for the $A_1\rho\pi$ system and K_{l3} decays come from $\langle VAP\rangle$ and $\langle VPP\rangle$, neither of which has anomalies in this model. Thus there is no ambiguity[5] in these calculations due to considerations of this kind. However, it is perhaps necessary to use a more realistic model in which the strangeness-changing vector currents are not conserved in order to finally answer questions about K_{l3} decays. Calculations of this are in progress.

Assuming that the general point of view expressed above about the mechanism of Ward identity violation applies to arbitrary n-point functions, it is easy to argue that no anomalies will be present for $n>3$. The reason for $n>4$ is simple. These n-point functions lead to Ward identities relating divergences of n-point functions to n- and $(n-1)$-point functions. Now when $n-1>3$, i.e., $n-1=4, 5, \cdots$, the integral erpresentation is at most logarithmically divergent ($n-1=4$). However, one may shift variables in logarithmically divergent integrals; thus, it should be possible to verify the reliability of all these Ward identities.

For $n=4$, one expects violation of the Ward identities only when the four- and three-point functions, occurring on the right-hand side, are vanishing due to a discrete symmetry. However, a vanishing four-point function will vanish in *any* integral representation, since it is at most logarithmically divergent. The only vanishing three-point functions are *VSP*, *ASS*, *APP*, *SSP*, and *PPP*. Now it may be verified that all these vanish in *any* integral representation, since they *are not* ambiguous, i.e., the shifting of variables is permitted.

PHYSICAL REVIEW VOLUME 181, NUMBER 5 25 MAY 1969

Form Factors, Kronecker-δ Terms, and the Absorptive Peripheral Model*

P. K. WILLIAMS
Florida State University, Tallahassee, Florida 32306
(Received 20 January 1969)

A prescription (OPE-δ) is introduced which enables one to obtain with little effort approximate results of absorption-model calculations for spin-space density matrix elements of unstable particles produced in high-energy quasi-two-body inelastic reactions. The OPE-δ method does not involve a partial-wave expansion. It consists of dropping those parts of the OPE amplitudes which would otherwise lead to Kronecker-δ contributions to the partial-wave amplitudes. The method is partially justified by studying results of one-pion-exchange-absorption calculations with and without form factors. Explicit calculations are presented for the reaction $\pi^-+p\rightarrow\rho^-+p$ at 3 BeV/c.

I. INTRODUCTION

THE success of the absorptive peripheral model[1] has rested on its ability to predict, with very few parameters, the size and shape of production differential cross sections for many inelastic two-body reactions over sizable ranges of incident momenta. In addition, at least as importantly, some detailed features of decays of unstable particles produced in these reactions can be correctly predicted. One finds, however, that calculations of density matrix elements for the decaying unstable particles are sensitive to certain input information, for example, the steepness of phenomenological vertex form factors (which affect mostly lower partial waves) and the amount of absorption in the lowest partial waves.[2–5] Apparently, spin properties of the various absorption models are most sensitive to details and modifications of the lower partial waves.

One of the most intriguing details concerns the occasional appearance in the low partial waves of large, sometimes unitarity-limit-violating Kronecker-δ (or "exceptional") terms of the general form $\delta_{JJ'}$.[6] We will examine in this paper the role of these Kronecker-δ terms in the one-pion-exchange (OPE) model. In particular, we will be concerned with understanding the effect of Kronecker terms in one-pion-exchange absorption-model (OPEA) calculations which have been done both with and without form factors,[2–5] in order to gain some insight into the mechanism of depolarization (i.e., deviation of density matrix elements from OPE predictions).

* Research supported in part by the U. S. Atomic Energy Commission.

[1] For a fairly complete survey of the absorption model, as applied to two-body reactions, see J. D. Jackson, J. T. Donohue, K. Gottfried, R. Keyser, and B. Svensson, Phys. Rev. **139**, B428 (1965), and references therein; also J. D. Jackson, in *Proceedings of the Thirteenth Annual International Conference on High-Energy Physics, Berkeley, 1966* (University of California Press, Berkeley, Calif., 1967).

[2] M. Bander and G. Shaw, Phys. Rev. **139**, B956 (1965).

[3] K. Gottfried and J. D. Jackson, Nuovo Cimento **34**, 735 (1964).

[4] P. Yock and D. Gordon, Phys. Rev. **157**, 1362 (1967).

[5] H. Hogaasen, J. Hogaasen, R. Keyser, and B. Svensson, Nuovo Cimento **42**, 323 (1966).

[6] Y. Chiu and L. Durand, Phys. Rev. **137**, B1530 (1965); S. Mandelstam, Nuovo Cimento **30**, 1148 (1963). The former authors have applied absorptive corrections to OPE-δ amplitudes with some resulting reduction in sensitivity to form factors, degree of absorption, etc.

Kronecker terms appear in the low partial waves because of extra polynomial t dependence of helicity amplitudes for production of particles with high spins in perturbation theory.[6] For future reference, we will indicate here how Kronecker terms arise. We will assume the following general property of two-body reaction amplitudes in the helicity representation: For the reaction $a+b\to c+d$, the single-particle-exchange amplitude may be written[6]

$$\langle\lambda_c\lambda_d|B(s,t)|\lambda_a\lambda_b\rangle = [\tfrac{1}{2}(1-x)]^{|\lambda-\nu|/2}[\tfrac{1}{2}(1+x)]^{|\lambda+\nu|/2}P(\lambda,\nu;x)/(z-x), \quad (1)$$

where we have displayed explicitly the dependence on x, the cosine of the c.m. scattering angle; $P(\lambda,\nu;x)$ is a polynomial in x; $\lambda=\lambda_c-\lambda_d$, $\nu=\lambda_a-\lambda_b$, $z=1+[(\mu^2-t)/2qq']_{x=1}$; and q, q' are initial and final c.m. momenta, respectively, and μ is the exchanged mass. The amplitude may also be multiplied by phenomenological form factors (normalized to unity at the exchanged particle pole: $x=z$) in order to take account of higher-order corrections to the interaction vertices. Form factors will be discussed later. For the moment let us consider the partial-wave decomposition of Eq. (1). The partial-wave amplitude corresponding to (1) is

$$\langle\lambda_c\lambda_d|B(s,J)|\lambda_a\lambda_b\rangle = \frac{1}{2}\int_{-1}^{1}\langle\lambda_c\lambda_d|B(s,x)|\lambda_a\lambda_b\rangle d_{\nu\lambda}{}^{J}(x)dx, \quad (2)$$

where the rotation functions $d_{\nu\lambda}{}^{J}(x)$ may be written

$$d_{\nu\lambda}{}^{J}(x)=[\tfrac{1}{2}(1-x)]^{-|\lambda-\nu|/2}[\tfrac{1}{2}(1+x)]^{-|\lambda+\nu|/2} \times L_{\lambda\nu}[P_l(x)], \quad (3)$$

where $L_{\lambda\nu}[P_l(x)]$ denotes a linear combination of Legendre polynomials. The partial-wave decomposition then always reduces to the form

$$\langle\lambda_c\lambda_d|B(s,J)|\lambda_a\lambda_b\rangle=\frac{1}{2}\int_{-1}^{1}\frac{P(\lambda,\nu;x)}{z-x}L_{\lambda\nu}[P_l(x)]dx. \quad (4)$$

Note that

$$P(\lambda,\nu;x)/(z-x)=P(\lambda,\nu;z)/(z-x)+R(\lambda,\nu;x), \quad (5)$$

where $R(\lambda,\nu;x)$ is a polynomial in x, and that

$$\frac{1}{2}\int_{-1}^{1}\frac{P_l(x)}{z-x}dx\equiv Q_l(z),$$

where $Q_l(z)$ denotes a Legendre function of the second kind, we obtain

$$\langle\lambda_c\lambda_d|B(s,J)|\lambda_a\lambda_b\rangle = P(\lambda,\nu;z)L_{\lambda\nu}[Q_l(z)]+\text{Kronecker terms}, \quad (6)$$

where the Kronecker terms arise from the integral $\int R(\lambda,\nu;x)L_{\lambda\nu}[P_l(x)]dx$. This then, is how Kronecker-δ terms arise in single-particle-exchange models. It is normally found that Kronecker terms appear, depending on the dynamics [i.e., degree of $P(\lambda,\nu;x)$], in different forms: $\delta_{JJ'}$, $\mu^2\delta_{JJ'}$, $\mu^4\delta_{JJ'}$, etc., where μ is the mass of the exchanged pion.

Simple phenomenological form factors of the type

$$F_1(t)=(M^2-\mu^2)/(M^2-t)$$

multiplying the single-particle-exchange amplitudes cancel, upon partial-wave decomposition, simple $\delta_{JJ'}$ terms in favor of terms continuable in J. This is the case for the part of $R(\lambda,\nu;x)$ which is independent of μ^2. Since

$$[1/(\mu^2-t)]F_1(t)=1/(\mu^2-t)-1/(M^2-t),$$

the μ-independent part of $R(\lambda,\nu;x)$ is canceled upon partial-wave decomposition and Eq. (6) becomes, instead,

$$\langle\lambda_c\lambda_d|B(s,J)|\lambda_a\lambda_b\rangle = P(\lambda,\nu;z)L_{\lambda\nu}[Q_l(z)]-P(\lambda,\nu;z')L_{\lambda\nu}[Q_l(z')] +[\mu^2\delta_{JJ'},\ M^2\delta_{JJ'},\ \mu^4\delta_{JJ'},\ M^4\delta_{JJ'},\ \text{etc., terms}], \quad (7)$$

where $z'=1+[(M^2-t)/2qq']_{x=1}$. One can easily devise reasonable form factors which cancel in an analogous way the $\mu^2\delta_{JJ'}$, $\mu^4\delta_{JJ'}$, etc., terms as well as $\delta_{JJ'}$ terms in favor of continuable terms (Sec. II). Consequently, suitable form factors can always be viewed as "spreading" or "smearing out" Kronecker-δ contributions to the partial-wave amplitudes so that one can test their influence in certain models. This has consequences in particular for the absorption model.

Specifically, we suggest that the spin properties of OPEA models imply that Kronecker terms should be very severely damped, while form factors interfere with the OPEA mechanism for doing this. In practice, it is not unreasonable to view absorption-model calculations as a way of (a) deleting Kronecker terms, and (b) introducing strong t-dependent (but helicity-independent) factors which serve only to collimate the production angular distributions. As an illustration, an explicit calculation of $\pi+N\to\rho+N$ will be presented (Sec. III) in which this approach is adopted. Without performing the usual absorption calculations, we simply drop all Kronecker-δ terms. This can be done in general without actually performing the partial-wave expansion of the OPE helicity amplitudes, and is a helicity-dependent modification. The resulting OPE-δ amplitudes then are found to give satisfactory agreement to experimental ρ-density matrix elements without further modification. We therefore believe that we are partially justified in suggesting the main helicity-dependent effect of absorption-model calculations to be the removal of these Kronecker-δ terms. The OPE-δ prescription might be used for more complicated reactions to gain some insight into spin properties without resorting to the more cumbersome machinery of the actual absorption calculations.

II. KRONECKER TERMS AND FORM FACTORS

It has been shown by Bander and Shaw[2] that the inclusion of a form factor in the OPEA model for the reaction $\pi+N\to\rho+N$ destroys the best fit to experimental ρ-density matrix elements. Similarly, Yock and Gordon[4] have shown that OPEA models for production of higher-spin particles (e.g., $\pi+N\to\rho+N^*$, $\pi+N\to f^0+N$) require fairly strong form factors in order to fit production angular distribution data, while predictions for density matrix elements differ significantly from the OPEA model without form factors. These results can be partially understood by considering what a form factor does to the single-particle-exchange amplitude, in particular to Kronecker-δ terms which appear in its partial-wave expansion.

In the OPE model for $\pi+N\to\rho+N$, the only Kronecker-δ terms are of the form $\delta_{J\frac{1}{2}}$. With the usual absorption-model prescription the $J=\frac{1}{2}$ terms are almost completely absorbed. However, as was noticed in Ref. 2, a form factor $F_1(t)$ multiplying the OPE amplitude replaces the Kronecker-δ term with something which is continuable in J and which reduces to the Kronecker term for large M^2, as in Eq. (7), thus spreading out the $\delta_{J\frac{1}{2}}$ term over several partial waves. Absorption then no longer completely cancels the effect of the replaced Kronecker terms, and the residual contributions tend to make the ρ-density matrix elements *revert* to their OPE values. Steeper form factors are found to produce more reversion,[2] so that any appreciably steep form factor is not desirable in the absorption model for this reaction.

For production of particles with higher spins, say, S and S', in the reaction $\pi+N\to S+S'$, typical OPE models produce Kronecker-δ terms in partial waves with $J\leq S+S'-1$. In the reaction $\pi+N\to\rho+N^*$, for example, terms appear of the form $\delta_{J\frac{1}{2}}$, $\delta_{J\frac{3}{2}}$, and $\mu^2\delta_{J\frac{1}{2}}$. The situation is essentially the same with regard to form factors and absorption. A form factor $F_1(t)$ spreads out the $\delta_{J\frac{1}{2}}$ and $\delta_{J\frac{3}{2}}$ terms as before, but we are left with a Kronecker term of the form $(\mu^2-M^2)\delta_{J\frac{1}{2}}$. However, absorption removes completely the $(\mu^2-M^2)\delta_{J\frac{1}{2}}$ term, but only a part of the spread out $\delta_{J\frac{1}{2}}$, $\delta_{J\frac{3}{2}}$ terms. The residual contributions tend to further collimate the production angular distributions and to make the density matrix elements revert to their OPE values.[4]

We notice that a "double-pole" form factor of the type

$$F_2(t)=\frac{M_1^2-\mu^2}{M_1^2-t}\,\frac{M_2^2-\mu^2}{M_2^2-t}$$

would spread out the $\mu^2\delta_{J\frac{1}{2}}$ term, in the previous example, as well as the $\delta_{JJ'}$ terms. In fact, a multiple-pole form factor

$$F_n(t)=\prod_{i=1}^{n}\frac{M_i^2-\mu^2}{M_i^2-t}$$

removes all Kronecker-δ terms:

$$(\mu^2)^i\delta_{JJ'},\quad i=0,1,\cdots(n-1).$$

The proof of this assertion is only an exercise in partial fractions. Thus we can always define a reasonable[7] form factor which spreads out all Kronecker-δ terms. If such form factors are employed in OPEA models, the reversion of density matrix elements to OPE values will be more nearly complete.

Form factors can be expected to have such effects in any absorption-model calculation. The maximum deviation of density matrix elements from OPE values should always be reduced by form factors. This dilution of absorptive effects will most likely lead to conflict with observation, as it has already with ρ-production experiments. The questionable use of both form factors and absorption is not restricted to OPE mechanisms alone. Maor and Yock[8] have found that certain two-pion-exchange (box-graph) mechanisms with absorption are very sensitive to the inclusion of vertex form factors, especially, for the purposes of this discussion, in the density matrix element predictions. On the other hand, Dürr and Pilkuhn[9] have argued that it is not unreasonable in a Born-term model to have form factors which decrease more rapidly than $F_1(t)$ at large $|-t|$ for vertices involving particles of higher spin, due to the angular momentum barriers. The point of view of these authors is similar to ours in that something should be done about large t-dependent effects in the Born term. However, as we have pointed out, steeper form factors do not improve density matrix element predictions when absorptive corrections have been included.

Intuitively, one might expect vertex form factors to be present and compatible with the unitary corrections contained in the absorption model. However, we have seen that OPEA models with form factors can lead to conflict with experiment. The best results of OPEA calculations to date indicate that Kronecker-δ contributions should be severely damped.

III. APPLICATION AND CONCLUSIONS

As an illustration of the point that Kronecker-δ terms should be severely damped, we adopt the assumption that the main helicity-dependent effect of absorption is to damp precisely these terms. For the process $\pi+N\to\rho+N$ at 3 BeV/c we take the OPE helicity amplitudes,[2,3] dropping those parts which would otherwise lead to Kronecker-δ terms. This is a helicity-dependent prescription which can be accomplished upon inspection without doing the partial-wave expansion by

---

[7] Multiple-pole form factors are reasonable in the sense that, for $n\to\infty$, if the product converges, the form factor decreases no faster than $e^{a\sqrt{-t}}$ for large $(-t)$, and therefore satisfies analytic bounds. See, e.g., J. D. Stack, Phys. Rev. **164**, 1904 (1967); A. Balachandran and J. Loeffel, *ibid.* **167**, 1322 (1968).

[8] U. Maor and P. Yock, Phys. Rev. **148**, 1542 (1966).

[9] H. P. Dürr and H. Pilkuhn, Nuovo Cimento **40A**, 899 (1965).

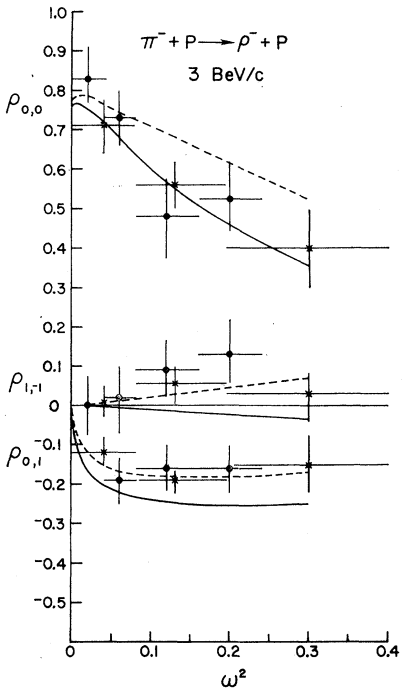

FIG. 1. Density matrix elements of the ρ for the reaction $\pi^-+p \rightarrow \rho^-+p$ at 3 BeV/c. Solid (dashed) curves are OPE-δ (OPEA) predictions. Data are from Ref. 10. (× Clear *et al.*; • Hagopian *et al.*) Horizontal scale: $\omega^2=2(1-x)$, where x is the cosine of the c.m. scattering angle.

dropping $R(\lambda,\nu;x)$ in Eq. (5), or, equivalently, evaluating $P(\lambda,\nu;x)$ at the pion pole: $x=z$, in Eq. (4). The resulting OPE-δ amplitudes are then used to calculate the ρ-density matrix elements. The results of this calculation are presented in Fig. 1, compared to experiment[10] and the usual absorption-model calculation. There is substantial agreement between the two methods of calculation. Note that the OPE-δ prescription gives results for the element $\rho_{0,0}$ which decrease with increasing $(-t)$ somewhat faster than the OPEA calculation. Also, the OPE-δ result for $\rho_{1,-1}$ is small and negative, whereas the OPEA result is small and positive. However, $\rho_{1,-1}$ is sensitive as to its sign (slide-rule calculations of $\rho_{1,-1}$ are inadequate). The main feature of $\rho_{1,-1}$ is that it should be small in magnitude. Finally, the OPE-δ result for $\rho_{1,0}$ is essentially the same as the OPEA result, with the OPE-δ values lying slightly farther from the OPE value (zero) than the OPEA values. It is clear from the gross comparison of OPE-δ and OPEA calculations that the main helicity-*independent* effect of absorption must be primarily to collimate the production angular distribution toward smaller angles.

We have given here a prescription (OPE-δ) for obtaining approximate results of lengthier absorption-model calculations for the density matrix elements of unstable particles in a particular reaction. However, because of the rather general nature of the remarks contained in this paper, we might also expect the OPE-δ prescription to give reasonable predictions for spin properties of other two-body reactions involving production of particles of higher spin.

ACKNOWLEDGMENTS

The author gratefully thanks Professor Marc Ross and Professor J. D. Kimel, Jr. for helpful conversations and critical reading of the manuscript.

[10] D. R. Clear *et al.*, Nuovo Cimento **49**, 399 (1967) (3 BeV/c); V. Hagopian *et al.*, Phys. Rev. **145**, 1128 (1966) (2.75 and 3.0 BeV/c combined).