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Divergent Vertices and Ward Identities*

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Schwinger and Adler have discovered that if the A_1-ω-ρ off-shell vertex function is linearly divergent, then the axial-vector Ward identity for the A_1-ω-ρ vertex contains an extra model-dependent term. It is shown here that no extra term occurs in the Ward identities for the A_1-ρ-π system when the A_1-A_1-ρ vertex is linearly divergent.

I. INTRODUCTION

ESSENTIAL to the hard-pion calculations of Schnitzer and Weinberg (SW)[1] are a set of Ward identities relating the A_1-A_1-ρ vertex to the π-A_1-ρ and π-π-ρ (off-shell) vertex functions. Investigations by Schwinger,[2] Adler,[3] and Bell and Jackiw[4] put these Ward identities into question. Specifically, one might expect the Ward identities to be modified if the vertices defined by Schnitzer and Weinberg are divergent. The A_1-A_1-ρ vertex is linearly divergent in quarklike models of current algebra (i.e., models in which the currents are bilinear products of free-fermion fields). In these cases, the A_1-A_1-ρ vertex function is given by a linearly divergent triangle graph similar to graphs discussed by Adler *et al.*[2–4]

The analysis of Adler *et al.* was concerned with the π-ω-ρ and A_1-ω-ρ vertex functions. They showed in effect that the Ward identities are irreversibly modified when the A_1-ω-ρ vertex is divergent. That is, there is no way one can regularize the A_1-ω-ρ vertex so that it satisfies the same Ward identities as the unregularized vertex. A contrary result will be obtained here for the A_1-A_1-ρ vertex. Namely, it *is* possible to regularize the A_1-A_1-ρ vertex so that it satisfies the original Ward identities of SW. That the A_1-A_1-ρ vertex behaves differently from the A_1-ω-ρ vertex is due to its different kinematic structure—at least on the surface. The author has not found any more fundamental reason for the difference. In this paper, the two vertices will be analyzed in parallel so that the contrast will be manifest.

It is assumed in this paper that vertices containing pions are convergent. This disagrees wiht the quarklike models for which the pion field is a bilinear product of fermion fields (in quarklike models the π-π-ρ vertex is also linearly divergent). However, this assumption simplifies the analysis; in addition, the assumption is true for the recently proposed non-Lagrangian models of current algebra[5] if the pion field has dimension less than three. The ρ and A_1 propagators are permitted to be quadratically divergent, as in quarklike models; propagators involving pions are assumed to converge. The analysis given in this paper does not assume specific triangle-graph models for the A_1-A_1-ρ or $A_1-\omega-\rho$ vertices; the only assumption is the qualitative one that both vertices are linearly divergent.

In Sec. II, regularized A_1-A_1-ρ and A_1-ω-ρ vertices will be defined and Ward identities obtained for these vertices. The regularization procedure is chosen for its simplicity only, and leads to modified forms of the Ward identities for both vertices. The most general forms are given for the extra terms in the Ward identities. In Sec. III, modifications of the regularization procedure are examined which might simplify the Ward identities. Modifications are permitted only if they do not change the physically relevant parts of the vertex functions. These modifications restore the SW form of the Ward identities for the A_1-A_1-ρ vertex, but leave an extra term in the axial-vector identity for the A_1-ω-ρ vertex. This extra term is the term found by Schwinger[2] and Adler.[3]

II. SUBTRACTED VERTEX FUNCTIONS

In this section, a set of subtracted vertex functions and propagators will be defined for the A_1-ρ-π and A_1-ρ-ω-π systems. Then Ward identities will be obtained for the subtracted functions. The method of subtraction is chosen for its simplicity, not to preserve the form of the Ward identities. The subtractions will later be modified (in Sec. III) to simplify the Ward identities.

The $SU(3)\times SU(3)$ currents will be denoted as follows: $V_a{}^\mu(x)$ is the ρ current, $A_a{}^\mu(x)$ is the A_1 current, $V^\mu(x)$ is the ω current; a is an isospin index. The pion field is $\phi_a(x)$ and the partial conservation of axial-vector current (PCAC) condition is[6]

$$\partial_\mu A_a{}^\mu(x)=F_\pi m_\pi{}^2\phi_a(x). \qquad (2.1)$$

Let

$$\epsilon_{abc}F^{\mu\nu\lambda}(x,y)=\langle TA_a{}^\mu(x)A_b{}^\nu(y)V_c{}^\lambda(0)\rangle \qquad (A_1\text{-}A_1\text{-}\rho\text{ vertex}), \quad (2.2)$$

$$\epsilon_{abc}F^{\nu\lambda}(x,y)=\langle T\phi_a(x)A_b{}^\nu(y)V_c{}^\lambda(0)\rangle \qquad (\pi\text{-}A_1\text{-}\rho\text{ vertex}), \quad (2.3)$$

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[1] H. J. Schnitzer and S. Weinberg, Phys. Rev. **164**, 1828 (1968); see also S. Brown and G. West, Phys. Rev. Letters **19**, 812 (1967); Phys. Rev. **168**, 1605 (1968); R. Arnowitt *et al.*, Phys. Rev. Letters **19**, 1085 (1967).

[2] J. Schwinger, Phys. Rev. **82**, 664 (1951), especially Eqs. (5.15)–(5.25). (I thank R. Jackiw for this reference.)

[3] S. L. Adler, Phys. Rev. **177**, 2426 (1969).

[4] J. S. Bell and R. Jackiw, Nuovo Cimento **60**, 47 (1969).

[5] K. G. Wilson, Phys. Rev. **179**, 1499 (1969).

[6] The metric used here is $(+\cdots)$ and the currents are normalized to satisfy the usual commutation relations.

$$\epsilon_{abc}F^{\lambda}(x,y)=\langle T\phi_a(x)\phi_b(y)V_c^{\lambda}(0)\rangle \quad (\pi\text{-}\pi\text{-}\rho \text{ vertex}), \quad (2.4)$$

$$\delta_{ab}F_A^{\mu\nu\lambda}(x,y)=\langle TA_a^{\mu}(x)V^{\nu}(y)V_b^{\lambda}(0)\rangle \quad (A_1\text{-}\omega\text{-}\rho \text{ vertex}), \quad (2.5)$$

$$\delta_{ab}F_A^{\nu\lambda}(x,y)=\langle T\phi_a(x)V^{\nu}(y)V_b^{\lambda}(0)\rangle \quad (\pi\text{-}\omega\text{-}\rho \text{ vertex}). \quad (2.6)$$

The vertex functions used by SW were defined as unsubtracted Fourier transforms of $F^{\mu\nu\lambda}(x,y)$, etc. When these Fourier transforms are divergent, one must introduce a regularization procedure. The triangle graphs of quarklike models can be regularized by subtracting their small-momentum behavior: To be precise, one subtracts through first order in a small-momentum expansion because the graphs are linearly divergent. If this subtraction method is used, then the subtracted Fourier transform of $F^{\mu\nu\lambda}(x,y)$ is

$$U^{\mu\nu\lambda}(q,p)=\int_x\int_y (e^{iq\cdot x}e^{-ip\cdot y}-1-iq\cdot x+ip\cdot y) \times F^{\mu\nu\lambda}(x,y), \quad (2.7)$$

where we use the notation $\int_x\equiv\int d^4x$. A function $U_A^{\mu\nu\lambda}(q,p)$ can be defined analogously by substituting $F_A^{\mu\nu\lambda}$ for $F^{\mu\nu\lambda}$. The assumption that these vertex functions are at most "linearly divergent" will be *defined* to mean that these subtracted integrals converge whether or not an unsubtracted Fourier transform exists.

The ρ and A_1 propagators will be assumed to be quadratically divergent (at most), as they are in quarklike models. This means one can define finite subtracted propagators. Let

$$\delta_{ab}G_\rho^{\nu\lambda}(x)=\langle TV_a^{\nu}(x)V_b^{\lambda}(0)\rangle, \quad (2.8)$$

$$\delta_{ab}G_A^{\nu\lambda}(x)=\langle TA_a^{\nu}(x)A_b^{\lambda}(0)\rangle, \quad (2.9)$$

and define

$$D_\rho^{\nu\lambda}(p)=\int_x [e^{ip\cdot x}-1-ip\cdot x+\tfrac{1}{2}(p\cdot x)^2]G_\rho^{\nu\lambda}(x), \quad (2.10)$$

and analogously for $D_A^{\nu\lambda}(p)$. The subtracted functions D_ρ and D_A are assumed to exist whether or not the unsubtracted propagators exist and furthermore they are to contain no noncovariant terms (noncovariant terms should be removed by the subtraction method).

Vertex functions and propagators containing pions are assumed in this paper to be convergent so they can be Fourier transformed without subtraction. Thus, one defines

$$U^{\nu\lambda}(q,p)=\int_x\int_y e^{iq\cdot x}e^{-ip\cdot y}F^{\nu\lambda}(x,y), \quad (2.11)$$

and likewise $U^{\lambda}(q,p)$ and $U_A^{\nu\lambda}(q,p)$ are unsubtracted. Also one defines

$$\delta_{ab}G_A^{\nu}(x)=\langle T\phi_a(x)A_b^{\nu}(0)\rangle, \quad (2.12)$$

$$D_A^{\nu}(p)=\int_x e^{ip\cdot x}G_A^{\nu}(x). \quad (2.13)$$

The pion propagator will not be needed in this paper.

The basic assumption being made when one assumes convergence of the subtracted integral in Eq. (2.7) is that the divergence of the unsubtracted integral occurs for x and y both going to zero. In the case of triangle graphs, it is easy to see that the divergence is of this form. For a triangle graph, $F^{\mu\nu\lambda}(x,y)$ is just the product of three fermion propagators, say $S^F(x)S^F(y)S^F(x\text{-}y)$, apart from γ matrices. If y is of order x and both are small, this product scales as x^{-9}. This singularity cannot be overcome by phase space (d^4xd^4y) and the unsubtracted Fourier transform diverges. However, with subtractions, the subtracted exponential behaves as x^2 for small x (with $y\sim x$) and this factor combined with phase space makes the subtracted integral converge. One might also look for divergences for $x\to 0$, for fixed y, or $y\to 0$ for fixed x, or $x\to y$. But in this case (at least for triangle graphs), the integrand behaves as x^{-3} or y^{-3} which is compensated by phase space without subtractions. It is assumed in this paper that these other limits do not cause trouble in non-free-field cases.

One can now obtain Ward identities for the subtracted vertex functions. Consider, for example, $q_\mu U^{\mu\nu\lambda}(q,p)$. One can write

$$q_\mu U^{\mu\nu\lambda}(q,p)=-i\int_x\int_y [e^{iq\cdot x}e^{-ip\cdot y}-1-iq\cdot x+ip\cdot y -\tfrac{1}{2}(iq\cdot x-ip\cdot y)^2]\nabla_\mu^x F^{\mu\nu\lambda}(x,y). \quad (2.14)$$

Note that one must subtract the exponential through order x^2 and xy before applying ∇_μ in order to have subtractions of order x and y after differentiating. Integrating by parts in the usual way, one gets[7]

$$q_\mu U^{\mu\nu\lambda}(q,p)=iF_\pi m_\pi^2\int_x\int_y [e^{iq\cdot x}e^{-ip\cdot y}-1-iq\cdot x+ip\cdot y -\tfrac{1}{2}(iq\cdot x-ip\cdot y)^2]F^{\nu\lambda}(x,y)-\int_y\{e^{i(q-p)\cdot y}-1 -i(q-p)\cdot y+\tfrac{1}{2}(q\cdot y-p\cdot y)^2\}G_\rho^{\nu\lambda}(y) +\int_y\{e^{-ip\cdot y}-1+ip\cdot y-\tfrac{1}{2}(ip\cdot y)^2\}G_A^{\nu\lambda}(y). \quad (2.15)$$

[7] One can ask whether it is mathematically legitimate to integrate by parts when there are singularities in the integrand. A careful analysis shows that the integration by parts is legitimate here. The analysis involves defining all integrals as the limit for $\epsilon\to 0$ of integrals with the regions $|x|\leq\epsilon$, $|x-y|\leq\epsilon$, or $|y|\leq\epsilon$, excluded (cf. Ref. 5).

The equal-time commutator terms exactly correspond to the definition of the subtracted propagators. However, the integral of $F^{\nu\lambda}(x,y)$ does not correspond to $U^{\nu\lambda}$ because of the subtractions. Let

$$P^{\nu\lambda}(q,p)=-iF_\pi m_\pi^2\int_x\int_y[1+iq\cdot x-ip\cdot y-\tfrac{1}{2}(q\cdot x-p\cdot y)^2]F^{\nu\lambda}(x,y);\quad(2.16)$$

then the Ward identity is

$$q_\mu U^{\mu\nu\lambda}(q,p)=P^{\nu\lambda}(q,p)+iF_\pi m_\pi^2U^{\nu\lambda}(q,p)-D_\rho^{\nu\lambda}(q-p)+D_A^{\nu\lambda}(-p).\quad(2.17)$$

The term $P^{\nu\lambda}(q,p)$ is caused by the presence of subtractions; it is a second-order polynomial in p and q. Such a term can occur only in connection with a partially conserved current; no such term occurs in Ward identities for the vector current. The analogous vector Ward identity is

$$(p-q)_\lambda U^{\mu\nu\lambda}(q,p)=-D_A^{\mu\nu}(p)+D_A^{\mu\nu}(q).\quad(2.18)$$

The analogous Ward identities for the $A_1-\omega-\rho$ vertex function are

$$q_\mu U_A^{\mu\nu\lambda}(q,p)=P_A^{\nu\lambda}(q,p)+iF_\pi m_\pi^2U_A^{\nu\lambda}(q,p),\quad(2.19)$$

$$p_\nu U_A^{\mu\nu\lambda}(q,p)=0,\quad(2.20)$$

$$(p-q)_\lambda U_A^{\mu\nu\lambda}(q,p)=0,\quad(2.21)$$

with

$$P_A^{\nu\lambda}(q,p)=-iF_\pi m_\pi^2\int_x\int_y[1+iq\cdot x-ip\cdot y-\tfrac{1}{2}(q\cdot x-p\cdot y)^2]F_A^{\nu\lambda}(x,y).\quad(2.22)$$

The Ward identities for the vector and axial-vector propagators in the presence of subtractions have the form

$$p_\nu D_\rho^{\nu\lambda}(p)=0,\quad(2.23)$$

$$p_\nu D_A^{\nu\lambda}(p)=Q^\lambda(p)+iF_\pi m_\pi^2D_A^\lambda(p),\quad(2.24)$$

where

$$Q^\lambda(p)=-iF_\pi m_\pi^2\int_x[1+ip\cdot x-\tfrac{1}{2}(p\cdot x)^2-\tfrac{1}{6}i(p\cdot x)^3]G_A^\lambda(x).\quad(2.25)$$

In addition, $D_\rho^{\nu\lambda}(p)$ and $D_A^{\nu\lambda}(p)$ are even in p and symmetric in ν and λ, due to Lorentz invariance.

The following Ward identities unchanged from SW are indeed:

$$p_\nu U^{\nu\lambda}(q,p)=-iF_\pi m_\pi^2U^\lambda(q,p)+D_A^\lambda(q),\quad(2.26)$$

$$(p-q)_\lambda U^{\nu\lambda}(q,p)=-D_A^\nu(p)+D_A^\nu(q).\quad(2.27)$$

The corresponding identities for the π-ω-ρ vertex are

$$p_\nu U_A^{\nu\lambda}(q,p)=0,\quad(2.28)$$

$$(p-q)_\lambda U_A^{\nu\lambda}(q,p)=0.\quad(2.29)$$

These identities involve only unsubtracted vertex functions and propagators.

The Ward identities just obtained differ from the SW Ward identities by polynomials in p and q such as $P^{\nu\lambda}(q,p)$. To complete this section the most general form for these polynomials consistent with Lorentz invariance will be given

$$P^{\nu\lambda}(q,p)=(A+Bp^2+Dq^2+Fp\cdot q)g^{\nu\lambda}+Cp^\nu p^\lambda+Eq^\nu q^\lambda+Gq^\nu p^\lambda+Hq^\lambda p^\nu,\quad(2.30)$$

$$P_A^{\nu\lambda}(q,p)=J\epsilon^{\nu\lambda\rho\sigma}q_\rho p_\sigma,\quad(2.31)$$

$$Q^\lambda(p)=Kp^\lambda+Lp^2p^\lambda,\quad(2.32)$$

where A–H, J, K, and L are constants. These forms incorporate the restriction that P and P_A be second order in p and q, while Q is third order in p.

III. SIMPLIFICATION OF THE WARD IDENTITIES

In Sec. II, regularized forms of the A_1-A_1-ρ and A_1-ω-ρ vertex functions were defined and Ward identities derived for these vertex functions. A regularized vertex function is usually an ambiguous vertex function, the ambiguity depending on the choice of regularization. The question now is whether one can use this ambiguity to simplify the Ward identities. To be specific, is it possible to redefine the regularized vertex functions and propagators so that the redefined functions satisfy the original SW Ward identities? The answer is that one can get back the original SW Ward identities in every case except one, the axial-vector identity for $U_A^{\mu\nu\lambda}(q,p)$.

The starting point of the proof is to assume the most general form for $P^{\nu\lambda}(q,p)$, etc., as given at the end of Sec. II. The constants A–H, etc., are to start with, completely arbitrary. But there are consistency conditions that the Ward identities must satisfy, and these consistency conditions reduce the number of arbitrary parameters. Then one can use the freedom of redefining the regularized vertex functions to eliminate further parameters.

The redefinitions will consist in adding linear combinations of p and q to $U^{\mu\nu\lambda}$ $U_A^{\mu\nu\lambda}$, and quadratic forms in p and q to the vector meson propagators. These redefinitions do not affect any physical quantity because the functions $U^{\mu\nu\lambda}(q,p)$ are complete vertex functions which have poles when any particle is on the mass shell, and all physical quantities obtainable from the vertex functions have at least one particle on the mass shell (the A_1 form factor, for example, has two particles on the mass shell). Changing the complete vertex function by a polynomial in p and q does not change the residue at any pole.

The consistency conditions are obtained by combining pairs of Ward identities. For example, from Eqs. (2.17), (2.27), and (2.23), one gets

$$(p-q)_\lambda q_\mu U^{\mu\nu\lambda}(q,p)=(p-q)_\lambda P^{\nu\lambda}(q,p)+iF_\pi m_\pi^2[-D_A{}^\nu(p)+D_A{}^\nu(q)]+(p-q)_\lambda D_A{}^{\nu\lambda}(-p)\,, \quad (3.1)$$

while from Eqs. (2.18) and (2.24), one has

$$q_\mu(p-q)_\lambda U^{\mu\nu\lambda}(q,p)=-q_\mu D_A{}^{\mu\nu}(p)+Q^\nu(q)+iF_\pi m_\pi^2 D_A{}^\nu(q)\,. \quad (3.2)$$

One can replace $D_A{}^{\nu\lambda}(-p)$ by $D_A{}^{\lambda\nu}(p)$ in the first equation and then subtract the two. Using Eq. (2.24) again, one has

$$(p-q)_\lambda P^{\nu\lambda}(q,p)=-Q^\nu(p)+Q^\nu(q)\,. \quad (3.3)$$

There are two other consistency conditions, involving $q_\mu p_\nu U^{\mu\nu\lambda}(q,p)$ and $p_\nu(p-q)_\lambda U^{\mu\nu\lambda}(q,p)$. These two conditions can be replaced by a single symmetry condition, which results from $\epsilon_{abc}F^{\mu\nu\lambda}(x,y)$ being symmetric to the exchange of the two axial-vector currents $A_a{}^\mu(x)$ and $A_b{}^\nu(y)$. This results in $U^{\mu\nu\lambda}(q,p)$ being antisymmetric:

$$U^{\mu\nu\lambda}(q,p)=-U^{\nu\mu\lambda}(-p\,,-q)\,. \quad (3.4)$$

This means that $p_\nu q_\mu U^{\mu\nu\lambda}(q,p)$ must be antisymmetric to the exchange $p\rightarrow -q$, $q\rightarrow -p$. If this condition is satisfied, as well as Eq. (3.3), then one can obtain the Ward identity for $p_\nu U^{\mu\nu\lambda}(q,p)$ by symmetry and all three consistency conditions will be satisfied. From Eqs. (2.17), (2.26), and (2.24), one has

$$p_\nu q_\mu U^{\mu\nu\lambda}(q,p)=p_\nu P^{\nu\lambda}(q,p)+F_\pi{}^2 m_\pi{}^4 U^\lambda(q,p)+iF_\pi m_\pi{}^2 D_A{}^\lambda(q)-p_\nu D_\rho{}^{\nu\lambda}(q-p)-Q^\lambda(-p)-iF_\pi m_\pi{}^2 D_A{}^\lambda(-p)\,. \quad (3.5)$$

$U^\lambda(q,p)$ must also be antisymmetric to the exchange $p\leftrightarrow -q$, so one must have

$$p_\nu P^{\nu\lambda}(q,p)-q_\nu P^{\nu\lambda}(-p,\,-q)=Q^\lambda(-p)+Q^\lambda(q)\,. \quad (3.6)$$

The corresponding consistency conditions for $P_A{}^{\nu\lambda}(q,p)$ are simply

$$p_\nu P_A{}^{\nu\lambda}(q,p)=(p-q)_\lambda P_A{}^{\nu\lambda}(q,p)=0\,. \quad (3.7)$$

The consistency conditions for $P_A{}^{\nu\lambda}(q,p)$ are all satisfied for the form (2.31). The first condition (3.3) on $P^{\nu\lambda}(q,p)$ limits the form (2.30) to be

$$P^{\nu\lambda}(q,p)=-Kg^{\nu\lambda}+B(p\cdot kg^{\nu\lambda}-k^\nu p^\lambda)-D(q\cdot kg^{\nu\lambda}-k^\nu q^\lambda)-L(q^2 g^{\nu\lambda}+p^\nu q^\lambda+p^\nu p^\lambda)\,, \quad (3.8)$$

where K and L are the constants of Eq. (2.32), and B and D are arbitrary; $k=p-q$. This restricted form satisfies the second consistency condition (3.6) identically.

Now one can try to simplify the Ward identities by redefining the regularized propagators and vertex functions. First, make the following redefinitions of the vector and axial-vector propagators:

$$D_\rho{}^{\nu\lambda}(p)\rightarrow D_\rho{}^{\nu\lambda}(p)+i(g_\rho{}^2/m_\rho{}^2)g^{\nu\lambda}\,, \quad (3.9)$$

$$D_A{}^{\nu\lambda}(p)\rightarrow D_A{}^{\nu\lambda}(p)-(K+Lp^2)g^{\nu\lambda}+i(g_\rho{}^2/m_\rho{}^2)g^{\nu\lambda}\,. \quad (3.10)$$

By adding these polynomials in p to the old propagators, one changes the Ward identities for the propagators; the new propagators satisfy

$$p_\nu D_A{}^{\nu\lambda}(p)=i(g_\rho{}^2/m_\rho{}^2)p^\lambda+iF_\pi m_\pi{}^2 D_A{}^\lambda(p)\,, \quad (3.11)$$

$$p_\nu D_\rho{}^{\nu\lambda}(p)=i(g_\rho{}^2/m_\rho{}^2)p^\lambda\,, \quad (3.12)$$

which are precisely the identities satisfied by the SW propagators.[8] The addition of a polynomial in p to a propagator does not change any physically measureable quantity; the physical observable is the absorptive part of the propagator which is unaffected by the change. Using the new propagators, the Ward identity (2.18) is changed to

$$(p-q)_\lambda U^{\mu\nu\lambda}(q,p)=L(q^2-p^2)g^{\mu\nu}-D_A{}^{\mu\nu}(p)+D_A{}^{\mu\nu}(q) \quad (3.13)$$

and the polynomial $P^{\nu\lambda}(q,p)$ in Eq. (2.17) is changed from Eq. (3.8) to

$$P^{\nu\lambda}(q,p)=B(p\cdot kg^{\nu\lambda}-k^\nu p^\lambda)-D(q\cdot kg^{\nu\lambda}-k^\nu q^\lambda)-L(q^2 g^{\nu\lambda}-p^2 g^{\nu\lambda}+p^\nu q^\lambda+p^\nu p^\lambda)\,. \quad (3.14)$$

The other Ward identities are unchanged.

The final step in simplifying the Ward identities is to redefine the vertex functions $U^{\mu\nu\lambda}(q,p)$ and $U_A{}^{\mu\nu\lambda}(q,p)$. In addition, a further modification of $D_\rho{}^{\mu\nu}(p)$ will be made. The modifications of $U^{\mu\nu\lambda}(q,p)$ and $U_A{}^{\mu\nu\lambda}(q,p)$ must be linear in q and p in order that the modifications to $P^{\nu\lambda}(q,p)$ and $P_A{}^{\nu\lambda}(q,p)$ be quadratic. The most general modification consistent with Lorentz invariance and the antisymmetry requirement on U is

$$U^{\mu\nu\lambda}(q,p)\rightarrow U^{\mu\nu\lambda}(q,p)+\alpha(q^\lambda+p^\lambda)g^{\mu\nu}+\beta(q^\nu g^{\mu\lambda}+p^\mu g^{\nu\lambda})+\gamma(q^\mu g^{\nu\lambda}+p^\nu g^{\mu\lambda})\,, \quad (3.15)$$

$$U_A{}^{\mu\nu\lambda}(q,p)\rightarrow U_A{}^{\mu\nu\lambda}(q,p)+\epsilon^{\mu\nu\lambda\sigma}(\delta p_\sigma+\eta q_\sigma)\,, \quad (3.16)$$

where $\alpha-\eta$ are arbitrary constants. The modification of $D_\rho{}^{\mu\nu}(q)$ will be of the form

$$D_\rho{}^{\mu\nu}(p)\rightarrow D_\rho{}^{\mu\nu}(p)+\theta(g^{\mu\nu}p^2-p^\mu p^\nu)\,, \quad (3.17)$$

which does not change Eq. (3.12).

The constants $\alpha-\theta$ are now to be chosen, if possible, to eliminate the L term in Eq. (3.13) and the polynomials $P^{\nu\lambda}(q,p)$ and $P_A{}^{\nu\lambda}(q,p)$ from Eqs. (2.17) and (2.19). To eliminate the L term from Eq. (3.13) one must take $\alpha=L$ and $\gamma=-\beta$. The function $P^{\nu\lambda}(q,p)$

[8] These identities are satisfied by the covariant propagators of SW in the pole approximation.

[given by Eq. (3.14)] is completely removed from Eq. (2.17) if one chooses $\beta=D-B-2L$ and $\theta=-(B+L)$. With these choices for α, β, γ, and θ, all the Ward identities in the $A_1-\rho-\pi$ system have their original SW form. In contrast, a modification of $U_A{}^{\mu\nu\lambda}$ according to Eq. (3.16) does not work. In order not to change Eq. (2.20), η must be zero, and in order not to change Eq. (2.21), δ must be zero also. So U_A cannot be modified, and the extra term $P_A{}^{\nu\lambda}(q,p)$ remains in the identity (2.19).

One could eliminate $P_A{}^{\nu\lambda}(q,p)$ from the Ward identity (2.19) by redefining the π-ω-ρ vertex $U_A{}^{\nu\lambda}(q,p)$, namely, by letting

$$U_A{}^{\nu\lambda}(q,p)\rightarrow U_A{}^{\nu\lambda}(q,p)-i(F_\pi m_\pi{}^2)^{-1}J\epsilon^{\nu\lambda\rho\sigma}q_\rho p_\sigma. \quad (3.18)$$

Unfortunately, this change is unacceptable. The reason is that the added term violates the smoothness condition that the π-ω-ρ vertex $U_A{}^{\nu\lambda}(q,p)$ must satisfy in order to calculate the $\pi_0\rightarrow 2\gamma$ decay rate by current algebra.[9] The current-algebra calculation requires that $U_A{}^{\nu\lambda}(q,p)$ have the following form, for small q and p

$$U_A{}^{\nu\lambda}(q,p)=(p^2-m_\pi{}^2)^{-1}\epsilon^{\nu\lambda\rho\sigma}q_\rho p_\sigma T(q,p), \quad (3.19)$$

where T changes only by a small percentage for changes of q and p of order m_π. But the redefinition of $U_A{}^{\nu\lambda}(q,p)$ is equivalent to changing T by

$$T(q,p)\rightarrow T(q,p)-iF_\pi[(p^2/m_\pi{}^2)-1]J. \quad (3.20)$$

Because of the factor $p^2/m_\pi{}^2$, the added term changes rapidly when p^2 changes from $m_\pi{}^2$ to 0. There is no reason to doubt that the original unsubtracted π-ω-ρ vertex satisfies the smoothness condition, so the redefined vertex violates the smoothness condition.[10]

The Ward identity for the A_1-ω-ρ vertex, with the extra term, is

$$q_\mu U_A{}^{\mu\nu\lambda}(q,p)=J\epsilon^{\nu\lambda\rho\sigma}q_\rho p_\sigma+iF_\pi m_\pi{}^2U_A{}^{\nu\lambda}(q,p). \quad (3.21)$$

The constant J can be related to the short-distance behavior of the product of three currents. The starting point is Eq. (2.22), which reduces by Lorentz invariance and parity requirement to

$$\epsilon^{\nu\lambda\rho\sigma}J=-iF_\pi m_\pi{}^2\int_x\int_y x^\rho y^\sigma F_A{}^{\nu\lambda}(x,y). \quad (3.22)$$

This integral has been analyzed elsewhere.[5] The J term in Eq. (3.21) can be regarded as a generalization of the equal-time commutator terms that appear in other Ward identities.

Because the subtracted vertices of the $A_1-\rho-\pi$ system can be defined to satisfy the Ward identities of SW, the SW hard-pion analysis applies to these subtracted vertices without change. The question whether the assumption of ρ and A_1 dominance holds for these vertices[11] is a low-energy question (energies of order the A_1 mass or less) and need not be affected by the problem of subtractions which is related to the high-energy behavior of the vertex functions.

It was assumed in this paper that all pion vertices are convergent. The analysis becomes considerably more complicated when the pion vertices diverge, because one must introduce subtractions in these vertices without violating the smoothness condition. The author has not carried through an analysis of this problem.

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[9] M. Veltman, Proc. Roy. Soc. (London) **A301**, 107 (1967); D. G. Sutherland, Nucl. Phys. **B2**, 433 (1967).

[10] The factor $(m_\pi{}^2)^{-1}$ would be cancelled if J were of order $m_\pi{}^2$ or smaller. But at least in the model discussed by Adler (Ref. 3) J is of order 1.

[11] See P. Horwitz and P. Roy, Phys. Rev. **180**, 1430 (1969).