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Compton Scattering From a Bound System of Two Charged Particles*

Giordano Bisiacchi†

Department of Physics and Astronomy of the University of Rochester, Rochester, New York 14627

and

Giorgio Calucci

Istituto di Fisica Teorica dell'Università di Trieste e *Istituto Nazionale di Fisica Nucleare, Sottosezione di Trieste, Italy*

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With reference to a previous paper by Fronsdal, the expression for Compton scattering on the fundamental state of the hydrogen atom is derived, taking into account the effect of the proton motion. An example of its extension to an inelastic case is shown. The result is discussed in connection with some problems of interpretation of the formalism of the infinite-component wave functions.

1. INTRODUCTION

The formalism of the infinite-component wave functions has often been proposed and used in recent years to describe systems having internal degrees of freedom.[1,2] One of the problems arising in this formulation is to show as clearly as possible the connection between the new mathematical techniques and the older ones. The system most commonly studied in this connection is the hydrogen atom, since there the older formulation is completely clear and the newer formulation can be built up explicitly. In particular, Fronsdal[3] recently used this formalism to calculate the scattering of photons by bound electrons without using the dipole approximation, but still keeping the proton mass infinite. Here we wish to go a step further and study the effect of the finite mass of the proton, which allows recoil effects and interaction of the proton with real photons. The main reason for including these effects is not that the finite-mass effect is important experimentally, but to study, in a concrete example, how one can describe, in the frame of the infinite-component formalism, the interaction of a composite system in which both the components interact. In particular, we shall see that the requirement of locality for the interaction Lagrangian is no longer valid and that some weaker condition should be substituted. The paper is, in a broad sense, a continuation of Ref. 3, although its particular purpose led us to choose a different starting point, i.e., we use the classical Schrödinger formulation and introduce the infinite-component functions after the problem has been completely formulated.

2. FORMULATION OF THE PROBLEM IN SCHRÖDINGER VARIABLES

Using Schrödinger variables the problem is a standard one. However, we think it useful to outline the main steps of the formulation. Starting from the two-body wave function $\Psi(\vec{r}_1, \vec{r}_2)$, where we use the canonical separation of c.m. and relative variables,

$$\Psi(\vec{r}_1, \vec{r}_2) = (2\pi)^{-3/2} e^{i\vec{P}\cdot\vec{R}} \varphi(\vec{r}),$$

where $c_{1,2} = m_{1,2}/m$, $m = m_1 + m_2$, $\mu = c_1 c_2 m$, $\vec{R} = c_1\vec{r}_1 + c_2\vec{r}_2$, $\vec{r} = \vec{r}_2 - \vec{r}_1$, we have to write down the currents of the particles "1" and "2" in order to express the two basic interactions with the photon. In this way, by taking the Fourier transform, a term corresponding to the atom with total momentum $\vec{p}$, emitting a photon of momentum $\vec{k}'$ from the particle "1", yields a matrix element

$$(e_1/m_1)\,\delta(\vec{p} - \vec{p}' - \vec{k}') \int \varphi_f^*(\vec{q})\,\varphi_i(\vec{q} - c_2\vec{k}')\,(c_1\vec{p} - \vec{q})\cdot\vec{e}(\vec{k}')\,d^3q \tag{2.1}$$

and analogously for the particle "2." Here the Coulomb gauge condition $\vec{k}'\cdot\vec{e}(\vec{k}') = 0$ has already been used. Since we are considering a two-photon process, we know that, besides the interaction of this vertex, we also have the contact term, corresponding to the original A^2 interaction:

$$(e_1^2/m_1)\,\delta(\vec{p} - \vec{p}' + \vec{k} - \vec{k}') \int \varphi_f^*(\vec{q})\,\varphi_i[\vec{q} - c_2(\vec{k}' - \vec{k})]\,d^3q\,\vec{e}(\vec{k})\cdot\vec{e}(\vec{k}') \tag{2.2}$$

To obtain the complete matrix element we have to connect any pair of first-order vertices with the perturbative denominator energy $(U - U_\nu)^{-1}$ and sum over all intermediate states ν, including the scattering states, and add the two contact terms. Denoting U the total nonrelativistic energy, including the c.m. kinetic energy K, and by E_ν the binding energy of the atom, the denominator can be written as $U - K - E_\nu$, where K does not depend, in the nonrelativistic case, on the internal quantum numbers ν but only on the total three momentum; E_ν on the contrary depends only on ν.

In order to sum over ν it is always possible to write

$$\sum_\nu \varphi_\nu \varphi_\nu^*/(U - K - E_\nu) = (U - K - H)^{-1}, \quad \text{where } H\varphi_\nu \equiv (q^2/2\mu - e^2/r)\,\varphi_\nu = E_\nu\varphi_\nu,$$

i.e. H is the relative Hamiltonian. Since we have to use the H operator acting on wave functions which are given as functions of $\vec{q} + \vec{l}$, instead of as functions of $\vec{q}$ [see Eq. (2.1)], we shall write

$$H(\vec{l})\,\varphi_\nu(\vec{q} + \vec{l}) \equiv [(\vec{q} + \vec{l})^2/2\mu - e^2/r]\;\varphi_\nu(\vec{q} + \vec{l}) \tag{2.3}$$

With these conventions the complete matrix element is

$$\begin{aligned}
\mathfrak{M} = {} & (e_1^2/m_1^2) \int \varphi_f^*(\vec{q})(c_1\vec{p}' - \vec{q})\cdot\vec{e}(\vec{k}')[U_D - K_D - H(-c_2\vec{k}']^{-1}(c_1\vec{p} - \vec{q} + c_2\vec{k}')\cdot\vec{e}(\vec{k})\,\varphi_i(\vec{q} - c_2\vec{k}' + c_2\vec{k})\,d^3q \\
& + (e_1e_2/m_1m_2) \int \varphi_f^*(\vec{q})(c_1\vec{p}' - \vec{q})\cdot\vec{e}(\vec{k}')[U_D - K_D - H(-c_2\vec{k}')]^{-1}(c_2\vec{p} + \vec{q} - c_2\vec{k}')\cdot\vec{e}(\vec{k})\,\varphi_i(\vec{q} - c_2\vec{k}' - c_1\vec{k})\,d^3q \\
& + (e_1^2/m_1^2) \int \varphi_f^*(\vec{q})(c_1\vec{p} - \vec{q})\cdot\vec{e}(\vec{k})[U_C - K_C - H(c_2\vec{k})]^{-1}(c_1\vec{p} - \vec{q} - c_2\vec{k})\cdot\vec{e}(\vec{k}')\,\varphi_i(\vec{q} - c_2\vec{k}' + c_2\vec{k})\,d^3q \\
& + (e_1e_2/m_1m_2) \int \varphi_f^*(\vec{q})(c_1\vec{p} - \vec{q})\cdot\vec{e}(\vec{k})[U_C - K_C - H(c_2\vec{k})]^{-1}(c_2\vec{p} + \vec{q} + c_2\vec{k})\cdot\vec{e}(\vec{k}')\,\varphi_i(\vec{q} + c_2\vec{k}' + c_1\vec{k})\,d^3q \\
& + (e_1^2/m) \int \varphi_f^*(\vec{q})\,\varphi_i(\vec{q} - c_2(\vec{k}' - \vec{k}))\,\vec{e}(\vec{k})\cdot\vec{e}(\vec{k}')\,d^3q + \{e_1 \to e_2, m_1 \to m_2, c_1 \to -c_2\}.
\end{aligned} \tag{2.4}$$

The substitution at the end of formula (2.4) corresponds clearly to the exchange of particle "1" with "2", the minus sign in $-c_2$ originates from the substitution $r \to -r$. In the first and second terms, corresponding to direct terms (see Fig. 1), we have

$$K_D = (\vec{p} + \vec{k})^2/2m = (\vec{p}' + \vec{k}')^2/2m, \quad U_D = m + p^2/2m + k = m + p'^2/2m + k'.$$

In the third and fourth terms, i.e., crossed terms, we have

$$K_C = (\vec{p} - \vec{k}')^2/2m = (\vec{p}' - \vec{k})^2/2m, \quad U_C = m + p^2/2m - k' = m + p'^2/2m - k.$$

To evaluate $\mathfrak{M}$ explicitly, we should perform an integration over d^3q, instead of the original sum and integration over ν, but we will transform it into a discrete, although infinite, sum.

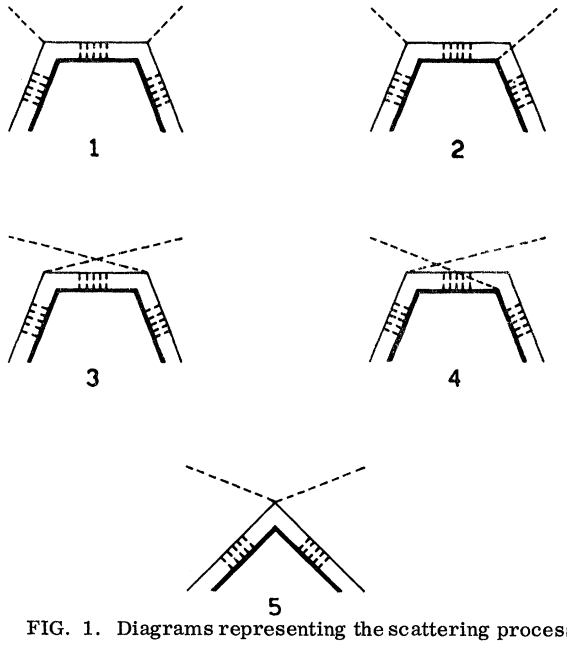

FIG. 1. Diagrams representing the scattering process. The light lines refer to particle "1," the heavy ones to particle "2," the dashed lines to photons. The other 5 terms are obtained by exchanging the light with the heavy lines.

3. THE INFINITE-COMPONENT FORMALISM

As done in previous papers,[2] and in the same way as in Ref. 3, the infinite-component wave functions and the "Majorana" matrices will be introduced through the operator correspondence

$$ar=\Gamma_0-\Gamma_4,\quad q_i=a(\Gamma_0-\Gamma_4)^{-1}\Gamma_i,\quad q^2=a^2(\Gamma_0-\Gamma_4)^{-1}(\Gamma_0+\Gamma_4)\,, \tag{3.1}$$

and with the following correspondence for the wave functions and their scalar product

$$\int\varphi^*(\vec{q})\,\Theta\,(\vec{r},\vec{q})\,\varphi(\vec{q}+\vec{l})\,d^3q=\psi^\dagger(0)(\Gamma_0-\Gamma_4)\,\Theta(\Gamma)\psi(\vec{l})\,. \tag{3.2}$$

The constant a is real and positive, but otherwise arbitrary, $\Theta(\Gamma)$ is the same operator as $\Theta(\vec{r},\vec{q})$, where the arguments have been transformed according to Eq. (3.1). The meaning of the argument in the ψ function will be clarified later.

The expression for $\mathfrak{M}$, given by Eq. (2.4) can then be translated in this new formalism and to this end we introduce the following five-vectors[4]

$$Q_A(\vec{l})=(2\mu)^{-1}\{2\mu(U_A-K_A)-l^2-a^2,\ -2a\vec{l},\ -2\mu(U_A-K_A)+l^2-a^2\}$$

$$\mathcal{E}_1(\vec{l})=a^{-1}\{(c_1\vec{p}-\vec{l})\cdot\vec{e},\ -a\vec{e},\ -(c_1\vec{p}-\vec{l})\cdot\vec{e}\},\qquad \mathcal{E}_2(\vec{l})=a^{-1}\{(c_2\vec{p}+\vec{l})\cdot\vec{e},\ a\vec{e},\ -(c_2\vec{p}+\vec{l})\cdot\vec{e}\} \tag{3.3}$$

with $A=C$, or D; $\vec{e}\equiv\vec{e}(\vec{k})$, $e'\equiv\vec{e}(\vec{k}')$; and $\mathcal{E}_1'$ and $\mathcal{E}_2'$, which are obtained by the substitution of $\vec{e}'$ for $\vec{e}$.

Note that $(Q_A)^0<0$ and $Q_A^2(\vec{l})=2a^2(K_A-U_A)/\mu$, i.e. $Q_A^2(l)$ is independent of $\vec{l}$, and $\mathcal{E}^2=-1$. Moreover $Q^2>0$ when the incoming photon is of sufficiently low energy while $Q^2<0$ for a high-energy photon, corresponding to the possibility of a real ionized state. The critical point, in the infinite-proton-mass case would clearly be $k=-E$, here the situation is a little more complicated by the presence of the c.m. kinetic energy. Apart from final considerations we shall consider the case $Q^2>0$, and we shall extend the validity of the results by analytic continuation into the case $Q^2<0$. With these auxiliary quantities, and the correspondences (3.1), (3.2), Eq. (2.4) can be written as

$$\mathfrak{M} = (e_1^2/m_1^2)\psi_f^{\dagger}(0)\,\mathcal{E}_1'(0)\cdot\Gamma a^2[Q_D(-c_2k')\cdot\Gamma+e^2a]^{-1}\,\mathcal{E}_1(-c_2\vec{k}')\cdot\Gamma\psi_i(c_2\vec{k}-c_2\vec{k}')$$

$$+(e_1e_2/m_1m_2)\,\psi_f^{\dagger}(0)\,\mathcal{E}_1'(0)\cdot\Gamma a^2[Q_D(-c_2k')\cdot\Gamma+e^2a]^{-1}\,\mathcal{E}_2(-c_2\vec{k}')\cdot\Gamma\psi_i(-c_1\vec{k}-c_2\vec{k}')$$

$$+(e_1e_2/m_1m_2)\psi_f^{\dagger}(0)\,\mathcal{E}_1(0)\cdot\Gamma a^2[Q_C(c_2k)\cdot\Gamma+e^2a]^{-1}\,\mathcal{E}_1'(c_2\vec{k})\cdot\Gamma\psi_i(c_2\vec{k}-c_2\vec{k}')$$

$$+(e_1^2/m_1^2)\psi_f^{\dagger}(0)\,\mathcal{E}_1(0)\cdot\Gamma a^2[Q_C(c_2k)\cdot\Gamma+e^2a]^{-1}\,\mathcal{E}_2'(c_2\vec{k})\cdot\Gamma\psi_i(c_1\vec{k}+c_2\vec{k}')$$

$$+(e_1^2/m_1)\psi_f^{\dagger}(0)(\Gamma_0-\Gamma_4)\psi_i(c_2\vec{k}-c_2\vec{k}')\vec{e}(\vec{k})\cdot\vec{e}(\vec{k}')+\left\{\begin{matrix}e_1\to e_2,\\ m_1\to m_2,\\ c_1\to -c_2,\end{matrix}\right\}, \tag{3.4}$$

and this equation can be put in close correspondence with Eqs. (7), (8) and (9) of Ref. 3 of which it is a generalization. As shown in the quoted references,[2] the ψ function satisfies an equation, linear in the Γ matrices, which is the translation of the Schrödinger equation (2.3). In fact (2.3) is transformed into

$$[(-2E/\mu)^{1/2}\lambda\cdot\Gamma-e^2]\,\psi(\vec{l})=0, \tag{3.5}$$

where $$\lambda\equiv\lambda(\vec{l})=(2a\mu)^{-1}(-\mu/2E)^{1/2}\{a^2+l^2-2\mu E,\,2a\vec{l},\,a^2-l^2+2\mu E\} \tag{3.6}$$

So it is clear that ψ depends essentially on λ, which in turn depends both on $\vec{l}$ and on the internal quantum number ν which defines E_ν; $\psi_\nu(\vec{l})$ would be more correctly written as $\psi(\lambda_\nu(\vec{l}))$.

We can define also the unit vector referring to Q as $-|Q|^{-1}Q$ and call it $\eta(\vec{l})$; as a pure formal fact, we note that $\mathcal{E}_1'(\vec{k})\cdot\mathcal{E}_1(\vec{k}')=-\vec{e}'\cdot\vec{e}$. Before proceeding in the calculation of $\mathfrak{M}$, which amounts essentially to the same steps and operations done in F, we have to make some remarks related to the fact that we have two charged particles of finite mass. Looking at Eqs. (2.4), (3.4) and remembering that the arguments of H or Q are connected to the momenta of the intermediate state, we see that we go, in the vertex, from one state to another changing the argument not by the three-momentum transfer, but by that quantity multiplied by a constant, say c_2. In the original formulation of the infinite-component theories it was intended that the change of the argument, from the initial to the final state in a vertex, was just that given by a simple kinematical transformation. This result can be obtained by just putting $c_2=1$, $c_1=0$, that is, by considering the proton mass to be infinite and discarding its electromagnetic (e. m.) interaction. In the more realistic case the situation is less simple, as the internal properties of the bound system are strictly related to the appearance of the coefficients c_1 and c_2 and to the "nonlocality" of the interaction.

In order to better clarify this point let us consider the simple case of a Coulomb interaction with an external field. In this case we have two graphs contributing to the first order in the e. m. field, and the corresponding matrix element would simply be

$$e_1\int\Psi_f^*(\vec{r}_1,\vec{r}_2)\Psi_i(\vec{r}_1,\vec{r}_2)A_0(\vec{r}_1)\,dr_1dr_2+e_2\int\Psi_f^*(\vec{r}_1,\vec{r}_2)\Psi(\vec{r}_1,\vec{r}_2)A_0(\vec{r}_2)\,dr_1dr_2$$

which can be rewritten as

$$(2\pi)^{-3}\int\varphi_f^*(\vec{r})\left(e_1e^{-ic_2\vec{k}\cdot\vec{r}}+e_2e^{ic_1\vec{k}\cdot\vec{r}}\right)\varphi_i(\vec{r})A_0(\vec{k}),$$

where as usual we defined

$$\Psi(\vec{r}_1,\vec{r}_2)=(2\pi)^{-3/2}e^{i\vec{p}\cdot\vec{R}}\varphi(\vec{r})$$

We see then that if we take into account the structure of the system we are led to define a "pseudolocal" current which algebraically reads

$$j_0(\vec{k})=\psi_f^{\dagger}(\Gamma_0-\Gamma_4)\left(e_1e^{-ic_2\vec{k}\cdot\vec{r}}+e_2e^{ic_1\vec{k}\cdot\vec{r}}\right)\psi_i$$

instead of the usual form

$$j_0(\vec{k}) = \psi_f^{\dagger}(\Gamma_0 - \Gamma_4) e^{i\vec{k}\cdot\vec{r}} \psi_i, \quad \text{where } r_i = (S_{i0} - S_{i4})/a, \quad \vec{p}_f - \vec{p}_i = \vec{k}\ .$$

4. EVALUATION OF THE MATRIX ELEMENT

It is evident that Eq. (3.4) reduces to Eq. (7) of Ref. 3 when the mass of the proton is allowed to go to infinity, in which case only the first, third, and fifth terms survive.[5] Moreover, the other terms are all of the same kind so that the evaluation goes exactly as in Ref. 3, to which we refer.

We wish only to sketch a point that can be of some interest. The main trick in the summation over the intermediate states is the introduction of the discrete set of mathematical states $\Psi_n(\lambda)$ which are eigenstates of $\lambda \cdot \Gamma$ with eigenvalues $n+1$. On the contrary the external states are the physical states, in direct correspondence to the ones of Schrödinger's through Eq. (3.2) and satisfying Eq. (3.5). It is convenient to find the relation between the two sets, for a given n and λ. The relation amounts to a change of normalization, since the mathematical states are normalized as $\Psi_m^{\dagger}(\lambda)\Psi_m(\lambda) = 1$. Comparing the two normalizations we find that, writing $\psi_m(\lambda) = c_m \Psi_m(\lambda)$, we must have

$$|c_m|^2 = e^2\mu/a(m+1)^2\ . \tag{4.1}$$

Another problem we have to solve, is the evaluation of the operation of the Γ matrices on the states Ψ. In the quoted reference 2 this action has been expressed through the tensor representations of the states[3,6] Ψ. The explicit expression we obtain, taking into account that we always have to perform contractions with traceless tensors, and so drop every term in g_{ab}, is

$$\Psi_{(0)}^{\dagger}(\lambda)\Gamma_c = [(2N+1)!!/(N+3)!][(2N+3)\lambda_c - \partial/\partial\lambda^c]\,\lambda^{a_1}\cdots\lambda^{a_N}\tilde{\varphi}^* \qquad a,b,c = 0,\ldots,4\ . \tag{4.2a}$$

for the fundamental state, where $\tilde{\varphi}$, using formulas (II–16, 17) of Ref. 6 is defined by $\tilde{\varphi}^*\tilde{\varphi} = 2(N+2)!/(2N+1)!!$. For the first excited state, we get analogously

$$\Psi_1^{\dagger}(\lambda)\Gamma_c = [(2N+1)!!/(N+3)!]\,\mathbf{S}\left\{[(N+3) - N(N-1)/(N+4)]\,\tfrac{1}{2}g_{bc} + \left(1 + \frac{(N-1)}{(N+4)}\right) 2\lambda_c \frac{\partial}{\partial\lambda^b} - \frac{2}{(N+4)}\frac{\partial}{\partial\lambda^b}\frac{\partial}{\partial\lambda^c}\right\}\lambda^{a_1}\cdots\lambda^{a_N}\tilde{\varphi}^{*b}\ . \tag{4.2b}$$

Here we write $\tilde{\varphi}^{*b} = \gamma^* \epsilon^b$, where $\sum_r \epsilon_{(r)}{}^a \epsilon_{(r)}{}^b = \vartheta^{ab} = \lambda^a\lambda^b - g^{ab}$, i.e., the $\epsilon_{(r)}{}^b$ are unit vectors in the space orthogonal to λ, and find

$$|\gamma|^2 = -4!!(N+3)!/N(2N+1)!!$$

In the particular case in which $N = -2$, corresponding to the $SO(4,1)$ representation we are using,[2,3] we get very simple coefficients; in particular, for the ground state we found that, in the tensor representation, the application of a Γ_c is equivalent to the application of the operator $-(\partial/\partial\lambda^c)|\lambda|$, where $|\lambda|$ is to be put equal to 1 after the derivative has been obtained [see Eq. (22) of Ref. 3]. With these elements, we obtain now the formula which is the exact parallel of Eqs. (7) and (27) of Ref. 3:

$$\sum_{1}^{2}{}_{\alpha,\beta}\{(e_\alpha{}^2/m_\alpha)(e^2\mu/a)\xi^a X_a(\lambda', \lambda_{\alpha\alpha}) - e^2\mu(e_\alpha e_\beta/m_\alpha m_\beta)$$

$$\times[\mathcal{E}'_\alpha{}^a(k)\ Z_{ab}(\lambda', \eta_{D,\alpha}, \lambda_{\alpha\beta})\mathcal{E}_\beta{}^b(k') + \mathcal{E}'_\beta{}^a(k')\ Z_{ab}(\lambda', \eta_{C,\beta}, \lambda_{\alpha\beta})\mathcal{E}_\alpha{}^b(k)]\} \tag{4.3}$$

where λ' is the unit vector referring to the final state,

$$\eta_{D,1} = -|Q_D(-c_2\vec{k})|^{-1}Q_D(-c_2\vec{k}), \quad \eta_{D,2} = -|Q_D(c_1\vec{k})|^{-1}Q_D(c_1\vec{k}),$$
$$\eta_{C,1} = -|Q_C(c_2\vec{k}')|^{-1}Q_C(c_2\vec{k}'), \quad \eta_{C,2} = -|Q_C(-c_1\vec{k}')|Q_C(-c_1\vec{k}'), \tag{4.4a}$$

the λ's are the unit vectors referring to the initial state with the following law [see Eq. (3.6)]

$$\lambda_{11}=\lambda(-c_2\vec{k}+c_2\vec{k}'),\quad \lambda_{12}=\lambda(-c_2\vec{k}-c_1\vec{k}'),$$
$$\lambda_{21}=\lambda(c_1\vec{k}+c_2\vec{k}'),\quad \lambda_{22}=\lambda(c_1\vec{k}-c_1\vec{k}'), \tag{4.4b}$$

$$\zeta=(1,\vec{0},1). \tag{4.4c}$$

Moreover

$$X_a=-2\left(\lambda'_a+\frac{\partial}{\partial\lambda'^a}\right)|\lambda'|^{-1}(|\lambda'|+\lambda\cdot\lambda')^{-1} \tag{4.5a}$$

$$Z_{ab}(\lambda',\eta,\lambda)=2ie^{-2}\left(\lambda'_a+\frac{\partial}{\partial\lambda'^a}\right)\left(\lambda_b+\frac{\partial}{\partial\lambda^b}\right)(\sin\Phi)^{-1}(|\lambda'||\lambda|)^{-1}[\lambda'\cdot\vartheta\cdot\lambda'\lambda\cdot\vartheta\circ\lambda]^{-1/2}$$
$$\times[{}_2F_1(1,-b;1-b;ye^{i\Phi})-{}_2F_1(1,-b;1-b;ye^{-i\Phi})] \tag{4.5b}$$

The parameter b and the auxiliary variables y and Φ are defined in the same way as in Ref. 3:

$$b=\frac{e^2\mu^{1/2}}{[2(K_A-U_A)]^{-1/2}},\quad y=\frac{(\lambda'\cdot\vartheta\cdot\lambda')^{1/2}(\lambda\cdot\vartheta\cdot\lambda)^{1/2}}{(|\lambda'|+\eta\cdot\lambda')(|\lambda|+\eta\cdot\lambda)},$$
$$\cos\Phi=\frac{\lambda'\circ\vartheta\circ\lambda}{(\lambda'\cdot\vartheta\circ\lambda')^{1/2}(\lambda\cdot\vartheta\circ\lambda)^{1/2}},\quad \vartheta_{ab}=\eta_a\eta_b-g_{ab}. \tag{4.6}$$

These expressions can be further simplified by the substitution of $(\partial/\partial\lambda)|\lambda|$ for $\lambda+\partial/\partial\lambda$; here this property has not been explicitly used because by using the other form it is more straightforward to obtain the expression for excited final states of the atom.

Comparing Eq. (4.2a) with Eq. (4.2b) and taking into account Eq. (4.1) we see that the amplitude for the case of the first excited state can be obtained by substituting

$$2^{-\frac{1}{2}}\left(\tfrac{1}{2}g_{ab}+2\lambda'_a\,\partial/\partial\lambda'^b+\partial^2/\partial\lambda'^a\partial\lambda'^b\right)\epsilon^b \quad \text{for } (\lambda'_a+\partial/\partial\lambda'^a),$$

and the calculation of the more excited states amounts to the generalization of Eq. (4.2b) (clearly the $\partial/\partial\lambda$ does not act on the vector $\vec{\varphi}^a$).

5. FINAL REMARKS

We see here that the method of calculation is sufficiently flexible so as to allow generalization of the calculation done in Ref. 3, both with respect to the recoil effect and with respect to inelastic processes; the further obvious generalization would be to the calculation of the three-body final state, with the ionization of the atom. This is less immediate than the case of the excited bound states but still workable within this formalism.

With respect to the other problem, which was the main interesting point for us, i.e. the connection between form factors and external properties of the wave function, we can repeat that, even in this very simple model, it is less strict than in the original formulation.[1,2] (The problem of the locality of the currents, within the infinite component formalism, has also been dealt with by Y. Nambu, in a recent paper[7]). Still in any case the Lorentz properties (here in fact the Galilei properties) of the states define the functional dependence of the form factor on the momentum transfer. At this point, the connection between wave function and form factor in more interesting, but still soluble, examples, namely tightly bound systems, relativistic internal kinematics and possibly three-body systems, is still an open problem. The analysis of this problem, as indicated by presently known models, could be relevant for high energy physics.

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Coupled-State Calculations of Proton-Helium Scattering*

J. van den Bos†

Joint Institute for Laboratory Astrophysics,‡ Boulder, Colorado 80302

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Using the impact-parameter method, calculations have been done on the excitation of helium by protons in the kinetic-energy region of 10–10 000 keV. The total wave function of the system is expanded in a set of helium eigenstates including the $n=1$, $n=2$, and $n=3$ (except for 3^1S) states. The resulting linear differential equations (up to 18) have been solved numerically and compared with existing calculations and experiment. It is shown that for higher impact energies the many-state cross sections tend to the Born cross sections, but that for non-allowed transitions discrepancies exist up to very high energy. For allowed transitions, the agreement between theory and experiment is reasonable. Sublevel cross sections are shown to be very sensitive to the number of states retained in the expansion; only at the highest energies for which calculations have been done are they in agreement with Born sublevel cross sections. This fact is also shown by a comparison of calculated and measured polarization fractions of the emitted light induced by the excitations.

I. INTRODUCTION

Up till now the excitation of helium by protons in the keV energy region has been calculated in the Born and distortion approximations. Born results and a review of other work have been presented by the author in a previous paper,[1] henceforth referred to as I. The aim of this paper is to expand the total wave function of the system in an increasing number of target eigenstates and investigate whether the resulting cross sections show any convergence and at which energy, and how the theory agrees with experimental results.

For reasons of mathematical simplicity no projectile eigenstates were included, although it is felt that this severely limits the usefulness of the method. The states which have been included are 1^1S, 2^1S, 2^1P, 3^1P, and 3^1D including the magnetic substates. The collision plane was chosen to be the XZ plane, and the real representation of the substates was used, so that this set yielded a maximum of nine states. The combinations for which expansions have been made include the following: $1S$-$2S$-$2P$ [4 state], $1S$-$2P$ [3 state], $1S$-$3P$ [3 state], $1S$-$3D$ [4 state], $1S$-$2P$-$3D$ [6 state ($2P$)], $1S$-$3P$-$3D$ [6 state ($3P$)], $1S$-$2P$-$3P$-$3D$ [8 state], and $1S$-$2S$-$2P$-$3P$-$3D$ [9 state].

II. THEORY

We shall only give a short description of the theory; a more complete treatment has been given by Bates.[2] Atomic units will be used throughout, unless otherwise mentioned.

We assume that the proton is moving along a rectilinear path with collision parameter ρ and constant velocity $\vec{v}$. The trajectory is parallel to the Z axis of a fixed coordinate system with origin located at the helium nucleus. The internuclear distance is denoted by $\vec{R}$. Defining the electronic wave function of the system as $X(\vec{r},t)$, where $\vec{r}$ stands for the electron coordinates and t denotes the time, we may expand X in helium eigenstates $\Psi(\vec{r})$ with eigenenergy E_n,