Reggeized Resonance Model for Arbitrary Production Processes*

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(Received 13 January 1969)

A Reggeized resonance model for a scalar n-point function is given in the form of an integral. Reggeization and a multiple-factorization theorem is proved. Feynman rules are developed for the parent resonance couplings and compared with the Regge results. An approximation scheme for simplifying the n-point function is suggested, and based on this approximation scheme, a Bethe–Salpeter-type equation is derived for the multiperipheral processes.

I. INTRODUCTION

RECENTLY, there has been a good deal of interest in the Reggeized resonance model. After the original four-point function by Veneziano,[1] five-,[2] six-, and seven[3]-point functions have been constructed. In the Sec. II we present a generalization of these results to the n-point amplitude, and derive some of the consequences of this model. One important result is the factorization property of the parent trajectory, which we derive in both the Regge and Feynman languages. In the last section, we propose a truncation scheme which replaces the complicated n-point function by simpler expressions. It is based on the idea of correlation of momenta in a multipheripheral graph, and gives rise to 2-, 3-, etc., particle correlation approximations. In the three-particle correlation approximation, we write down a Bethe–Salpeter-type equation in the manner of Amati, Fubini, Stanghellini, and Tonin.[4] It is hoped that this equation is an improvement over similar equations[5–7] used in recent literature, especially for the high-multiplicity and low-energy part of the multipheripheral model.

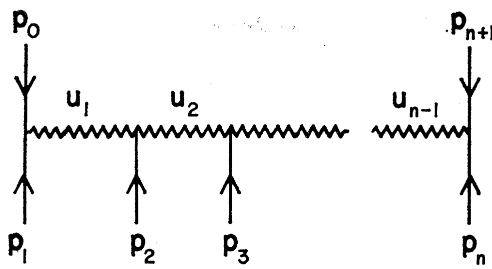

FIG. 1. The general production amplitude.

II. GENERAL PRODUCTION AMPLITUDE

From the explicit forms for 5-, 6-, and 7-point functions given in Refs. 2 and 3, we posit the following form for the $(n+2)$-point function;

$$
\begin{aligned}
B_n(p_0,p_1,\cdots,p_n,p_{n+1}) = \int_0^1 & du_1\cdots du_{n-1}u_1^{-\alpha_{0,1}-1}\cdots \\
& \times u_{n-1}^{-\alpha_{0,n-1}-1}(1-u_1)^{-\alpha_{12}-1}\cdots(1-u_{n-1})^{-\alpha_{n-1,n}-1} \\
& \times(1-u_1u_2)^{-2b(p_1\cdot p_3)+a+b}\cdots \\
& \times(1-u_{n-2}u_{n-1})^{-2b(p_{n-2}\cdot p_n)+a+b}(1-u_1u_2u_3)^{-2b(p_1\cdot p_4)}\cdots \\
& \times(1-u_1u_2\cdots u_{n-1})^{-2b(p_1\cdot p_n)}, \quad (1)
\end{aligned}
$$

where $\alpha_{i,j}=b(p_i+\cdots+p_j)^2+a$ is the Regge trajectory in all channels, and all masses are taken to be $1=p_0^2=\cdots=p_n^2=p_{n+1}^2$. The choice of $(n+2)$- rather than n-point function is for notational simplicity. The notation is further explained by Fig. 1.

Each momentum is considered to be incoming, and the particles labeled 0 and p_{n+1} are singled out. With each internal line (Reggeon) we associate a suitable variable u. In formula (1), the factors involving variables u_i, $(1-u_i)$ and $(1-u_iu_{i+1})$ have special exponents and they have explicitly written out in (1). The factors become uniform when the number of u's exceeds two, and they can be expressed as

$$
\text{Typical factor} = (1-u_iu_{i+1}\cdots u_{i+k})^{-2bp_i\cdot p_{i+k+1}},
$$

where $k \geqq 2$. The symmetry of formula (1) under a cyclic permutation of indices 0, 1, $\cdots$, $n+1$ and the existence of multiple resonances allowed by the Feynman rules are shown in Refs. 8 and 9, where slightly different but essentially equivalent forms of Eq. (1) are independently derived. Instead, we now prove that (1) Reggeizes in the multi-Regge limit.

* Research supported in part by the Air Force Office of Scientific Research, Office of Aerospace Research, United States Air Force, under Grant No. AF-AFOSR-68-1471.

† Alfred P. Sloan Foundation Fellow.

‡ Under contract to CICP.

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Consider the limit

$$s_{12}\to\infty,\quad s_{23}\to\infty,\quad \cdots,\quad s_{n-1,n}\to\infty,$$

with

$$s_{01},\quad s_{02},\quad \cdots,\quad s_{0,n-1}=\text{const} \tag{2a}$$

and

$$(s_{i,i+1}s_{i+1,i+2})/s_{i,i+2}=-\kappa_i=\text{const},$$

with

$$i=1,\,2,\,\cdots,\,n-2,\quad s_{i-j}=(p_i+p_{i+1}+\cdots+p_j)^2.$$

In this limit, the κ's factorize,[10] so that

$$(s_{i,i+1}\cdots s_{i+k,i+k+1})/s_{i,i+k+1}=(-\kappa_i)\cdots(-\kappa_{i+k-1}). \tag{2b}$$

The minus sign introduced in the definition of the κ's is for notational convenience. Now, in Eq. (1), we make the following change of variables:

$$1-u_i=\exp(x_i/\alpha_{i,i+1}),\quad i=1,\,2,\,\cdots,\,n-1. \tag{3}$$

In the terms in (1) of the form $(1-u_i\cdots u_{i+k})^{-2bp_i\cdot p_{i+k+1}}$, we expand the exponential in Eq. (3) and keep only the first two terms, which are the leading terms for large values of $\alpha_{12},\cdots$, etc. For a justification of this heuristic procedure for $n=3$, we refer the reader to Ref. 2. The final result is

$$\begin{aligned}B_n\sim&(-\alpha_{12})^{\alpha_{01}}(-\alpha_{23})^{\alpha_{02}}\cdots(-\alpha_{n-1,n})^{\alpha_{0,n-1}}\\&\times\int_0^\infty\cdots\int_0^\infty dx_1\cdots dx_{n-1}(x_1)^{-\alpha_{01}-1}\cdots(x_{n-1})^{-\alpha_{0,n-1}-1}\\&\times\exp(-x_1-x_2-\cdots-x_{n-1})\left(1+\frac{x_1x_2}{\alpha_{12}\alpha_{23}}\right)^{-\alpha_{13}}\cdots\\&\times\left(1+\frac{x_{n-2}x_{n-1}}{\alpha_{n-2,n-1}\alpha_{n-1,n}}\right)^{-\alpha_{n-2,n}}\cdots\\&\times\left(1+(-1)^{n-1}\frac{x_1\cdots x_{n-1}}{\alpha_{12}\cdots\alpha_{n-1,n}}\right)^{-\alpha_{1,n}},\end{aligned} \tag{4}$$

where the high-energy limits

$$\exp\left(\frac{x_i}{\alpha_{i,i+1}}\right)\approx1+\frac{x_i}{\alpha_{i,i+1}}$$

and $2b(p_i\cdot p_j)=\alpha_{i,j}-\alpha_{i+1,j}-\alpha_{i,j-1}+\alpha_{i+1,j-1}\approx\alpha_{i,j}$ have been used. We take the limit under the intergal sign, using

$$\lim_{\alpha\to\infty}[1+(\gamma/\alpha)]^\alpha=e^\gamma,$$

and obtain

$$B_n\to(-\alpha_{12})^{\alpha_{0,1}}(-\alpha_{23})^{\alpha_{0,2}}\cdots(-\alpha_{n-1,n})^{\alpha_{0,n-1}}G_n, \tag{5}$$

where

$$\begin{aligned}G_n=&\int_0^\infty\cdots\int_0^\infty dx_1\cdots dx_{n-1}(x_1)^{-\alpha_{01}-1}\cdots(x_{n-1})^{-\alpha_{0,n-1}-1}\\&\times\exp\left[-\left(\sum_{i=1}^{n-1}x_i+\sum_{i=1}^{n-2}\frac{x_ix_{i+1}}{\kappa_i}+\sum_{i=1}^{n-3}\frac{x_ix_{i+1}x_{i+2}}{\kappa_i\kappa_{i+1}}+\cdots\right.\right.\\&\left.\left.+\frac{x_1\cdots x_{n-1}}{\kappa_1\cdots\kappa_{n-2}}\right)\right].\end{aligned}$$

[10] N. F. Bali, G. F. Chew, and A. Pignotti, Phys. Rev. Letters **19**, 614 (1967).

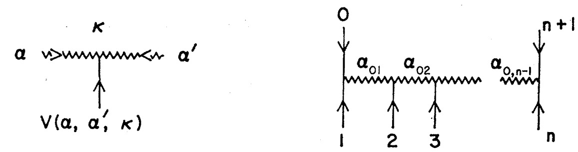

FIG. 2. Reggeon–Reggeon-scalar coupling.

Now define

$$\begin{aligned}V(\alpha,\alpha',\kappa)=&\frac{1}{\Gamma(-\alpha)}\frac{1}{\Gamma(-\alpha')}G_3(\alpha,\alpha',\kappa)\\=&\frac{1}{\Gamma(-\alpha)}\frac{1}{\Gamma(-\alpha')}\int_0^\infty\int_0^\infty dx_1dx_2x_1^{-\alpha-1}x_2^{-\alpha'-1}\\&\times\exp\left[-\left(x_1+x_2+\frac{x_1x_2}{\kappa}\right)\right],\end{aligned} \tag{6}$$

where G_3 of Ref. 2 gives the coupling of two Reggeons to a scalar, as indicated in Fig. 2.

In Eq. (6), the Γ functions appearing in the definition of V are to be thought of as propagators for Reggeons, so that V is free of external line poles. Factorization for a multi-Regge exchange graph illustrated in Fig. 2 is expressed by the equation,

$$\begin{aligned}G_n(\alpha_{01},\cdots,\alpha_{0,n-1};\kappa_1,\cdots,\kappa_{n-2})=&\Gamma(-\alpha_{0,1})V(\alpha_{0,1},\alpha_{0,2},\kappa_1)\\&\times\Gamma(-\alpha_{0,2})V(\alpha_{0,2},\alpha_{0,3},\kappa_2)\cdots\Gamma(-\alpha_{0,n-2})\\&\times V(\alpha_{0,n-2},\alpha_{0,n-1},\kappa_{n-2})\Gamma(-\alpha_{0,n-1}),\end{aligned} \tag{7}$$

where we have a Γ for each Reggeon propagator and a V for each vertex. To prove (7), we use induction. For $n=3$, it is trivially true, and assuming it for $n-1$, we have to show that

$$G_n=G_{n-1}\times V(\alpha_{0,n-2},\alpha_{0,n-1},\kappa_{n-2})\Gamma(-\alpha_{0,n-1}), \tag{8}$$

which is equivalent to the formula

$$\begin{aligned}G_{n-1}V\Gamma=&\int_0^\infty\cdots\int_0^\infty dx_1\cdots dx_{n-2}(x_1)^{-\alpha_{0,1}-1}\cdots\\&\times(x_{n-2})^{-\alpha_{0,n-2}-1}\exp\left[-\left(\sum_{i=1}^{n-2}x_i+\cdots+\frac{x_1\cdots x_{n-2}}{\kappa_1\cdots\kappa_{n-3}}\right)\right]\\&\times\int_0^\infty dx_{n-1}(x_{n-1})^{-\alpha_{0,n-1}-1}\left(1+\frac{x_{n-1}}{\kappa_{n-2}}\right)^{\alpha_{0,n-2}}e^{-x_{n-1}}=G_n.\end{aligned} \tag{9a}$$

Here, we use the identity

$$V\Gamma=\int_0^\infty dx_{n-1}x_{n-1}^{-\alpha_{0,n-1}-1}\left(1+\frac{x_{n-1}}{\kappa_{n-2}}\right)^{\alpha_{0,n-2}}e^{(-x_{n-1})}. \tag{9b}$$

The last step in Eq. (9a) follows from a change of variable

$$x_{n-2}\to x_{n-2}[1+(x_{n-1}/\kappa_{n-2})],$$

which converts the integral in question into formula (5).

Factorization is clearly important if one wants to calculate couplings of various high-spin resonances,

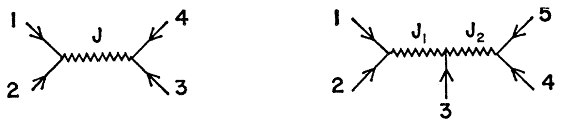

FIG. 3. Resonance exchange in four- and five-point functions.

using the n-point function. It assures us that, independent of n, we shall get the same answer for the couplings of parent resonances. Also, if one constructs Veneziano-type formulas for the scattering of high-spin objects from the n-point function, the result is guaranteed to be independent of the various ways of doing it by the factorization property. This situation presumably does not hold in general for the daughters.

In the next section, the results of the present section will be reexpressed in the usual Feynman language and compared with the Regge-Toller formula (5). There is, of course, complete agreement, but also this comparison, in our opinion, clarifies the role of the Toller variable κ and relates it to orbital angular momentum. Furthermore, the rules for the coupling may be more useful in the usual Feynman language.

III. FEYNMAN RULES FOR THE PARENT RESONANCES

First, we consider the original Veneziano model, where external momenta are labeled in Fig. 3.

A typical pole in the $12 \leftrightarrow 34$ channel has the form

$$\frac{1}{J-\alpha_{12}}\binom{\alpha_{23}+J}{J}, \tag{10}$$

where J is the angular momentum of the resonance exchanged and

$$\binom{\alpha_{23}+J}{J} \equiv \frac{(\alpha_{23}+J)(\alpha_{23}+J-1)\cdots(\alpha_{23}+1)}{J!}$$

is the generalized binomial coefficient. This residue contains all values of angular momentum $\leqq J$, but the coefficient of the highest-spin resonance (parent) comes from the highest power of $(p_2\cdot p_3)$ in (10). This coefficient is $(2b)^J/J!$. We wish to write a Feynman graph for the resonance with mass $\alpha_{12}=J$, and we are only interested in the coupling on the mass shell, since off-mass-shell coupling is essentially arbitrary.

We denote by

$$\Delta_{\nu_1\nu_2\cdots\nu_J}{}^{\mu_1\mu_2\cdots\mu_J}(p)$$

the covariant projection operator for spin J at mass $s_J=(J-a)/b=p^2$. It is symmetric and traceless in the vector indices $\mu_1\cdots\mu_J$ and $\nu_1\cdots\nu_J$ separately, and, dotted into p, gives zero. For simplicity, we denote it by the shorthand notation $\Delta_\nu{}^{\mu(J)}(p)$, and we shall never need its explicit form. In Fig. 3(a), we then use the propagator

$$(J-\alpha_{12})^{-1}\Delta_\nu{}^{\mu(J)}(p_1+p_2)$$

and the couplings at each vertex,

$$T^{\mu(J)}=\frac{(2b)^{J/2}}{(J!)^{1/2}}(p_2)^{\mu(J)}$$

and

$$T^{\nu(J)}(p_3,p_4)=\frac{(2b)^{J/2}}{(J!)^{1/2}}(p_3)^{\nu(J)}. \tag{11}$$

Here, $(p_2)^{\mu(J)}$ stands for the product $p_2{}^{\mu_1}p_2{}^{\mu_2}\cdots p_2{}^{\mu_J}$. Instead of (11), one could use the Hermitian coupling $\frac{1}{2}i(p_2-p_1)$ and $\frac{1}{2}i(p_3-p_4)$, which would give the same result, but in view of later developments there is no advantage in doing this. It should be realized that the various ways of writing this coupling are the same as far as the highest-angular-momentum component (parent) is concerned.

We now write the graph for the five-point function of Fig. 3(b), with angular momenta J_1 and J_2 exchanged in the internal lines. The pole corresponding to this resonance exchange has the form[2]

$$\sum_{k=0}^{k=\min(J_1,J_2)}\frac{1}{J_1-\alpha_{12}}\frac{1}{J_2-\alpha_{45}}\binom{\alpha_{23}+J_1-k}{J_1-k} \times\binom{\alpha_{34}+J_2-k}{J_2-k}\binom{2b(p_1\cdot p_5)-b-a+k-1}{k}. \tag{12}$$

Again, the highest components of spin come from the leading powers in the momenta. To reproduce (12), the coupling of spin J_1 and J_2 to a scalar must be given by

$$T^{\mu(J_1),\nu(J_2)}(p_3)=\sum_k^{k=\min(J_1,J_2)}\frac{(J_1!J_2!)^{1/2}}{(J_1-k)!(J_2-k)!k!} \times(2b)^{(J_1+J_2/2)-k}(p_3)^{\mu(J_1-k)}(p_3)^{\nu(J_2-k)}g_\nu{}^{\mu(k)}. \tag{13}$$

The symbolic expression $(p_3)^{\mu(J_1-k)}(p_3)^{\nu(J_2-k)}g_\nu{}^{\mu(k)}$ stands for the tensor,

$$g_{\nu_1}{}^{\mu_1}g_{\nu_2}{}^{\mu_2}\cdots g_{\nu_k}{}^{\mu_k}p_3{}^{\mu_{k+1}}\cdots p_3{}^{\mu_{J_1}}p_3{}^{\gamma_{k+1}}\cdots p_3{}^{\nu_{J_2}}.$$

Here, $|J_1-J_2|+2k$ is the orbital angular momentum between the Reggeons. One can similarly calculate the coupling of the parent to $(n+1)$ scalars by expanding the integrand in Eq. (1) with respect to the last variable u_{n-1} and looking at the highest power of momentum p_n. This situation is indicated in Fig. 4. For the parent resonance, only the highest powers of the various momentum transfers $(p_n\cdot p_{n-1}),\cdots(p_n\cdot p_1)$count. We find

$$B_{(J)}{}^{\mu(J)}=(J!)^{1/2}\sum_{k_1+k_2+\cdots+k_{n-1}=J}(p_1)^{\mu_1(k_1)}\cdots \times(p_{n-1})^{\mu_{n-1}(k_{n-1})}B_{n-1}(\alpha_{01}-k_1,\alpha_{02}-k_1-k_2,\cdots, \times\alpha_{0,n-2}-k_1\cdots-k_{n-2};-2b(p_1\cdot p_3),\cdots) \times1/(k_1!\cdots k_{n-2}!). \tag{14}$$

In the definition of the B_{n-1} function appearing in Eq. (14), the exponents involving $\alpha_{01},\cdots,\alpha_{0,n-2}$ in Eq. (1) are shifted by k_1, $k_1+k_2,\cdots$, etc., and all other exponents stay unchanged. One can now further expand the integrand involved in the definition of B_{n-1} function of Eq. (14) in terms of the variable u_{n-2}, and obtain the coupling given by (13) in an alternative way. The agreement of the two different ways of calculating the Reggeon-Reggeon-scalar coupling provides a more direct proof of factorization. Of course, the couplings given by Eqs. (13) and (14) must be multiplied by the appropriate spin-projection operators acting on the external legs to project our unwanted lower-spin components. For example, (14) has to be multiplied by $\Delta_\nu^{\mu(J)}(p_n)$, where $p_n=-p_0-p_1-\cdots-p_{n-1}$.

Finally, we compare directly the three-point coupling given by Eq. (6) with the one given by Eq. (13). This will establish a correspondence between the Toller variable κ and the k that appears in Eq. (13), and in our opinion will help clarify the role played by the Toller variable. Integrating the right-hand side of Eq. (6) with respect to x_1, we obtain[8]

$$V(\alpha,\alpha',k)=\frac{1}{\Gamma(-\alpha')}\int_0^\infty dx_2\left(1+\frac{x_2}{\kappa}\right)^\alpha x_2^{-\alpha'-1}e^{-x_2}. \quad (15a)$$

Now set $\alpha=J_1=$ integer and expand the first factor in (15a) in power series,

$$V(J_1,\alpha',\kappa)=\frac{1}{\Gamma(-\alpha')}\sum_{m=0}^{J_1}\frac{J_1!}{m!(J_1-m)!}\frac{\Gamma(-\alpha'+m)}{(\kappa)^m}. \quad (15b)$$

Now set $\alpha'=J_2$,

$$V(J_1,J_2,\kappa)=\sum_{m=0}^{\min(J_1,J_2)}\frac{J_1!J_2!}{m!(J_1-m)!(J_2-m)!}\frac{1}{(-\kappa)^m}. \quad (15c)$$

Comparing (15c) with (13), we see that they agree up to a factor of $(J_1!J_2!)^{1/2}$, which can be absorbed in the external-line normalization if one identifies the variable k in (13) with variable m in (15c). Therefore, the Toller variable $(-1/\kappa)$ is conjugate to the orbital angular momentum l; different powers of $(-1/\kappa)$ give the different orbital angular momentum states of the two resonances of spin J_1 and J_2.

IV. MULTIPERIPHERAL MODELS

In the original multipheripheral model of Amati, Fubini, Stanghellini, and Tonin,[4] the absorptive part of elastic two-particle scattering amplitude is written as a sum over an infinite number of terms in the following way:

$$2\pi iA_s(s,t)=\sum_{n=1}^{\infty}A_n(s,t),$$

$$A_n(s,t)=\int d^4q_1\cdots d^4q_n\delta^+(q_1^2-1)\cdots\delta^+(q_n^2-1)$$
$$\times K_n(p_1,p_2;q_1,\cdots,q_n)K_n^*(p_3,p_4;q_1,\cdots,q_n), \quad (16)$$

where $A_s(s,t)$ is the absorptive part of the two-particle scattering amplitude in the s channel, and $K(p_1,p_2;q_1,\cdots,q_n)$ is the amplitude for $2\leftrightarrow n$-particle

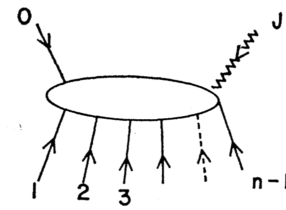

FIG. 4. Production amplitude with an external particle having spin.

process. The relevant diagrams are given in Fig. 5. For simplicity, all the masses are taken to be 1.

For K_n, Amati *et al.* took the multiple-particle-exchange graph given in Fig. 5. The major defect of such a model is that it does not Reggeize, hence it cannot be expected to be valid in the high-energy, comparatively low-multiplicity region, where the subenergies can be quite high. A recently proposed model replaces the multiple-particle-exchange graph of Amati *et al.* by a multi-Regge exchange graph.[5–7] However, in the high-multiplicity, low-individual-subenergy region where direct s-channel resonances are expected to be important, the validity of such a model is doubtful. Chan *et al.* constructed a model which extrapolates between high- and low-energy regions and apparently gives good results.

In this section, we propose an alternative model for the multiple-particle production amplitude K_n which both has high-energy Reggeism and low-energy resonance features correctly built in without double counting. An ideal solution would be to use the amplitude given by Eq. (1) for K_n; unfortunately this leads to an infinite number of coupled Bethe–Salpeter-type equations which appear unmanageable. So we define a set of truncations of the function B_n which yield a manageable set of equations. To this end, we define

$$K_{(n)}{}^{(m)}(p_1,p_2;q_1,\cdots,q_n)$$
$$\equiv\int_0^1\int_0^1\cdots du_1\cdots du_{n-1}u_1^{-\alpha(p_1,q_1)-1}u_2^{-\alpha(p_1,q_2)-1}\cdots$$
$$\times u_{n-1}^{-\alpha(p_1,q_{n-1})-1}(1-u_1)^{-\alpha_{12}-1}\cdots$$
$$\times(1-u_{n-1})^{-\alpha_{n-1,n}-1}\cdots(1-u_1u_2\cdots u_m)^{-2bq_1\cdot q_{m+1}}\cdots$$
$$\times(1-u_{n-m}\cdots u_{n-1})^{-2bq_n\cdot q_{n-m}}, \quad (17)$$

where $m\leq n-1$, $\alpha(p_1,q_k)\equiv\alpha((p_1+q_1+\cdots+q_k)^2)$, $\alpha_{12}\equiv\alpha((q_1+q_2)^2)$ as before. Equation (17) is identical to Eq. (1) for the factors that involve products of up to m u's; all the factors involving more than m u's are omitted. For $m=n-1$, (17) coincides with (1). For $m=1$, it

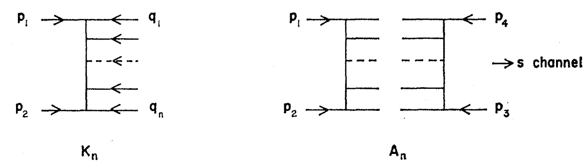

FIG. 5. Multiperipheral graphs.

[11] Chan Hong-Mo *et al.*, Nuovo Cimento **49**, 157 (1967).

reduces to a product of simple Veneziano models, so that

$$K_{(n)}^{(1)}(p_1,p_2;\,q_1,\cdots,q_n)=B_2(-\alpha(p_1,q_1),\,-\alpha_{12})\,B_2(-\alpha(p_1,q_2),\,-\alpha_{23})\cdots B_2(-\alpha(p_1;q_{n-1}),\,-\alpha_{n-1,n}),\tag{18}$$

where B_2 is the Veneziano function. This function clearly Reggeizes in all the final momenta $q_1,\cdots,q_n$; it also contains the two-particle resonances in the same channel as dictated by duality. However, it has multiple poles that are not allowed. Moreover, the model has no three-particle correlation at all; for example, there is no dependence on the Toller variable in multi-Reggeization. We shall see later that these defects are removed when one goes one step further in the truncation scheme.

To derive a Bethe–Salpeter-type equation of $A_s(s,t)$ in this approximation we define the following function:

$$\begin{aligned}A_n(p_1,p_4;\,P,p_2,p_3)=\int d^4q_1\cdots d^4q_n\delta^+(q_1^2-1)\cdots\\ \times\delta^+(q_n^2-1)K_{(n+1)}(p_1,\,-p_1-\textstyle\sum q-P,\,q_1,\,\cdots,\,q_n,\,P)\\ \times K_{(n+1)}(p_4,\,-p_4+\textstyle\sum q+Q;\,-q_1,\,\cdots,\,-q_n,\,-P),\end{aligned}\tag{19a}$$

which satisfies the recursion relation

$$\begin{aligned}A_{n+1}(p_1,p_4;\,P,p_2,p_3)\\ =\int d^4P'\delta^+(P'^2-1)A_n(p_1,\,p_4;P';\,p_2+P,\,p_3-P)\\ \times B_2(\alpha((p_2+P)^2,\alpha(P+p)^2))\\ \times B_2(\alpha((p_3-P)^2,\alpha(P+P')^2)),\end{aligned}\tag{19b}$$

which leads to the equation

$$\begin{aligned}A_s(p_1,p_4,P;\,p_2,p_3)=A_1(p_1,p_4,P,p_2,p_3)\\ +\int d^4P'\delta^+(p'^2-1)\frac{\Gamma(-\alpha(p_2+P)^2)\Gamma(-\alpha(P+P')^2)}{\Gamma(-\alpha((p_2+P)^2)-\alpha(P+P')^2)}\\ \times\frac{\Gamma(-\alpha(p_3-P)^2)\Gamma(-\alpha(P+P')^2)}{\Gamma(-\alpha(p_3-P)^2-\alpha(P+P')^2)}\\ \times A_s(p_1,p_4;\,P;\,p_2,p_3),\end{aligned}\tag{20}$$

where $A_s(s,t)=A_s(p_1,p_4,0,p_2,p_3)$ with $s=(p_1+p_2)^2$ and $t=(p_1+p_4)^2$. In the next-order truncation, we have

$$\begin{aligned}K_{(n)}^{(2)}(p_1,p_2;\,q_1,\cdots,q_n)\\ =\int_0^1 du_1\cdots du_{n-1}u_1^{-\alpha(p_1,q_1)-1}\cdots u_{n-1}^{-\alpha(p_1,q_{n-1})-1}\\ \times(1-u_1)^{-\alpha_{12}-1}\cdots(1-u_{n-1})^{-\alpha_{n-1,n}-1}\\ \times(1-u_1u_2)^{-2b(q_1\cdot q_3)+a+b}\cdots\\ \times(1-u_{n-2}u_{n-1})^{-2b(q_{n-2}\cdot q_n)+a+b}.\end{aligned}\tag{21}$$

First of all, the two-particle poles are correctly given by (21), so that there are no simultaneous poles in variables $(q_i+q_{i+1})^2$ and $(q_{i+1}+q_{i+2})^2$ which share a momentum, as is required by Feynman rules. This feature was of course absent in the previous order of truncation. We now show that (21) Reggeizes in the limit $\alpha_{12}\to\infty\cdots$, $\alpha_{n-1,n}\to\infty$, $-\kappa_1\equiv(\alpha_{12}\alpha_{23})/\alpha_{13}$, etc. If one changes variables by

$$\begin{aligned}1-u_1=e^{k_1/\alpha_{12}}\\ \cdots\\ 1-u_{n-1}=e^{x_{n-1}/\alpha_{n-1,n}}\end{aligned}\tag{22}$$

in a way similar to Eq. (3), upon expending various terms in lowest order in x_1/α_{12}, etc., we have

$$\begin{aligned}K_n^{(2)}(p_1,p_2;\,q_1,\cdots,q_n)\approx\int_0^\infty dx_1\cdots dx_{n-1}(-\alpha_{12})^{\alpha(p_1,q_1)}\cdots(-\alpha_{n-1,n})^{\alpha(p_1,q_{n-1})}\exp[-(x_1+x_2+\cdots+x_{n-1})]\\ \times x_1^{-\alpha(p_1,q_1)-1}\cdots x_{n-1}^{-\alpha(p_1,q_{n-1})-1}\left(1-\frac{x_1x_2}{\alpha_{12}\alpha_{23}}\right)^{-2b(q_1\cdot q_3)+a+b}\cdots\left(1-\frac{x_n\_x_{n-1}}{\alpha_{n-2,n-1}\alpha_{n-1,n}}\right)^{-2b(q_{n-2}\cdot q_n)+a+b},\end{aligned}\tag{23}$$

Taking the high-energy limit as before, we have

$$\begin{aligned}K_n^{(2)}(p_1,p_2;\,q_1,\cdots,q_n)\sim(-\alpha_{12})^{+\alpha(p_1,q_1)}\cdots(-\alpha_{n-1,n})^{\alpha(p_1,q_{n-1})}\int_0^\infty dx_1\cdots dx_{n-1}(x_1)^{-\alpha(p_1,q_1)-1}\cdots(x_{n-1})^{-\alpha(p_1,q_{n-1})-1}\\ \times\exp\left[-\left(\frac{x_1x_2}{\kappa_1}+\frac{x_2x_3}{\kappa_2}+\cdots+\frac{x_{n-2}x_{n-1}}{\kappa_{n-2}}+x_1+\cdots+x_{n-1}\right)\right],\end{aligned}\tag{24}$$

which proves multi-Reggeization. Although this order of truncation Reggeizes correctly and has the correct two-particle resonances, it does not contain three-and-more-particle resonances correctly. It can be easily shown that the kth-order truncation contains k-particle resonances correctly, but no higher. Instead of discussing higher orders of truncation, we shall now derive a multiperipheral equation based on (21), which we believe to be an improvement over (20).

Define

$$K_{(n)}{}^{(2)}(p_1,p_2;\,q_1,\cdots,q_n,Q,u)\equiv\int_0^1 du_1\cdots du_{n-1}u_1{}^{-\alpha(p_1,q_1)-1}\cdots u_{n-1}{}^{-\alpha(p_1,q_{n-1})-1}$$
$$\times u^{-\alpha[(p_2+Q)2]-1}(1-u_1)^{-\alpha_{12}-1}\cdots(1-u_{n-1})^{-\alpha_{n-1,n}-1}(1-u)^{-\alpha[(q_n+Q)^2]-1}$$
$$\times(1-u_1u_2)^{-2b(q_1\cdot q_3)+a+b}\cdots(1-u_{n-2}u_{n-1})^{-2b(q_{n-2}\cdot q_n)+a+b}(1-u_{n-1}u)^{-2b(q_{n-1}\cdot Q)+a+b},\quad(25)$$

and

$$A_n{}^{(2)}\equiv\int d^4q_1\cdots d^4q_{n-1}\delta^+(q_1{}^2-1)\cdots\delta^+(q^2{}_{n-1}-1)$$
$$\times K_n{}^{(2)}(p_1,p_2;\,q_1,\cdots,q_{n-1},q,Q,u)K_n{}^{(2)}(p_3,\,p_4;\,-q_1,\,\cdots,\,-q_{n-1},\,-q,\,-Q,\,v),\quad(26)$$

which satisfies the recursion relation

$$A_{(n+1)}{}^{(2)}(p_1,p_2;\,p_3,p_4,q,Q,u,v)=\int d^4q'\delta^+(q'^2-1)\int_0^1 du'dv'$$
$$\times A_n{}^{(2)}(p_1,\,p_2+Q;\,p_3-Q,\,p_4,\,u'v',\,q',\,Q'=q)u^{-\alpha[(p_2+Q+q)^2]-1}v^{-\alpha[(p_3-Q-q)^2]-1}(1-u)^{-\alpha[(q'+q)^2]-1}$$
$$\times(1-u'u)^{-2b(q'\cdot Q)+a+b}(1-v'v)^{-2b(q'\cdot Q)+a+b},\quad(27)$$

which leads to the equation

$$A_s{}^{(2)}(p_1,p_2;\,p_3p_4,q,Q,u,v)=\sum_{n=2}^{\infty}A_n{}^{(2)}(p_1,p_2,p_3,p_4,q,Q,u,v)=A_2{}^{(2)}(p_1,p_2,p_3,p_4,q,Q,u,v)+\int d^4q'\delta^+(q'^2-1)\int_0^1 du'dv'$$
$$\times u^{-\alpha[(p_2+Q+q)^2]-1}v^{-\alpha[(p_3-Q-q)^2]-1}(1-u)^{-\alpha[(q'+q)^2]-1}(1-v)^{-\alpha[(q'+q)^2]-1}(1-u'u)^{-2b(q'\cdot Q)+a+b}$$
$$\times(1-v'v)^{-2b(Q\cdot q')+a+b}A_s{}^{(2)}(p_1,\,p_2+Q;\,p_3-Q,\,p_4,\,u',\,v',\,q',\,q),\quad(28)$$

where

$$A_2{}^{(2)}(p_1,p_2,p_3,p_4,Q,q,u,v)\equiv\int d^4q_1\delta^+(q_1{}^2-1)\int_0^1 du_1\int_0^1 dv_1u_1{}^{-\alpha[(p_1+q_1)^2]-1}v_1{}^{-\alpha[(p_4-q_1)^2]-1}$$
$$\times(1-u_1)^{-\alpha[(q+Q)^2]-1}(1-v_1{}^{-\alpha(q+Q)^2]-1}(1-u_1u)^{-2b(q_1\cdot Q)}(1-v_1v)^{-2b(Q\cdot q_1)},$$

and

$$A_s(p_1,p_2,p_3,p_4)$$
$$\equiv A_s(p_1,\,p_2;\,p_3,\,p_4,\,Q=0,\,q=0.\,v=0,\,u=0).$$

The derivation of equations using higher truncations is now straightforward; these equations will always involve the same number of variables of integration (d^4q' and $dv'du'$) but a larger number of "hanging" variables.

V. CONCLUSIONS

We have presented Reggeized resonance models for arbitrary production processes, with only the provision that the external particles carry spin zero. One can then easily compute similar expressions for external particles with spin, by letting the scalar external particles form resonances in pairs (or in higher multiplicity). Different ways calculating the same process must give the identical result if there is a general factorization theorem. We have demonstrated this property of factorization for the coupling of an arbitrary-spin resonance to n scalars and also for arbitrary-spin–arbitrary-spin–scalar vertex, and only for the parent resonance. It would be interesting to extend factorization to more complicated processes.

Finally, the various truncations of the n-point function discussed in the last section may prove useful in the case of elastic processes, especially for the calculation of the Pomeranchon trajectory.

ACKNOWLEDGMENTS

The authors thank Dr. Chan Hong-Mo, Dr. J. F. L. Hopkinson, and Dr. E. Plahte for useful discussions and helpful comments. They also thank Professor W. Thirring and Professor J. Prentki for the hospitality at the theoretical study division at CERN, where this work was done.