$$A^0=\frac{ef}{2m}\left[2M\left(\frac{1}{M^2-s}+\frac{1}{M^2-u}\right)-\frac{1}{M}(\kappa_p+\kappa_n)\right],\quad B^0=\frac{ef}{2m}\left[\frac{-8M}{(M^2-s)(M^2-u)}\right],$$

$$C^0=\frac{ef}{2m}\left[(\kappa_p+\kappa_n)\left(\frac{1}{M^2-u}-\frac{1}{M^2-s}\right)\right],\quad D^0=\frac{ef}{2m}\left[(\kappa_p+\kappa_n)\left(\frac{1}{M^2-u}+\frac{1}{M^2-s}\right)\right].$$

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An $I=1$ $\pi\pi$ Sum Rule and the Derivatives of the S-Wave $\pi\pi$ Amplitudes*

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A sum rule for the $I=1$ $\pi\pi$ amplitude is written at the symmetry point, $s=t=u=\frac{4}{3}m_\pi^2$, under the assumption that $\partial A^1(t,V)/\partial V$ satisfies an unsubtracted fixed-t dispersion relation, where t and V are the energy and momentum-transfer variables. From the symmetry-point conditions, $\partial A^0/\partial t$ and $\partial A^2/\partial t$ are obtained and compared with the current-algebra estimates.

HERE we present a sum rule for the $I=1$ $\pi\pi$ amplitude from which one can obtain sum rules for the derivative of the $I=0$ and $I=2$ $\pi\pi$ amplitudes at the symmetry point, $s=t=u=\frac{4}{3}m_\pi^2$. The sum rule has the property that the contributions of high-mass resonances drop off very rapidly.

The derivative relations at the symmetry point for the amplitudes $A^I(t,V)$, where t is the energy variable and $V=\frac{1}{2}(s-u)$ are (we take $\hbar=c=m_\pi=1$)[1]

$$\partial A^0/\partial t(\tfrac{4}{3},0)=-2(\partial A^2/\partial t)(\tfrac{4}{3},0) \tag{1}$$

$$=\partial A^1/\partial V(\tfrac{4}{3},0). \tag{2}$$

Now if $A^I(t,V)\to\beta(t)V^{\alpha_I(t)}$ as $V\to\infty$, then since $\alpha_I(t)<1$ for $I=1$ and $t<m_\rho^2$ we can write an unsubtracted, fixed-t, dispersion relation in the neighborhood of $t=\frac{4}{3}$,

$$\frac{\partial A^1}{\partial V}(t,V)=\frac{1}{\pi}\int_{V_0}^{\infty}dV'A_V{}^1(t,V')\times\left(\frac{1}{(V'-V)^2}+\frac{1}{(V'+V)^2}\right). \tag{3}$$

Using crossing symmetry for the absorptive part $A_V{}^1$ we get at $t=\frac{4}{3}$ and $V=0$,

$$\frac{\partial A^1}{\partial V}(\tfrac{4}{3},0)=\frac{2}{\pi}\left\{\int_{V_0}^{N}\frac{dV'}{V'^2}[\tfrac{1}{3}\,\mathrm{Im}A^0(V',t)+\tfrac{1}{2}\,\mathrm{Im}A^1(V',t)-\tfrac{5}{6}\,\mathrm{Im}A^2(V',t)]+c(t)\frac{N^{\alpha_1(t)-1}}{1-\alpha_1(t)}\right\}, \tag{4}$$

where $\mathrm{Im}A^I(V,t)$ are the imagninary parts in the s channel, and the second term on the right-hand side of (4) is the Regge term.

An important point to notice in relation (4) above is that the $I=0$ and $I=1$ contributions, as well as the Regge contribution, are all *positive*. Since no resonance in $I=2$ has been observed, the derivative $\partial A^1/\partial V$ is positive and, therefore, from (2) $\partial A^0/\partial t$ is positive at the symmetry point.

In Table I we have given the contributions of different $\pi\pi$ resonances to the right-hand side of the sum rule (4). In addition to ρ and f_0, we have also included higher resonances such as g (the recurrence of ρ) and h (a possible recurrence of f^0). The experimental value of the $\pi\pi$-partial width of the g meson is not known precisely. It varies between 30 and 75 MeV.[2] The values in Table I correspond to $\Gamma_g=100$ MeV, $\Gamma_g=30$ MeV, and $m_g^2=2.6$ BeV^2. Experimentally very little is known about h, but we expect the f^0 trajectory to be roughly parallel to the ρ trajectory, so that a recurrence of f^0 should exist around $m_h^2=3.6$ BeV^2. The values given in Table I correspond to $\Gamma_h=100$ MeV and $\Gamma_h=30$ MeV. The main point we want to emphasize about the sum rule (4) is that the contributions of higher resonances fall off very rapidly and, therefore, the value of $\partial A^0/\partial t$ obtained through (2), is *not* sensitive to the contributions of g, h, and other higher resonances.

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[1] S. Y. Chu and B. R. Desai, Phys. Rev. Letters **21**, 54 (1968).

[2] For the upper limit of the g partial width we have used 50% of the value of g total width [A. H. Rosenfeld *et al.*, University of California Radiation Laboratory Report No. UCRL-8030 (rev.) 1968 (unpublished)] where it is indicated that the 2π decay mode is dominant. For the lower limit we have taken the partial widths given by T. F. Johnston *et al.*, Phys. Rev. Letters **20**, 1414 (1968); D. J. Crennell *et al.*, *ibid.* **18**, 323 (1967); and N. N. Biswas *et al.*, *ibid.*. **21**, 50 (1968).

TABLE I. Resonance contributions to $\partial A^0(\frac{4}{3},0)/\partial t$ in BeV^{-2} obtained from the $I=1$ sum rule (4) and the $I=2$ derivative sum rule (5).

$\pi\pi$ resonances	$\partial A^0(\frac{4}{3},0)/\partial t$ (BeV^{-2}) $I=1$ Sum rule (4)		$I=2$ Sum rule (5)	
ρ	0.90		2.60	
f^0	0.20		−2.40	
	$\Gamma=100$ MeV	$\Gamma=30$ MeV	$\Gamma=100$ MeV	$\Gamma=30$ MeV
g	0.17	0.05	3.80	1.14
h	0.10	0.03	−4.10	−1.23

The above sum rule for the $I=1$ amplitude should be compared with the sum rule for $I=2$ given by Chu and Desai in an earlier paper (CD).[1] The derivative sum rule is given by

$$\frac{\partial A^2}{\partial t}(\tfrac{4}{3},0)=-\frac{2}{\pi}\frac{\partial}{\partial t}\left\{\int_{V_0}^{N}\frac{dV'}{V'}\left[\tfrac{1}{3}\operatorname{Im}A^0(V',t)-\tfrac{1}{2}\operatorname{Im}A^1(V',t)+\tfrac{1}{6}\operatorname{Im}A^2(V',t)\right]\right\}. \quad (5)$$

This sum rule together with relation (1) would also give the value of $\partial A^0/\partial t$ at the symmetry point. However, in (5) the $I=0$ and $I=1$ contributions appear with *opposite* signs, and higher resonances are quite important. In Table I we have also given the contributions to the sum rule (5) with the same assumption for the g and h mesons as before. We find that the contributions of individual resonances remain quite large as we go up in energy. This is so because in the right-hand side of (5) there are terms of the type $\partial A^I(V',t)/\partial t$, which for each resonance have a factor

$$(d/dz)P_l(z)\approx\tfrac{1}{2}l(l+1) \quad (6)$$

for $z\approx 1$ or for $t=\frac{4}{3}\ll m_R^2$ (m_R being the corresponding resonance energy). Thus, the higher l-resonance contributions are enhanced. In fact for $\Gamma_g=100$ MeV and $\Gamma_h=100$ MeV, the contributions are larger than those of ρ and f^0.[3]

Thus, the sum rule (5) is not as useful as sum rule (4) because it is sensitive to high-mass resonances and it is not as strongly convergent.[4]

If we use the S-wave phase shifts of Tryon[5,6] who used current algebra estimates of Weinberg,[7] then the integral on the right-hand side of (4) gives[8] $\partial A^0/\partial t(\frac{4}{3},0)\approx 2.2\ \mathrm{BeV}^{-2}$, which is close to Weinberg's estimate of $2.5\ \mathrm{BeV}^{-2}$. It should be noted that the $I=0$ S-wave phase shifts used above correspond to a wide resonance or a very large phase shift in the region between 700 and 1000 MeV.[5,6]

The negative scattering lengths for $I=0$ discussed by CD are also consistent with (4) and will be the subject of a later publication.

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[3] A typical contribution to the right-hand side of (5) from a resonance (assuming linear Regge trajectories) is

$$\sim\frac{l}{m_R^2}(m_R\Gamma_R)\frac{l^2}{m_R^2}\sim l(m_R\Gamma_R).$$

[4] Recently Tryon (see Ref. 6) has claimed that by putting the g meson in (5) with $\Gamma_g=30$ MeV but neglecting the h meson, the correct current algebra estimate of $\partial A^0/\partial t$ is obtained. However, to use g without h in sum rule (5) could be quite dangerous in view of the importance of high-mass resonances as given by (6) (also see Ref. 3) and in view of the fact that g and h appear with opposite signs. Thus, his estimate of Γ_g cannot be reliable.

[5] E. P. Tryon, Phys. Rev. Letters **20**, 769 (1968).

[6] E. P. Tryon, Phys. Rev. Letters **22**, 110 (1969).

[7] S. Weinberg, Phys. Rev. Letters **17**, 616 (1966).

[8] We ignore the (positive) Regge contribution in (4) since N is quite large. One can obtain a crude estimate of $c(t)$ if one assumes, in the sense of finite-energy sum rules [see R. Dolen, D. Horn, and C. Schmid, Phys. Rev. **166**, 1768 (1968)], that the Regge term describes the average of the resonances. If this criterion is applied in the neighborhood of the g resonance, then the Regge contribution is $\approx 0.4\ N^{-1/2}$ for $\alpha_1\approx 0.5$ and $\Gamma_g=30$ MeV. For a value of N just above the h meson (i.e., $N\simeq 4\ \mathrm{BeV}^2$) the Regge contribution is $\simeq 0.2\ \mathrm{BeV}^{-2}$, about 10% of the total value.