional momenta κ. We here suggest that this may be regarded as a further point abstracted from the ladder-diagram model, similar to the assumption concerning the behavior of the coupling functions.

Consider, for example, the set of ladder insertions that would be made in Fig. 1. These are drawn in Fig. 4, where it is understood that sums are taken over the numbers of rungs in the ladders. It would be possible to parametrize all the internal momenta by expressions of the form of (2.2). In each propagator the two-dimensional momenta κ then appear separated from p and p' (though not from q). Suppose now that Feynman parameters are introduced by an expression like (4.2) for each propagator. The two-dimensional integration over the κ will then produce in the integral a factor that is an exponential involving q^2 but not s, the κ having effectively been replaced by finite values.[12] Thus in a sense only finite values of the κ are relevant to the evaluation of the asymptotic behavior in s.