Regge Poles and an S-Matrix Theory of the Lamb Shift

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The Lamb shift in hydrogen is calculated by means of dispersion theory. Relativistic invariance is maintained throughout the calculation, and electron spin is included, although the spin of the proton is neglected. Divergence problems which have plagued other calculations are eliminated by the exploitation of the Regge behavior of the amplitudes which appear in the dispersion integrals. The preliminary results are in excellent agreement with the lowest-order values given by Yennie and Erickson.

SINCE the experimental determination by Lamb and Retherford of a difference in energy between the $2S_{1/2}$ and $2P_{1/2}$ levels of hydrogen,[1] and the subsequent theoretical explanation by Bethe,[2] the calculation of the Lamb shift has become an important part of the theoretical physicist's repertoire. One of the major successes of quantum electrodynamics (QED) has been the extremely accurate evaluation of the Lamb shift in hydrogen and in other atoms.[3,4] It would be desirable if this important quantity could be calculated with comparable accuracy by means of dispersion theory, since this would provide both an independent check of the perturbation calculation and a convenient proving ground for dispersion techniques. While there have been several attempts recently to develop a dispersion theory of the Lamb shift which would present an acceptable alternative to QED,[5–7] for the most part, these efforts have met with indifferent success. At best, they have indicated the possibility of a dispersion treatment of the Lamb shift, but so far the quantitative results have not been particularly encouraging.

The Lamb shift is due principally to the effect of the inelastic electromagnetic contributions to the right-hand cut (RHC) of the electron-proton scattering amplitude. In evaluating them, one needs explicitly the photoelectric effect (PE) amplitudes. The simplest approximation to the PE amplitude, a single s-channel pole, yields a shift of the $2S$ state of nearly $+1600$ Mc/sec.[5] A nonrelativistic calculation of the Lamb shift employing a dipole approximation to the PE amplitude yields a value of $+1523$ Mc/sec, which represents a small improvement, although here a cutoff is apparently needed to give finite results. Both of these numbers, however, are rather far from the observed shift of $+1058$ Mc/sec.

A problem which has prevented any real comparison with QED or with experiment has been an infrared divergence associated with the left-hand cut (LHC). We shall show in this paper that the LHC divergence is eliminated and the accuracy of the calculation greatly improved if the Regge behavior of the amplitudes which appear in the dispersion integrals is properly exploited.

Relativistic covariance will be maintained throughout the calculation which follows. In addition, we will include electron spin, although the spin of the proton will be neglected. We will give here only the results of a first-order calculation. In the future we hope to have available the results of a more accurate evaluation.

In order to calculate the Lamb shift in hydrogen, we will use essentially the bound-state perturbation theory introduced by Dashen and Frautschi.[5–8] Thus, if the lth unperturbed partial-wave amplitude $f_0(s)=N_0(s)/D_0(s)$ has a pole at $s=s_n$ due to a zero of $D_0(s)$, then the change in the position of the pole δs_n due to inclusion of additional (electromagnetic) intermediate states in the unitarity, is given to first order in the energy shift by

$$\delta s_n = h(s_n)/R_0 D_0'^2(s_n)\,, \tag{1}$$

where R_0 is the residue of f_0 at the pole, and $D_0'=dD/ds$ is the derivative of the unperturbed D function. If we write the perturbed amplitude $f(s)=f_0(s)+\delta f(s)=N(s)/D(s)$, then

$$h(s)=e^{-2\pi\eta(s)}(ND_0-DN_0)\,, \tag{2}$$

where $\eta(s)$ is defined below.[9] To first order, $h(s)$ satisfies the following dispersion relation:

$$h_l(s)=\frac{1}{\pi}\int_{\text{cuts}}\frac{\operatorname{Im}h_l(s')}{s'-s}ds'\,, \tag{3}$$

where the cuts are essentially those of $\delta f_l(s)$.

The quantity $[R_0 D_0'^2(s_n)]^{-1}$ may be obtained from the solution of the Dirac equation with a Coulomb potential. If we substitute the correct two-particle

* This work was completed while the author was a NASA trainee at the University of Florida.

[1] W. E. Lamb and R. C. Retherford, Phys. Rev. **72**, 241 (1947).

[2] H. A. Bethe, Phys. Rev. **72**, 339 (1947).

[3] D. R. Yennie and G. W. Erickson, Ann. Phys. (N. Y.) **35**, 271 (1965).

[4] The discrepancy between the theoretical and experimental values for the Lamb shift in hydrogen is now, however, larger than three standard deviations. See, for example, R. C. Barrett, S. J. Brodsky, G. W. Erickson, and M. H. Goldhaber, Phys. Rev. **166**, 1589 (1968).

[5] J. McEnnan (unpublished).

[6] X. Artru, J. L. Basdevant, and R. Omnes, Phys. Rev. **150**, 1387 (1966).

[7] H. D. I. Abarbanel, Ann. Phys. (N .Y.) **39**, 177 (1966).

[8] R. F. Dashen and S. C. Frautschi, Phys. Rev. **135**, B1190 (1964).

[9] $h_l(s)$ is essentially the function $B(k^2)$ of Ref. 6, except for kinematical corrections.

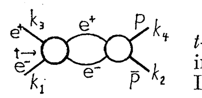

FIG. 1. This term should be added to the t-channel unitarity to give the perturbation in that channel. It will contribute only to the LHC of the partial-wave amplitude.

kinematical factors for electron-proton scattering, $\mu(s)=s-m_e^2-m_p^2/2W$, which is equal to the reduced mass at threshold and $q(s)$, the c.m. relative momentum then the parity-conserving partial-wave amplitudes may be written

$$f_{l,\pm}(s)=(2\pi W/2iq)(e^{2i\delta(l,\pm)}-1)\,, \tag{4}$$

where

$$\delta(l,\pm;s)-\chi(s)=\arg\Gamma(J-i\eta)+\tan^{-1}\varphi_\pm+\pi(j+\tfrac{1}{2}-J)\,, \tag{5}$$

$J=[(j+\frac{1}{2})^2-\alpha^2]^{1/2}$, and $\eta(s)=\alpha\mu(s)/q(s)$;

$$e^{2i\varphi_\pm}=(j+\tfrac{1}{2}\pm i\eta')/(J+i\eta)$$

and $\eta'=(\eta^2-\alpha^2)^{1/2}$. Note that with the phase shift (5), we have multiplied the usual Dirac amplitude by an angle-independent phase, $e^{2i\chi}=\Gamma(\frac{1}{2}+i\eta)/\Gamma(\frac{1}{2}-i\eta)$. This has been done in order to reduce the singularity of the Coulomb D function at threshold; it will have no observable effect in the physical region, and $D_0(s)$ will still have the proper zeros. With the phase shift given by (5), we can define

$$D_0(s)=e^{-i\delta(l,\pm)}\,|\Gamma^{-1}(J-i\eta)\Gamma^{-1}(\tfrac{1}{2}+i\eta)|\,, \tag{6}$$

so that

$$[R_0D_0'^2(s_n)]^{-1}=(-1)^{j-1/2}(w_0^2/2s_nk!\alpha)\times\Gamma(2J+k)\zeta^4(1+\zeta^2)^{-2}. \tag{7}$$

For $j=\frac{1}{2}$, (7) reduces to $(w_0^2/2s_n)\alpha^3/n^3$, neglecting higher powers of α. In (7), $n=k+j+\frac{1}{2}$ is the principal quantum number $w_0=2m_em_p$ and $\zeta=\alpha/k+J$. We see that (7) is already of about the correct order of magnitude, has the correct $1/n^3$ dependence on the principal quantum number and the correct sign. One difficulty of a previous calculation[7] was an irregular sign dependence which is removed by our particular definition of the unperturbed D function. For future reference, we also note that for s near threshold, s_0,

$$D_j(s)\simeq(q/\mu)^{J-1/2}\sin\pi(J-i\eta)\,. \tag{8}$$

It is believed that the amplitudes represented by Eqs. (4) and (5) are exact except for (i) the effects of the transverse components of the electromagnetic field (s channel), and (ii) the effects of massive particle-antiparticle pair exchange (t channel). We will take as perturbation the least massive two-body corrections to the unitarity. For the LHC, this means we will include an electron-positron pair in the intermediate state (Fig. 1). For the RHC, we will sum over intermediate states comprising one photon and one hydrogen atom in an excited state E_k (Fig. 2).

We consider first the LHC, which should be equivalent to the anomalous magnetic moment and vacuum-polarization contributions of perturbation theory. We write the unitarity equation corresponding to Fig. 1 for the invariant spinor amplitude in the t channel

$$\delta M-\delta M^\dagger=i\rho_2(t)\int d\Omega' M_{e^+e^-\to e^+e^-}D^{1/2}(B^{-1})\otimes D^{1/2}(\bar{B}^{-1})M_{e^+e^-\to p^+p^-}{}^\dagger. \tag{9}$$

$D^{1/2}(K)$ is an element of the two dimensional irreducible representation of the homogeneous Lorentz group, and B is the square of the "boost" function from the rest frame to the c.m. frame.[10] δM is the change in the electron-proton scattering amplitude due to the new t-channel intermediate state. It would be convenient if the amplitudes on the right-hand side of Eq. (9) could be approximated by their pole terms. However, the u pole in the elastic electron-positron scattering amplitude causes the integral to diverge unless the photon is given a small mass. This is due to the fact that the Born term for the Coulomb amplitude has no partial-wave expansion. In perturbation theory, this infrared divergence is canceled by a similar term in the bremsstrahlung contribution, but this avenue of escape is closed to dispersion theory. It is, however, possible to avoid this divergence in a manner which is considerably more satisfactory then introducing an arbitrary cutoff. If we consider the electron-positron amplitude, we realize that all of its Regge-pole trajectory functions are known explicitly (and, except for radiative corrections which are assumed to be small, exactly). If, in addition, we remember that the full nonrelativistic scattering amplitude can be obtained exactly in closed form by solving the Schrödinger equation with a Coulomb potential, and that it exhibits typical Regge asymptotic behavior with *only* the leading trajectory contributing, then we might surmise that the exact relativistic amplitude should be very closely approximated by its leading term. With this in mind, we replace the simple pole terms in the electron-positron amplitude by the leading Regge-pole terms; i.e.,

$$\begin{aligned}\frac{1}{t}&\to\frac{-1}{2iq_u^2}\frac{\Gamma(1-i\beta(u))}{\Gamma(1+i\beta(u))}\left(\frac{-t}{2q_u^2}\right)^{-1+i\beta(u)},\\ \frac{1}{u}&\to\frac{-1}{2iq_t^2}\frac{\Gamma(1-i\beta(t))}{\Gamma(1+i\beta(t))}\left(\frac{-u}{2q_t^2}\right)^{-1+i\beta(t)},\end{aligned} \tag{10}$$

where $-1+i\beta=-1+i\alpha\mu/q$ is the leading positronium trajectory. The amplitude obtained by the replacement (10) is *not* the exact relativistic amplitude, but it does exhibit the correct asymptotic behavior in the t and u channels, the proper positronium Regge poles, and it will be antisymmetric with respect to exchange of identical electrons. Moreover, the amplitude generated by (10) will have a partial-wave expansion so that the integral (9) will exist. If we write the left-hand side of

[10] We have found it convenient to use the two-component spinor formalism developed by A. O. Barut, *The Theory of the Scattering Matrix* (The Macmillan Co., New York, 1967).

(9) as

$$\delta M-\delta M^{\dagger}=2i\,\mathrm{Im}_t\delta A(s,t,u)Y_1(K)+2i\,\mathrm{Im}_t\delta B(s,t,u)Y_2(K)\,,\quad(11)$$

where the $Y_i(K)$ are the two-component forms of the spinor-basis functions appropriate to spin-$\frac{1}{2}$–spin-0 scattering used by Chew, Goldberger, Low, and Nambu (CGLN),[11] then we find from (9), using (10) and neglecting recoil

$$\mathrm{Im}_t\delta A(s,t,u)=\rho_2(t)[4\pi^3 m_e m_p\alpha^2/tq_t^2\beta(t)]\times e^{-\epsilon/q}[m_e^2(\cos\delta_1-\cos\delta_2)]\,,$$

$$\mathrm{Im}_t\delta B(s,t,u)=\rho_2(t)[\pi^3 m_e m_p\alpha^2/3tq_t^2\beta(t)]\times e^{-\epsilon/q}[2(t+2m_e^2)\cos\delta_0+4q_t^2(3\cos\delta_1+\cos\delta_2)]\,,\quad(12)$$

where

$$\rho_2(t)=q_t/(2\pi)^2\sqrt{t}\,,\quad 4q_t^2=t-4m_e^2\,,$$

$$\delta_l(t)=2\arg\Gamma(l+1-i\beta)+\beta\ln 2\,.$$

The convergence factor $e^{-\epsilon/q}$ is necessary to damp out the oscillation of the $\cos\delta$ terms at threshold. At the end of the calculation, the limit $\epsilon\to 0^+$ may be taken, and the result will be finite and independent of ϵ.

The discontinuity functions represented in (12) are rather complicated; it would simplify matters considerably if one could make an expansion of (12) in powers of α, the fine structure constant. However, when one attempts to project out of the expansion of (12) the LHC contribution to the partial-wave amplitudes [cf. Eq. (15)], one finds that, after the lowest-order term, all the integrals are infinite; even though the sum—represented by the integral of (12)—is finite. This is due to the essential singularity at $q_t=0$ which appears in the $\cos\delta$ terms. In (12), the cosine merely oscillates; but the expansion yields an infinite series in q_t^{-1}, each term of which diverges at threshold in (15). Actually, this should not be surprising, as it is known from perturbation theory that the Lamb shift cannot be developed in a simple power series in α, since the expansion contains terms of the form $\alpha^n\ln^k\alpha$. Thus, if one desires more than first-order accuracy, it is necessary to deal with (12) explicitly and do a rather difficult numerical integration. However, even though the higher-order terms in the expansion of (12) give rise to (spurious) divergences, the lowest-order term is finite, and can be given a rigorous mathematical definition. Moreover, the resulting expressions are considerably simpler. For the purposes of this paper,

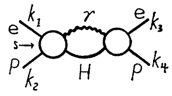

FIG. 2. The major part of the Lamb shift is due to the inelastic two-body contributions to the s-channel unitarity. These terms contribute only to the RHC of the partial-wave amplitudes.

[11] G. F. Chew, M. L. Goldberger, F. Low, and Y. Nambu, Phys. Rev. **106**, 1337 (1957).

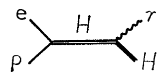

FIG. 3. A simple s-channel pole approximation to the PE amplitude has the effect of treating the hydrogen atom as if it were an elementary particle; whereas, it should appear as a Regge pole.

a lowest-order calculation will suffice. We find that $\mathrm{Im}_t\delta A(s,t,u)\simeq 0$,

$$\mathrm{Im}_t\delta B(s,t,u)\simeq\pi m_p\alpha t^{-3/2}(t-2m_e^2)\,.\quad(13)$$

Using (13), the LHC contribution to the Lamb shift can be written

$$\delta s(n,l\pm)_{\mathrm{LHC}}=\frac{1}{R_0D_0'^2}\left\{\frac{E_n+m_e}{m_e}\frac{W_n-m_e}{m_p}\times\frac{1}{\pi}\int_{-\infty}^{0}\frac{ds}{s-s_n}e^{-2\pi\eta}D_j^2\,\mathrm{Im}_L\delta B_{j\mp 1/2}+\frac{E_n-m_e}{m_e}\frac{W_n+m_e}{m_p}\times\frac{1}{\pi}\int_{-\infty}^{0}\frac{ds}{s-s_n}e^{-2\pi\eta}D_j^2\,\mathrm{Im}_L\delta B_{j\pm 1/2}\right\}\,,\quad(14)$$

where

$$\mathrm{Im}_L\delta B_{j\pm 1/2}(s)=(-1/4q_s^2)\int_{4m_e^2}^{-4q_s^2}dt\,P_{j\pm 1/2}(1+t/2q_s^2)\times\mathrm{Im}_t\delta B(s,t,u)\,.\quad(15)$$

From (13)–(15), we find that the change in the c.m. binding energy of the $2S_{1/2}$ state is -51 Mc/sec, and of the $2P_{1/2}$ state, -17 Mc/sec. We see that these values are of the correct sign and order of magnitude for the magnetic-moment and vacuum-polarization contributions to the Lamb shift.

Future work on the dispersion calculation of the Lamb shift should see a more accurate evaluation of the dispersion integrals as well as inclusion of the electron anomalous magnetic moment, which should yield the higher-order contributions to the LHC. It is possible that additional trajectories may have to be included in the amplitudes appearing in (9), but it is not anticipated that this will lead to any additional theoretical difficulties.

As we have said, it is the RHC which contributes the major part of the Lamb shift in hydrogen. Referring to Fig. 2, we see that it is necessary to know the amplitudes $M_k(s,z)$ for the process $e+p\to H_k^*+\gamma$, essentially the photoelectric effect. On the basis of pole-dominance assumptions common to dispersion theory, one might predict that a reasonable first-order calculation could be made using only a simple s-channel pole approximation to the PE amplitudes (Fig. 3). However, if this is done, the resulting Lamb shift is about 1600 Mc/sec, 60% larger than that which is observed.[5] In Ref. 6, Artru *et al.* have indicated that a nonrelativistic calculation of the Lamb shift may be made using the ordinary dipole approximation to the PE amplitude. Presumably, one could extend this to

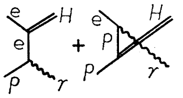

FIG. 4. The elementary-particle poles of the PE amplitude correspond to the proton and electron, and appear in the t and u channels, respectively.

the relativistic case. However, apparently the high-energy behavior of the dipole approximation is such that a cutoff is necessary to keep the dispersion integrals finite. Moreover, the result quoted, 1523 Mc/sec, is larger than the observed energy shift by about 50%. We are forced to conclude that neither of the approximations outlined above affords a satisfactory value for the Lamb shift in hydrogen. It is obvious that another approach must be explored.

Our s-channel pole approximation to the PE amplitude treated the hydrogen atom as if it were an elementary particle; however, we know this to be incorrect. Hydrogen is a bound state of the electron and proton; it should appear in the scattering amplitude as a Regge pole, rather than as a fixed pole. On the other hand, if we examine the PE amplitude we note that it *does* have elementary particle poles in the t and u channels at the masses of the proton and electron, respectively[12] (Fig. 4). As a matter of fact, we can go a step further. Since we know explicitly the hydrogen atom Regge-pole trajectory functions (except for radiative corrections), we thereby know precisely the asymptotic behavior in the t and u channels. (We note that the hydrogen poles are the only bound states of the PE amplitudes.) The t- and u-channel poles can be made to furnish a natural basis for the Reggeization of the PE amplitudes in exactly the same way that the photon poles of the electron-positron amplitude provided a point of departure for the Reggeization of that amplitude [cf. Eq. (10)]. Thus, if we assume that, as with the Coulomb amplitude, only the leading trajectory contributes appreciably, we can write

$$M_k(s,z)=A_k(s,z)U_k(K)+B_k(s,z)W_k(K)\,, \tag{16}$$

where $U_k(K)$ and $W_k(K)$ are the appropriate spinor-basis functions, and

$$\begin{aligned} A_k(s,z)&=\frac{g_p{}^k}{2qq_k}\frac{\Gamma(1-i\gamma(s))}{\Gamma(1+i\gamma(s))}\left(\frac{m_p{}^2-t}{2qq_k}\right)^{-1+i\gamma(s)},\\ B_k(s,z)&=\frac{g_e{}^k}{2qq_k}\frac{\Gamma(1-i\gamma(s))}{\Gamma(1+i\gamma(s))}\left(\frac{m_e{}^2-u}{2qq_k}\right)^{-1+i\gamma(s)}. \end{aligned} \tag{17}$$

$g_i{}^k$ is the residue at the pole, and is given by $|g_i{}^k/m_i|^2=4\pi m_e m_k \alpha f_k{}^2$, where $f_k{}^2=R_C/m_k{}^2$, and R_C is the residue of the Coulomb electron-proton scattering amplitude. $-1+i\gamma=-1+i\alpha\mu/q$ is the leading Regge trajectory, q is the relative c.m. momentum of the initial electron-proton pair, and q_k is the relative momentum of the final photon-hydrogen atom pair. We note that the amplitudes (17) have the correct bound-state poles, the appropriate asymptotic behavior in t and u; and, in lowest order in α, the correct fixed electron and proton poles. Moreover, the total cross section calculated from (17) is essentially the same as the total cross section obtained from the usual quantum-mechanical treatment.[13] We shall find that the amplitudes (16) and (17) provide a suitable basis for the calculation of the Lamb shift.

The unitarity equations corresponding to Fig. 2 may be written

$$\delta M-\delta M^\dagger=i\sum_k \rho_k(s)\int d\Omega' M_k(s,z')D^1 \otimes D^{S_k}M_k{}^\dagger(s,z'')\,, \tag{18}$$

where δM is the change in the electron-proton scattering amplitude due to the new electromagnetic states included in the unitarity, $\rho_k=q_k/(2\pi)^2W$, and the sum is over the lowest-lying members of each hydrogen-atom Regge trajectory. If we write the left-hand side of (18) in the form

$$\delta M-\delta M^\dagger=2i\,\mathrm{Im}_s\delta A(s,t,u)Y_1(K)+2i\,\mathrm{Im}_s\delta B(s,t,u)Y_2(K)\,, \tag{19}$$

where, as before, the notation is essentially that of Ref. 11, then we find, neglecting recoil,

$$\mathrm{Im}_s\delta A(s,z)=\mathrm{Im}_s\delta B(s,z)=\tfrac{1}{8}\sum_k\rho_k(s)\sum_l(2l+1)P_l(z)\times\left\{\frac{k_1\cdot k_3}{m_e{}^2}|a_l{}^k(s)|^2+\frac{k_2\cdot k_3}{m_em_p}[a_l{}^{k*}b_l{}^k+a_l{}^kb_l{}^{k*}]+\frac{k_2\cdot k_4}{m_p{}^2}|b_l{}^k(s)|^2\right\}, \tag{20}$$

where $a_l{}^k(s)$ and $b_l{}^k(s)$ are the partial-wave projections of (17), and are discussed in detail in the Appendix, and $P_l(z)$ is the Legendre function of the first kind. The change in the RHC of the parity-conserving partial-wave amplitudes may be found from the partial-wave projections of (20), with which the RHC contribution to the Lamb shift can be written

$$\delta s(n,l\pm)=\frac{1/\pi}{R_0D_0'^2}\int_{s_0}^\infty\frac{ds}{s-s_n}e^{-2\pi\eta}|D_j|^2\times\left\{\frac{E_n+m_e}{m_e}\left[\mathrm{Im}_R\delta A_{j\mp1/2}+\frac{W_n-m_e}{m_p}\mathrm{Im}_R\delta B_{j\mp1/2}\right]+\frac{E_n-m_e}{m_e}\left[-\mathrm{Im}_R\delta A_{j\pm1/2}+\frac{W_n+m_e}{m_p}\mathrm{Im}_R\delta B_{j\pm1/2}\right]\right\}. \tag{21}$$

[12] The electron and proton are certainly elementary particles if we remain within the framework of electrodynamics. Whether they may themselves be composite in some global bootstrap scheme does not seem to be particularly relevant here.

[13] See, for example, H. A. Bethe, *Intermediate Quantum Mechanics* (W. A. Benjamin ,Inc., New York 1964).

In (21) we have neglected the short integration from s_k to threshold s_0, since the integrand is strongly damped there. Because of the threshold behavior of $e^{-2\pi\eta}|D_j|^2$ [Eq. (8)], and of the partial-wave amplitudes (A4), there is no infrared problem. Moreover, the integrand behaves asymptotically as $s^{-2}\ln s$, so that there is no need for a cutoff. As with the LHC, however, we cannot develop the discontinuity functions in a power series in α, since all terms after the first contain additional powers of q^{-1} which give rise to (spurious) infrared divergences in (21). We hope in the future to be able to give the results of a numerical calculation involving the amplitudes (17), which should give higher-order corrections to the Lamb shift; but, because of the difficulty of that calculation we have only attempted, thus far, a lowest-order evaluation of (21). (As we noted before, the lowest-order term can be defined mathematically, and does give a finite, reasonable result.) We find, using (A5), that (21) yields a shift in the $2S_{1/2}$ state of $+1090$ Mc/sec, and of the $2P_{1/2}$ state, $+4$ Mc/sec. If we add to this the LHC contributions, we find that the difference in energy between the $2S_{1/2}$ and $2P_{1/2}$ states of hydrogen, to lowest order, is equal to 1052 Mc/sec. This is, within our limits of error, precisely the first-order result obtained by Yennie and Erickson.[3]

As we have seen, the divergence difficulties which have been associated with the dispersion theory of the Lamb shift can be removed by exploiting the known Regge behavior of the amplitudes which appear in the dispersion integrals. Moreover, we have found that the accuracy of the calculation is thereby improved to the point that a meaningful comparison with perturbation theory may be made. Work is now proceeding on the inclusion of the electron anomalous magnetic moment and reduced-mass and recoil effects, in addition to the more accurate numerical evaluation of the dispersion integrals. It is also possible that, in addition to the leading trajectory, other trajectories may have to be included to bring agreement with the experimental situation. On the basis of work done, we are confident that these refinements may be included without introducing any additional theoretical difficulties. We thus feel that this work will result in a convenient and accurate means of calculating the Lamb shift, and one which is entirely independent of the usual perturbation treatment.

The author would like to thank Mrs. J. Lipofsky for the numerical evaluation of $h(s)$, and Professor A. A. Broyles and M. McEnnan for their helpful comments and suggestions.

APPENDIX

The partial-wave projections of the amplitudes (18) may be found using the fundamental (integral) definition of the hypergeometric function.[14] We find

$$a_l{}^k(s)=\frac{2^l g_p{}^k}{2qq_k}\frac{\Gamma(l+1-i\gamma)\Gamma(l+1)}{\Gamma(1+i\gamma)\Gamma(2l+2)}\left(1+\frac{1}{v_p}\right)^{-l-1+i\gamma} \times F\left(l+1-i\gamma,\, l+1;\, 2l+2;\, \frac{2v_p}{1+v_p}\right). \quad \text{(A1)}$$

$b_l{}^k(s)$ is obtained from $a_l{}^k(s)$ by multiplying it by $(-1)^l$ and replacing p by e everywhere; we will thus omit $b_l{}^k(s)$ from the discussion which follows. In (A1), $v_p=q/E_p$ is the relativistic velocity of the incident proton in the c.m. frame. Eq. (A1) can be rewritten in a more convenient form using the relation between the hypergeometric function and the generalized Legendre function of the second kind[15]:

$$F\left(1+\nu-\mu,\, 1+\nu;\, 2+2\nu;\, \frac{2}{1+z}\right) =\frac{2\Gamma(2\nu+2)}{\Gamma(1+\nu)\Gamma(1+\nu+\mu)}\left(\frac{1+z}{2}\right)^{\nu+1}\left(\frac{z-1}{z+1}\right)^{\mu/2} \times e^{-i\pi\mu}Q_\nu{}^\mu(z). \quad \text{(A2)}$$

We find

$$a_l{}^k(s)=\frac{g_p{}^k}{2qq_k}\frac{\Gamma(l+1-i\gamma)e^{\pi\gamma}}{\Gamma(l+1+i\gamma)\Gamma(1+i\gamma)} \times\left(\frac{1}{v_p{}^2}-1\right)^{i\gamma/2} Q_l{}^{i\gamma}(1/v_p). \quad \text{(A3)}$$

The threshold behavior of $a_l{}^k(s)$ is given essentially by the asymptotic behavior of the Legendre function. We find that for s near threshold, s_0,

$$a_l{}^k(s)\to q^l e^{i\infty}, \quad \text{(A4)}$$

and similarly for $b_l{}^k(s)$. Thus, these amplitudes have the usual threshold behavior except for an infinite phase which is characteristic of electromagnetic processes. Finally, if we expand $a_l{}^k(s)$ in a power series in α, the fine-structure constant, we find that the lowest-order term is given by

$$a_l{}^k(s)=(g_p{}^k/2qq_k)Q_l(1/v_p), \quad \text{(A5)}$$

where $Q_l(z)$ is the ordinary Legendre function of the second kind.

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