Electromagnetic Radiation in Accelerated Systems

J. L. Anderson and J. W. Ryon
Stevens Institute of Technology, Hoboken, New Jersey 07030
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A generally covariant set of electromagnetic field equations and associated constitutive relations is developed to deal with electromagnetic radiation in arbitrarily moving media. The equations are sufficiently general to include dispersive as well as nonisotropic media. Several special cases are investigated to illustrate the method and to demonstrate the consistency of the formulation.

I. INTRODUCTION

Several recent papers on the subject of electromagnetic radiation in uniformly rotating systems have stirred interest in the general problem of radiation in arbitrarily moving systems. Such interest has two aspects: First is the problem of extending the theory so that it is logically consistent and, in the appropriate limit, it reduces to the usual form. Second is the possibility that experiments involving laser beams may prove of such precision to permit a choice between competing theoretical results.

The original rotating interferometer experiments were of three kinds. The details of these experiments have been amply reported in the review article of Post,[1] so that only their general nature will be outlined here. All the experiments involve a closed optical path incorporating both a light source and an interferometric detector. The light path may or may not, depending upon the type of experiment, be contained within a dielectric medium. The experiments of Sagnac,[2] Harras,[3] and Pogany[4] were all alike in that the interferometer and the dielectric medium rotated together. These investigators found a fringe shift between the clockwise and counterclockwise beams. The experiment of Dufour and Prunier[5] was of another type in that the medium was stationary while the interferometer was rotating. Finally, the experiment of Kantor[6] was of yet another type in that the medium rotated while the interferometer was stationary. A fringe shift due to rotation was also observed in these modified experiments, but its magnitude was found to depend upon the type of experiment.

The early theoretical investigations of the Sagnac-Harras-Pogany experiment were based upon kinematical and geometrical optical considerations and were provided by Harzer,[7] Einstein,[8] von Laue,[9] and Langevin.[10] Recent investigations have been based upon physical optical considerations and generally fall into two broad categories. Heer[11] has applied the machinery of general relativity to the problem of resonant frequencies in a rotating electromagnetic cavity. The chief inadequacy of his work lies in the *ad hoc* nature of the constitutive relations he employs. Post, Yildiz, and Tang[1,12,13] have used the naturally covariant form of the electromagnetic field equations to investigate all three types of rotational interferometer experiment. For reasons of computational efficiency, their approach is clearly superior to the general relativistic treatment. However, their method of selectively transforming the free-space portion and the matter portion of the constitutive tensor in different ways is an *ad hoc* and noncovariant procedure and can be shown to lead to inconsistencies.

The immediate motivation of the present paper is to develop a systematic procedure for handling the constitutive relations which will eliminate the inconsistencies of Post, Yildiz, and Tang (PYT) and which will be applicable to arbitrarily moving observers and media. In addition, equations capable of describing electromagnetic radiation in arbitrarily moving dispersive media will be obtained as a natural extension of the method. Lastly, the consistency of the present method will be demonstrated in the special case of uniform linear acceleration for which, in the limit of zero acceleration, the result is known.

The general procedure to be followed may be concisely stated: First, the naturally covariant form of the electromagnetic field equations and constitutive relations will be presented. Second, the generalized Minkowski constitutive relations will be explicitly developed. Third, Maxwell's equations will be written down and solved for various cases of uniform rotation and linear acceleration. Finally, the results will be checked for consistency and compared to those of PYT.

II. NATURALLY COVARIANT ELECTROMAGNETIC THEORY

Following Cartan,[14] Weyl,[15] and Post,[16] we write Maxwell's equations in a form which is covariant under

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[15] H. Weyl, *Space-Time-Matter* (Dover Publications Inc., New York, 1951), pp. 110 and 220.
[16] E. J. Post, *Formal Structure of Electromagnetics* (North-Holland Publishing Co., Amsterdam, 1962).

all holonomic coordinate transformations.

$$F_{[\mu\nu,\rho]}=0, \tag{1}$$

$$\mathcal{G}^{\mu\nu}{}_{,\nu}=0. \tag{2}$$

Here $F_{\mu\nu}$ and $\mathcal{G}^{\mu\nu}$ are the antisymmetric tensors representing **E**, **B**, and **D**, **H**, respectively. Observe the occurrence of the ordinary partial derivatives rather than the covariant derivative in Eqs. (1) and (2). The equations as written will preserve their form under any coordinate transformation of the form

$$F_{\mu'\nu'}=A_{\mu'}{}^{\rho}A_{\nu'}{}^{\sigma}F_{\rho\sigma}, \tag{3}$$

$$|A^{\lambda'}{}_{\kappa}|\,\mathcal{G}^{\mu'\nu'}=A^{\mu'}{}_{\rho}A^{\nu'}{}_{\sigma}\mathcal{G}^{\rho\sigma}, \tag{4}$$

where $A^{\mu'}{}_{\rho}$ and the condition of holonomicity are given, respectively, by

$$A^{\mu'}{}_{\rho}=\partial x'^{\mu}/\partial x^{\rho}, \tag{5}$$

$$A^{\mu'}{}_{[\rho,\sigma]}=0. \tag{6}$$

It should be clear from Eq. (4) that $\mathcal{G}^{\mu\nu}$ is a density of weight $+1$.

We should point out here that while Eqs. (1) and (2) hold for arbitrary coordinate systems, the problem is to discover the physical significance of such frames of reference. In inertial frames, the coordinates are unambiguously associated with inertial measuring devices. This is not the case for accelerated frames of reference. In Sec. IV we shall discuss this problem in detail.

The necessary connection between the two tensors $F_{\mu\nu}$ and $\mathcal{G}^{\mu\nu}$ is provided by the constitutive tensor. Thus, one writes the most general linear, instantaneous, and local relation between the two tensors as

$$\mathcal{G}^{\mu\nu}=\tfrac{1}{2}\chi^{\mu\nu\rho\sigma}F_{\rho\sigma}. \tag{7}$$

Relation (7) is capable of describing the general linear, nondispersive medium and so in particular neither the frame of reference nor the medium need be inertial. The constitutive tensor $\chi^{\mu\nu\rho\sigma}$ must transform as a density of weight $+1$ in order to ensure consistency of the formulation:

$$|A^{\lambda'}{}_{\kappa}|\,\chi^{\mu'\nu'\rho'\sigma'}=A^{\mu'}{}_{\alpha}A^{\nu'}{}_{\beta}A^{\rho'}{}_{\tau}A^{\sigma'}{}_{\epsilon}\chi^{\alpha\beta\tau\epsilon}. \tag{8}$$

The subsequent development crucially depends upon the explicit construction of the constitutive tensor which will be valid for an arbitrarily moving medium in an arbitrary coordinate system. With this goal in view, one introduces the following vectors and vector densities:

$$E_{\mu}=F_{\mu\nu}u^{\nu},\quad \mathcal{B}^{\sigma}=\tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}u_{\rho}, \tag{9}$$

$$\mathcal{D}^{\mu}=\mathcal{G}^{\mu\nu}u_{\nu},\quad H_{\sigma}=\tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}\mathcal{G}^{\mu\nu}u^{\rho}, \tag{10}$$

where u^{μ} is the local four-velocity of the medium and $\epsilon^{\mu\nu\rho\sigma}$ is the Levi-Civita density. Only if the medium is stationary in an inertial frame will the spatial components of E_{μ}, $\mathcal{B}^{\mu}$, $\mathcal{D}^{\mu}$, and H_{μ} reduce to the ordinary vector fields **E**, **B**, **D**, and **H**. Evidently, one has

$$E_{\mu}u^{\mu}=0,\quad \mathcal{B}^{\sigma}u_{\sigma}=0, \tag{11}$$

$$\mathcal{D}^{\mu}u_{\mu}=0,\quad H_{\sigma}u^{\sigma}=0. \tag{12}$$

One can easily show that Eqs. (9) and (11) together imply that

$$F_{\mu\nu}=\delta^{\alpha\beta}{}_{\mu\nu}E_{\alpha}u_{\beta}-\epsilon_{\mu\nu\alpha\beta}\mathcal{B}^{\alpha}u^{\beta}, \tag{13}$$

and similarly one obtains from Eqs. (10) and (12)

$$\mathcal{G}^{\mu\nu}=\delta^{\mu\nu}{}_{\alpha\beta}\mathcal{D}^{\alpha}u^{\beta}-\epsilon^{\mu\nu\alpha\beta}H_{\alpha}u_{\beta}. \tag{14}$$

If Eqs. (13) and (14) are substituted into Eqs. (1) and (2), then an alternative form of the electromagnetic field equations results:

$$(\delta^{\mu\nu}{}_{\alpha\beta}\mathcal{B}_{\alpha}u^{\beta})_{,\nu}-(\epsilon^{\mu\nu\alpha\beta}E_{\alpha}u_{\beta})_{,\nu}=0, \tag{15}$$

$$(\delta^{\mu\nu}{}_{\alpha\beta}\mathcal{D}^{\alpha}u^{\beta})_{,\nu}-(\epsilon^{\mu\nu\alpha\beta}H_{\alpha}u_{\beta})_{,\nu}=0. \tag{16}$$

Both Eqs. (15) and (16) take the form of a vanishing divergence of an antisymmetric second-rank density of weight $+1$. Thus, these equations are form-invariant under any holonomic coordinate transformation and offer a set of equations equally as valid as Eqs. (1) and (2).

III. CONSTITUTIVE RELATIONS

A. Nondispersive Media

The quantities E_{μ}, H_{μ}, $\mathcal{B}^{\mu}$, and $\mathcal{D}^{\mu}$ each have three independent components by virtue of Eqs. (11) and (12), and so these vectors have the important property of completely characterizing the electromagnetic field. This fact suggests that one can find suitable consitutive relations between these vectors. The simplest assumptions concerning these relations are first, that they are linear in the field quantities and second, that, in any inertial frame in which a region of the medium is locally and instantaneously at rest, they reduce to their nonrelativistic form. These requirements are met by the Minkowski[17] form of constitutive relations suitably reinterpreted to account for the lack of isotropy and homogeneity of the medium owing both to its structure and to its acceleration. Accordingly, one takes the constitutive relations to be

$$\mathcal{D}^{\mu}=\epsilon^{\mu\nu}E_{\nu}, \tag{17}$$

$$H_{\mu}=\mu^{-1}{}_{\mu\nu}\mathcal{B}^{\nu}. \tag{18}$$

Here, both the generalized dielectric constant $\epsilon^{\mu\nu}$ and the generalized inverse magnetic permeability $\mu^{-1}{}_{\mu\nu}$ may be functions of the coordinates x^{μ}, the local medium four-velocity u^{μ}, and derivatives of u^{μ}. We shall completely ignore derivatives of u^{μ} in the following development because of the absence of experimental results requiring such derivatives for an explanation. In the case of a medium stationary in an inertial frame, we shall

[17] H. Minkowski, Gött. Nachr. **53**; Math. Ann. **68**, 472 (1910).

see that relations (17) and (18) reduce to the usual result,

$$\mathfrak{D}^r=\epsilon^{rs}E_s\,, \tag{19}$$

$$H_r=\mu^{-1}{}_{rs}\mathfrak{B}^s. \tag{20}$$

It is possible to exhibit the constitutive relations in the form of Eq. (7) by introducing Eqs. (17) and (18) into (14) and employing the definitions given by Eq. (9). The result is

$$\mathcal{G}^{\mu\nu}=\tfrac{1}{2}(\delta_{\alpha\beta}{}^{\mu\nu}u^\beta\epsilon^{\alpha\kappa}u^\lambda\delta_{\kappa\lambda}{}^{\rho\sigma}-\epsilon^{\mu\nu\alpha\beta}u_\beta\mu^{-1}{}_{\alpha\lambda}u_\kappa\epsilon^{\kappa\lambda\rho\sigma})F_{\rho\sigma}. \tag{21}$$

The constitutive tensor $\chi^{\mu\nu\rho\sigma}$ is thus seen to be

$$\chi^{\mu\nu\rho\sigma}=\delta_{\alpha\beta}{}^{\mu\nu}u^\beta\epsilon^{\alpha\kappa}u^\lambda\delta_{\kappa\lambda}{}^{\rho\sigma}-\epsilon^{\mu\nu\alpha\beta}u_\beta\mu^{-1}{}_{\alpha\lambda}u_\kappa\epsilon^{\kappa\lambda\rho\sigma}. \tag{22}$$

In order for $\chi^{\mu\nu\rho\sigma}$ to satisfy the required symmetry properties[16]

$$\chi^{\mu\nu\rho\sigma}=\chi^{\rho\sigma\mu\nu}=-\chi^{\mu\nu\sigma\rho}=-\chi^{\nu\mu\rho\sigma}\,, \tag{23a}$$

$$\chi^{[\mu\nu\rho\sigma]}=0\,, \tag{23b}$$

it is necessary that $\epsilon^{\mu\nu}$ and $\mu^{-1}{}_{\mu\nu}$ be symmetric:

$$\epsilon^{\mu\nu}=\epsilon^{\nu\mu}\,, \tag{24a}$$

$$\mu^{-1}{}_{\rho\sigma}=\mu^{-1}{}_{\sigma\rho}. \tag{24b}$$

Equations (12), (17), (18), and (24) together imply the following relations:

$$u_\mu\epsilon^{\mu\nu}=\epsilon^{\nu\mu}u_\mu=0\,, \tag{25a}$$

$$u^\rho\mu^{-1}{}_{\rho\sigma}=\mu^{-1}{}_{\sigma\rho}u^\rho=0. \tag{25b}$$

If the medium is stationary in an inertial frame, Eqs. (25) require that

$$\epsilon^{0\mu}=\epsilon^{\mu 0}=0\,, \tag{26a}$$

$$\mu^{-1}{}_{0\rho}=\mu^{-1}{}_{\rho 0}=0\,, \tag{26b}$$

and so Eqs. (19) and (20) are obtained as a consequence.

In all the special cases to be investigated subsequently we shall be dealing with homogeneous, isotropic media for which

$$\epsilon^{\mu\nu}=(-g)^{1/2}\epsilon(g^{\mu\nu}-u^\mu u^\nu)\,, \tag{27a}$$

$$\mu^{-1}{}_{\rho\sigma}=(-g)^{-1/2}\mu^{-1}(g_{\rho\sigma}-u_\rho u_\sigma)\,, \tag{27b}$$

where ϵ and μ are scalar quantities and g is the determinant of the metric tensor $g_{\mu\nu}$. The constitutive relations (21) may then be written in the simplified form

$$\mathcal{G}^{\mu\nu}=(-g)^{1/2}[\epsilon\delta_{\alpha\beta}{}^{\mu\nu}u^\beta F^{\alpha\sigma}u_\sigma -\tfrac{1}{2}\mu^{-1}\epsilon^{\mu\nu\alpha\beta}u_\beta\mu^\kappa\epsilon_{\kappa\alpha\rho\sigma}F^{\rho\sigma}]. \tag{28a}$$

By making use of the properties of $\epsilon^{\mu\nu\rho\sigma}$ and $\delta_{\rho\sigma\beta}{}^{\mu\nu\alpha}$ we can cast (28a) into the following convenient form:

$$\mathcal{G}^{\mu\nu}=(-g)^{1/2}[\mu^{-1}F^{\mu\nu}+(\epsilon-\mu^{-1})\times(F^{\mu\sigma}u_\sigma u^\nu-F^{\nu\sigma}u_\sigma u^\mu)]. \tag{28b}$$

It is evident that $\mathcal{G}^{\mu\nu}$ has the correct transformation properties.

One must emphasize the generality of the explicit form of $\chi^{\mu\nu\rho\sigma}$ as given by Eq. (22). The field vectors E_μ, $\mathfrak{B}^\mu$, $\mathfrak{D}^\mu$, and H_μ were defined for an arbitrary system of reference and in terms of the local four-velocity of an arbitrarily moving medium. Furthermore, the constitutive relations as introduced in Eqs. (17) and (18) are the most natural for any coordinate system. One is drawn to conclude that Eq. (22) is the most general constitutive tensor for a linear, nondispersive, homogeneous, isotropic medium so long as derivatives of the local four-velocity are excluded. If derivatives of u^μ are permitted to occur in the consitutive relations then, of course, there are many other possibilities.

B. Dispersive Media

In order to extend the preceding analysis to account for dispersion, we must modify Eqs. (17) and (18) to include the noninstantaneous contribution of the fields to the polarization of the medium. The development is most easily carried out by means of the co-moving Lagrangian coordinates $(\tau,\xi)=\xi^\mu$ fixed in the medium. Then in any other frame of reference the world line of a given point of the medium will be given by

$$x^\mu=x^\mu(\tau,\xi). \tag{29}$$

We shall assume that in the co-moving coordinate system the relation between the fields and the polarization is purely local in space. This choice is made for convenience and simplicity and may easily be changed to include nonlocal as well as noninstantaneous effects.

The natural generalization of the constitutive relations (17) and (18) *in the co-moving system* is taken to be the integral relations

$$\mathfrak{D}^\mu(\tau,\xi)=\int d\tau'\;\epsilon^{\mu\nu}(\tau,\tau')E_\nu(\tau',\xi)\,, \tag{30}$$

$$H_\mu(\tau,\xi)=\int d\tau'\;\mu^{-1}{}_{\mu\nu}(\tau,\tau')\mathfrak{B}^\nu(\tau',\xi). \tag{31}$$

The procedure of integrating the above expressions along the world line, $\xi=$const, is relativistically covariant provided $\epsilon^{\mu\nu}(\tau,\tau')$ and $\mu^{-1}{}_{\rho\sigma}(\tau,\tau')$ transform appropriately.

$$|A^{\kappa'}{}_\lambda(\tau)|\,\epsilon^{\mu'\nu'}(\tau,\tau')=A^{\mu'}{}_\rho(\tau)A^{\nu'}{}_\sigma(\tau')\epsilon^{\rho\sigma}(\tau,\tau')\,, \tag{32}$$

$$|A^\kappa{}_{\lambda'}(\tau')|\,\mu^{-1}{}_{\mu'\nu'}(\tau,\tau')=A_{\mu'}{}^\rho(\tau)A_{\nu'}{}^\sigma(\tau')\mu^{-1}{}_{\rho\sigma}(\tau,\tau'). \tag{33}$$

If Eqs. (32) and (33) hold, it is straightforward to show that in any other frame of reference the constitutive relations take the form

$$\mathfrak{D}^\mu(x(\tau,\xi))=\int d\tau'\epsilon^{\mu\nu}(x(\tau,\xi),x(\tau',\xi))E_\nu(x(\tau',\xi))\,, \tag{34}$$

$$H_\mu(x(\tau,\xi))=\int d\tau'\mu^{-1}{}_{\mu\nu}(x(\tau,\xi),x(\tau',\xi))\mathfrak{B}^\nu(x(\tau',\xi))\,, \tag{35}$$

where we have suppressed the superscript on the coordinate x^μ. If the co-moving reference frame is inertial and the medium is isotropic then Eqs. (30) and (31) take the familiar form

$$\mathfrak{D}^r(\tau)=\int d\tau'\ \epsilon(\tau,\tau')E^r(\tau'), \quad (36)$$

$$H_r(\tau)=\int d\tau'\ \mu^{-1}(\tau,\tau')\mathfrak{B}_r(\tau'). \quad (37)$$

However, we do not obtain quite this simple form if the co-moving reference frame is noninertial. The reason for this can be traced to the implied parallel transport of the vectors E_μ and $\mathfrak{B}^\mu$ in the constitutive integrals. To see how this comes about let us look at Eq. (34) specialized to a noninertial, isotropic medium viewed from an inertial frame. For this case, Eq. (34) becomes

$$\mathfrak{D}^\mu(x(\tau,\xi))=\int d\tau'\ \epsilon(\tau,\tau')\eta^{\mu\nu}E_\nu(x(\tau',\xi)), \quad (38)$$

where $\eta^{\mu\nu}$ is the metric tensor for an inertial frame with nonvanishing components $\eta^{00}=-\eta^{11}=-\eta^{22}=-\eta^{33}=1$. After we transform to the co-moving reference frame, we have instead of Eq. (38)

$$\begin{aligned}\mathfrak{D}^{\mu'}(\tau,\xi)&=\int d\tau'[-g(\tau,\xi)]^{1/2}\epsilon(\tau,\tau')A^{\mu'}{}_\rho(\tau,\xi)\\&\qquad\times\eta^{\rho\sigma}A^{\nu'}{}_\sigma(\tau',\xi)E_{\nu'}(\tau',\xi),\\&=\int d\tau'[-g(\tau,\xi)]^{1/2}\epsilon(\tau,\tau')\\&\qquad\times T^{\mu'\nu'}(\tau,\tau',\xi)E_{\nu'}(\tau',\xi),\end{aligned} \quad (39)$$

where we have defined a transport tensor

$$T^{\mu'\nu'}(\tau,\tau',\xi)=A^{\mu'}{}_\rho(\tau,\xi)\eta^{\rho\sigma}A^{\nu'}{}_\sigma(\tau',\xi) \quad (40)$$

which parallel-transports the vector $E_{\nu'}(\tau',\xi)$ from the space-time point (τ',ξ) to the point (τ,ξ). This is proved in the Appendix.

The constitutive relations may be expressed in a form analogous to Eq. (7) if we insert Eqs. (34) and (35) into Eq. (14) and employ the definitions of Eq. (9). The result is

$$\mathcal{G}^{\mu\nu}(x)=\frac{1}{2}\int d\tau'\chi^{\mu\nu\rho\sigma}(x,x')F_{\rho\sigma}(x'), \quad (41)$$

where

$$\begin{aligned}\chi^{\mu\nu\rho\sigma}(x,x')=&-\delta_{\alpha\beta}{}^{\mu\nu}u^\alpha(x)\epsilon^{\beta\kappa}(x,x')u^\lambda(x')\delta_{\kappa\lambda}{}^{\rho\sigma}\\&-\epsilon^{\mu\nu\alpha\beta}u_\alpha(x)\mu^{-1}{}_{\beta\kappa}(x,x')u_\lambda(x')\epsilon^{\kappa\lambda\rho\sigma}.\end{aligned} \quad (42)$$

For an isotropic medium Eq. (42) reduces to the following:

$$\begin{aligned}\chi^{\mu\nu\rho\sigma}(x,x')=&-\epsilon(x,x')\delta_{\alpha\beta}{}^{\mu\nu}u^\alpha(x)T^{\beta\kappa}(x,x')u^\lambda(x')\delta_{\kappa\lambda}{}^{\rho\sigma}\\&-\mu^{-1}(x,x')\epsilon^{\mu\nu\alpha\beta}u_\alpha(x)T_{\beta\kappa}(x,x')u_\lambda(x')\epsilon^{\kappa\lambda\rho\sigma}.\end{aligned} \quad (43)$$

Here again we see that parallel transport is built in to the constitutive relations.

IV. COORDINATE TRANSFORMATIONS

A. General Remarks

We have developed a general mathematical formalism for describing electromagnetic radiation in arbitrarily moving media by means of arbitrary reference frames, the motion of which need not coincide with the motion of the medium. Before this formalism is applied to specific cases it is necessary to decide upon the appropriate description of the motion of either an accelerated medium or an accelerated reference frame. This is most conveniently done from the point of view of an observer in an inertial reference frame. In the first case, the inertial observer specifies the world line of each point of the medium as a function of the proper time τ along each trajectory. In the second case the inertial observer specifies the world line of each clock in the noninertial reference frame as a function of the coordinate time t' recorded by the accelerated clock. Let ξ^r specify a given point of the medium and x'^r specify a given clock in the noninertial frame. Then if x^μ are coordinates in the inertial frame, we have the following equations for the world lines:

$$x^\mu=x^\mu(\tau,\xi^r), \quad (44a)$$

$$x^\mu=x^\mu(t',x'^r). \quad (44b)$$

It is important to realize that in neither case can the equations for the world lines be given *a priori* as they can be when we deal with unaccelerated media or reference frames. The trajectories of points in a medium are determined by the equations of motion once we know the relevant forces which act on the medium. In general, we can expect the motion of the medium to be quite complex. Thus in the case of a rotating solid there must be compression and/or shear motion,[18] particularly if the system is large, to prevent the outer regions from exceeding the speed of light. The motion of the noninertial clocks may also be very complex because each clock may, within reasonable limits, move independently. What we shall do to avoid much of this complexity is to choose simple families of trajectories for both medium points and coordinate clocks which seem to be reasonable candidates for the description of observed types of accelerated motion. The correspondence of the description with actual systems can only be judged by observation.

Once we choose a family of world lines, i.e., noninertial reference frame, there are three kinds of observations we shall consider:

Case I, observer inertial, medium accelerated;
Case II, observer accelerated, medium inertial;
Case III, observer and medium coaccelerated.

[18] See, for example, B. Kursunoglu, Proc. Comb. Phil. Soc., 47, 177 (1961).

B. Constant Linear Acceleration

We consider now the types of coordinate transformation of the form $x^\mu = x^\mu(x'^\nu)$, which describe constant linear acceleration. In general, there are an infinite number of inequivalent descriptions which fit the condition of constant acceleration. By constant acceleration we mean that the proper acceleration of each point x'^r does not depend upon the proper time τ. For one-dimensional motion along the x direction in the inertial frame the condition for constant acceleration is

$$a^\mu a_\mu = \left(\frac{\partial^2 t}{\partial\tau^2}\right)^2 - \left(\frac{\partial^2 x}{\partial\tau^2}\right)^2 = -a^2(x'). \tag{45}$$

This may be satisfied in the usual way by the assignment

$$\partial^2 t/\partial\tau^2 = a(x') \sinh a(x')\tau, \tag{46a}$$

$$\partial^2 x/\partial\tau^2 = a(x') \cosh a(x')\tau. \tag{46b}$$

The boundary conditions at $\tau=0$ are taken to be

$$t(0)=0, \quad x(0)=x',$$
$$\left.\frac{\partial t}{\partial\tau}\right|_{\tau=0} = 1, \quad \left.\frac{\partial x}{\partial\tau}\right|_{\tau=0} = 0, \tag{47}$$

and are easily satisfied by the relations

$$t = a^{-1}(x') \sinh a(x')\tau, \tag{48a}$$

$$x = a^{-1}(x') \cosh a(x')\tau + x' - a^{-1}(x'). \tag{48b}$$

These equations, along with

$$y = y', \tag{48c}$$

$$z = z', \tag{48d}$$

are general transformation equations for constant, one-dimensional acceleration.

The function $a(x')$ may be chosen arbitrarily, but only two choices are of immediate physical interest. These are

$$a(x') = a = \text{const}, \tag{49a}$$

$$a(x') = 1/x'. \tag{49b}$$

Of these two alternatives, the first, (49a), has the virtue that the velocity of any point x'^r depends only upon τ for any inertial observer, while the second, (49b), has the virtue that x'^r and τ yield directly meter-stick and clock measurements in the noninertial system.

Which of the two alternatives we choose depends upon the physical conditions of the problem. For example, if the medium were a gas of noninteracting particles, each of which were independently accelerated by an applied force field, then the proper choice would be (49a). On the other hand, if the medium were a solid dielectric and we wished to preserve the Born condition for a rigid body, then the choice would be (49b). Because the condition (49a) implies a velocity field independent of x', the field equations can be solved exactly in closed form. For this reason we shall take the transformation equations to be those appropriate to (49a):

$$t = a^{-1} \sinh a\tau, \tag{50a}$$

$$x = x' + a^{-1}(\cosh a\tau - 1), \tag{50b}$$

$$y = y', \tag{50c}$$

$$z = z'. \tag{50d}$$

Observe that (50b) may be rewritten with the aid of (50a) in the following form:

$$x = x' + a^{-1}[(1+a^2t^2)^{1/2} - 1]. \tag{51}$$

Also, by virtue of (50a), the planes of simultaneity in the inertial frame are planes of simultaneity in the accelerated frame. This means we can relabel the time measure in the accelerated frame to obtain a simpler set of transformation equations. We take the new time measure to be

$$t' = a^{-1} \sinh a\tau, \tag{52}$$

so that the transformation equations take the form

$$t = t', \tag{53a}$$

$$x = x' + a^{-1}[(1+a^2t'^2)^{1/2} - 1], \tag{53b}$$

$$y = y', \tag{53c}$$

$$z = z'. \tag{53d}$$

One should note that t' no longer has the significance of proper time. Equations (53) represent no loss in generality and a considerable gain in computational simplicity. At the end of our calculation we can always transform back to the situation represented by (50).

C. Uniform Rotation

We turn now to a choice of coordinate transformation representing uniform rotation. Because our present interest in this type of motion is simply to compare the results of the present method with those of PYT, we adopt the same transformation as those authors. In cylindrical coordinates, the transformation is

$$t = t', \tag{54a}$$

$$r = r', \tag{54b}$$

$$\theta = \theta' + \Omega t', \tag{54c}$$

$$z = z', \tag{54d}$$

where the primed coordinates refer to the rotating system. In Cartesian coordinates the transformation is

$$t = t', \tag{55a}$$

$$x = x' \cos\Omega t' - y' \sin\Omega t', \tag{55b}$$

$$y = x' \sin\Omega t' + y' \cos\Omega t', \tag{55c}$$

$$z = z'. \tag{55d}$$

TABLE I. Constitutive relations for all three cases of *constant linear acceleration* calculated according to Eq. (28b).

Case I	$\mathbf{D}=\epsilon\mathbf{E}+(\epsilon-\mu^{-1})\gamma^2[\mathbf{v}\times\mathbf{B}-\mathbf{v}\times(\mathbf{v}\times\mathbf{E})]$
	$\mathbf{H}=\mu^{-1}\mathbf{B}+(\epsilon-\mu^{-1})\gamma^2[\mathbf{v}\times\mathbf{E}+\mathbf{v}\times(\mathbf{v}\times\mathbf{B})]$
Case II	$\mathbf{D}=\epsilon\mathbf{E}-\epsilon\mathbf{v}\times\mathbf{B}$
	$\mathbf{H}=\mu^{-1}\mathbf{B}-\epsilon\mathbf{v}\times\mathbf{E}+\epsilon\mathbf{v}\times(\mathbf{v}\times\mathbf{B})$
Case III	$\mathbf{D}=\epsilon\mathbf{E}-\mu^{-1}\mathbf{v}\times\mathbf{B}-(\epsilon-\mu^{-1})\gamma^2\mathbf{v}\times(\mathbf{v}\times\mathbf{E})$
	$\mathbf{H}=\mu^{-1}\mathbf{B}-\mu^{-1}\mathbf{v}\times\mathbf{E}+\mu^{-1}\mathbf{v}\times(\mathbf{v}\times\mathbf{B})$

V. SPECIAL CASES OF NONDISPERSIVE MEDIA

A. Procedure

In this section we shall apply the formalism developed in the preceding sections to various special cases of homogeneous, isotropic, and nondispersive media. We require the medium to be at rest in either an inertial frame or one of the accelerated frames discussed in the last section. Either observer can conduct experiments with the medium which may or may not be at rest in his frame of reference. Of the four possible experiments there are the three previously mentioned cases which we shall consider. Since the specification of the case number (I, II, or III) serves to specify both the motion of the observer and the motion of the medium, it is superfluous to further distinguish coordinate systems by primes or other labels on tensor components. Accordingly, we shall dispense with such labels and treat all three cases together with a unified notation.

The procedure to be followed is identical in all cases. The first step is the explicit evaluation of the consitutive relations which proceed from the definition of $F_{\mu\nu}$. One then calculates $F^{\mu\nu}$ and $F^{\mu\nu}u_\nu$ using the metric tensor and medium four-velocity appropriate to the particular case. The results of this calculation are substituted in the constitutive relation given by Eq. (28b), which yields the desired relations between the ordinary fields **E**, **B**, **D**, and **H**. The expressions for **D** and **H** are next substituted into the field equations (2) which must then be solved, along with Eq. (1), for the quantities **E** and **B**.

B. Constant Linear Acceleration

The constitutive relations, calculated by means of the transformation equations (53) for all three cases of constant linear acceleration, are given in Table I. For comparison purposes Table II gives the equivalent expressions calculated according to the PYT prescription. The various quantities appearing in Table I and Table II are listed here:

$$\mathbf{v}(t)=(v(t),0,0), \tag{56}$$

$$v(t)=dx/dt=at(1+a^2t^2)^{-1/2}, \tag{57}$$

$$\gamma=[1-v^2(t)]^{-1/2}. \tag{58}$$

The quantity $\mathbf{v}(t)$ is the velocity of any point of the accelerated reference frame as measured by the inertial observer. The velocity of any point of the inertial reference frame as measured by the accelerated observer is given by $-\mathbf{v}(t')$.

The significant differences between Table I and Table II are as follows:

(1) Factors of γ appear only in Table I.
(2) The term $\mathbf{v}\times(\mathbf{v}\times\mathbf{E})$ appears only in Table I.
(3) The quantity μ_0 appears only in Table II.

Even though the first two of these differences involve terms only of second order in $v(t)$, we shall see that these terms are most important in establishing the connection between the results of the present formalism and those for inertial media and inertial observers.

We shall look for solutions of the field equations that correspond to plane waves traveling in the x direction. This means the fields depend only upon x and t. With this condition in mind, we substitute the contents of Table I into the field equations (2) which, along with equations (1), become

$$\partial_x B_x=0,\quad \partial_t B_x=0, \tag{59a}$$

$$\partial_x E_x=0,\quad \partial_t E_x=0, \tag{59b}$$

$$\partial_t B_y=\partial_x E_z,\quad \partial_t(\alpha E_z+\lambda B_y)=\partial_x(\beta B_y-\lambda E_z), \tag{59c}$$

$$\partial_t B_z=-\partial_x E_y,\quad \partial_t(\alpha E_y-\lambda B_z)=-\partial_x(\beta B_z+\lambda E_y). \tag{59d}$$

The quantities α, β, and λ are defined in Table III for each of the three cases. Notice that we have introduced the index of refraction

$$n=(\epsilon\mu)^{1/2}$$

and that α, β, and λ are functions of the time through the velocity.

It is evident from (59a) and (59b) that E_x and B_x are constants and consequently may be taken equal to zero. Furthermore, there is only a difference in the relative sign of the field components between (59c) and (59d). Thus we concentrate on a set of equations of the form

$$\partial_t B+\partial_x E=0, \tag{60a}$$

$$\partial_t(\alpha E-\lambda B)+\partial_x(\beta B+\lambda E)=0, \tag{60b}$$

where α, β, and λ are functions of t but not of x.

To obtain a solution to (60), we first differentiate (60b) with respect to x and replace $\partial_x E$ by $-\partial_t B$, in accordance with (60a). The resulting equation for

TABLE II. Constitutive relations for all three cases of *constant linear acceleration* calculated according to PYT.

Case I	$\mathbf{D}=\epsilon\mathbf{E}+(\epsilon-\mu_0^{-1})\mathbf{v}\times\mathbf{B}$
	$\mathbf{H}=\mu^{-1}\mathbf{B}+(\epsilon-\mu_0^{-1})[\mathbf{v}\times\mathbf{E}+\mathbf{v}\times(\mathbf{v}\times\mathbf{B})]$
Case II	$\mathbf{D}=\epsilon\mathbf{E}-\epsilon\mathbf{v}\times\mathbf{B}$
	$\mathbf{H}=\mu^{-1}\mathbf{B}-\epsilon\mathbf{v}\times\mathbf{E}+\epsilon\mathbf{v}\times(\mathbf{v}\times\mathbf{B})$
Case III	$\mathbf{D}=\epsilon\mathbf{E}-\mu_0^{-1}\mathbf{v}\times\mathbf{B}$
	$\mathbf{H}=\mu^{-1}\mathbf{B}-\mu_0^{-1}\mathbf{v}\times\mathbf{E}+\mu_0^{-1}\mathbf{v}\times(\mathbf{v}\times\mathbf{B})$

$B(x,t)$ is

$$\partial_t(\alpha\partial_t B)+2\lambda\partial_t\partial_x B+\dot{\lambda}\partial_x B-\beta\partial_x^2 B=0, \tag{61}$$

where

$$\dot{\lambda}=\partial_t\lambda=d\lambda/dt. \tag{62}$$

Let us look for a solution of (61) of the form

$$B(x,t)=B(t)e^{ikx}. \tag{63}$$

The equation for $B(t)$ is then

$$\partial_t(\alpha\partial_t B)+ik(\dot{\lambda}B+2\lambda\partial_t B)+k^2\beta B=0. \tag{64}$$

The substitution

$$B(t)=e^{-i\phi(t)}, \tag{65}$$

$$\phi(t)=\int_0^t dt'\frac{\theta(t')+k\lambda(t')}{\alpha(t')}, \tag{66}$$

transforms (64) an equation for $\theta(t)$,

$$-i\alpha\dot{\theta}-\theta^2+k^2(\alpha\beta+\lambda^2)=0. \tag{67}$$

Using the explicit forms given in Table III, we find that

$$\alpha\beta+\lambda^2=1/n^2=\text{const} \tag{68}$$

for all three cases. The equation for θ is then

$$-i\alpha\dot{\theta}-\theta^2+k^2/n^2=0. \tag{69}$$

It is clear that one solution of (69) is certainly

$$\theta=\pm k/n, \tag{70a}$$

$$\dot{\theta}=0. \tag{70b}$$

The simple solution for $\theta(t)$ provides us with an exact solution of the field equations

$$B(x,t)=e^{ik[x-y(t)]}, \tag{71a}$$

$$y(t)=\int_0^t dt'\frac{\lambda(t')\pm 1/n}{\alpha(t')}. \tag{71b}$$

The velocity of the wave is given by

$$u(t)\equiv dx/dt=dy/dt, \tag{72a}$$

$$u(t)=[\lambda(t)\pm n^{-1}]/\alpha(t). \tag{72b}$$

Substitution of the expressions for α and λ from Table III into Eq. (72b) gives the velocity of the wave for all three cases.

(Case I) $u(t)=[v(t)\pm n^{-1}]/[1\pm n^{-1}v(t)]$, (73a)

(Case II) $u(t)=-v(t)\pm n^{-1}$, (73b)

(Case III) $u(t)=\pm n^{-1}[1-v^2(t)]/[1\pm n^{-1}v(t)]$. (73c)

We recognize the result for Case I as the relativistic velocity addition law which is known to hold when v is independent of t. In fact, Case I is nothing but the Fizeau experiment with the addition of an accelerated dielectric.

TABLE III. Definition of the coefficients α, β, λ which appear in Eq. (59) for all three cases of constant linear acceleration.

	α	β	λ
Case I	$1+(1-n^{-2})\gamma^2v^2$	$n^{-2}-(1-n^{-2})\gamma^2v^2$	$(1-n^{-2})\gamma^2v$
Case II	1	$n^{-2}-v^2$	$-v$
Case III	$1+(1-n^{-2})\gamma^2v^2$	$n^{-2}(1-v^2)$	$-n^{-2}v$

The emergence of the velocity addition law is compelling evidence that the formalism is correct. This result is quite sensitive to the precise form of the constitutive relations and any change in the quantities α, β, or λ would alter the result appreciably. In particular the PYT prescription will *not* yield the velocity addition law but something quite different, even in the limit of constant v.

We should point out that the wave velocity for Case III (medium and observer coaccelerated) which one expects to be just $\pm 1/n$ is quite different, due to our choice of coordinates. The expression for $u(t)$ represents the velocity calculated from coordinate distance and time, not physical distance and time. In fact $u(t)$ for Case III may be derived from the expression for $u(t)$ for Case I by means of the coordinate transformation equations (53). This is further evidence of the consistency of the formalism.

The expression for $B(x,t)$ as given by (71a) does not have a time dependence of the form $e^{-i\omega t}$, which means we cannot regard the solution as the result of driving the medium with a constant frequency generator. Let us require that the observer drive the medium at the point $x=0$ with a generator having an output signal of the form $e^{-i\omega t}$ between the times $-T$ and $+T$. Then, to find the response of the medium we look for a superposition of solutions of the form (71a) which matches the generator signal at $x=0$. Thus, we seek an $A(k)$ such that the expression

$$B(x,t)=\int_{-\infty}^{\infty} dk\, A(k)e^{ik[x-y(t)]} \tag{74}$$

will, at $x=0$, reduce to the prescribed result

$$B(0,t)=\theta(T+t)\theta(T-t)e^{-i\omega t}, \tag{75}$$

where $\theta(t)$ is the unit step function.

Now if $y(t)$ is a monotonically varying function, we may solve for t by inverting the function $y(t)$:

$$t=Y(y). \tag{76}$$

Then y may be taken as an independent variable and the Fourier inversion theorem may be used to obtain an expression for $A(k)$:

$$A(k)=\int_{\infty}^{\infty}\frac{dy}{2\pi}\,\theta[T+Y(y)]\theta[T-Y(y)]e^{i[ky-\omega Y(y)]}. \tag{77}$$

TABLE IV. Constitutive relations for all three cases of *uniform rotation* calculated according to Eq. (28b).

Case I	$\mathbf{D}=r\epsilon\mathbf{S}\cdot\mathbf{E}+r(\epsilon-\mu^{-1})\gamma^2[\mathbf{v}\times\mathbf{B}-r^2\mathbf{v}\times(\mathbf{v}\times\mathbf{E})]$
	$\mathbf{H}=(r\mu\mathbf{S})^{-1}\cdot\mathbf{B}+r(\epsilon-\mu^{-1})\gamma^2[\mathbf{v}\times\mathbf{E}+\mathbf{v}\times(\mathbf{v}\times\mathbf{B})]$
Case II	$\mathbf{D}=r\epsilon\mathbf{S}\cdot\mathbf{E}-r\epsilon\mathbf{v}\times\mathbf{B}$
	$\mathbf{H}=(r\mu\mathbf{S})^{-1}\cdot\mathbf{B}-r\epsilon\mathbf{v}\times\mathbf{E}+r\epsilon\mathbf{v}\times(\mathbf{v}\times\mathbf{B})$
Case III	$\mathbf{D}=r\epsilon\mathbf{S}\cdot\mathbf{E}-r\mu^{-1}\mathbf{v}\times\mathbf{B}-r^3(\epsilon-\mu^{-1})\gamma^2\mathbf{v}\times(\mathbf{v}\times\mathbf{E})$
	$\mathbf{H}=(r\mu\mathbf{S})^{-1}\cdot\mathbf{B}-r\mu^{-1}\mathbf{v}\times\mathbf{E}+r\mu^{-1}\mathbf{v}\times(\mathbf{v}\times\mathbf{B})$

When (77) is substituted into (74) and the order of integration interchanged, the following result is readily obtained:

$$B(x,t)=\theta[T-Y(y(t)-x)]\theta \times[T+Y(y(t)-x)]e^{-i\omega Y(y(t)-x)}. \quad (78)$$

The rather complex appearing expression for $B(x,t)$ is actually quite simple to interpret. First we see that $B(x,t)$ vanishes for values of x and t such that $Y(y(t)-x)$ is outside the range $-T$ to $+T$. The values of x and t for which we have

$$Y(y(t)-x)=\pm T$$

are the coordinates of the edges of wave packet and satisfy

$$dx/dt=dy/dt=u(t)=[\lambda(t)+n^{-1}]/\alpha(t). \quad (79)$$

Thus, the wave packet $B(x,t)$ does not spread in time but moves with a time-dependent speed given by (79) or equivalently (72b). Inside the region of nonvanishing $B(x,t)$ we have the simple expression

$$B(x,t)=e^{-i\omega Y(y(t)-x)}. \quad (80)$$

We see from (80) that the phase velocity of the wave is also given by (79). Let us expand $Y(y(t)-x)$ about the point $x=0$. In doing so, we use the fact that $Y(y(t))\equiv t$ and $d/dy=(1/u)d/dt$. We obtain

$$Y(y(t)-x)=t-\frac{x}{u}+\tfrac{1}{2}x^2\frac{1}{u}\frac{d}{dt}\left(\frac{1}{u}\right)+\cdots. \quad (81)$$

For sufficiently small values of x, we have approximately from (80) and (81)

$$B(x,t)=e^{i[k(t)x-\omega t]}, \quad (82)$$

which is a plane wave with a time-varying wave vector

$$k(t)=\omega/u(t). \quad (83)$$

We can find a better approximation than (82) with a little more effort. We use the explicit form of $v(t)$ in the expression for $u(t)=u(v(t))$ and then expand as a power series in t; substitute the series into the expansion for $Y(y(t)-k)$ and collect the successive powers of t.

For Case III (medium and observer coaccelerated) Eq. (76) holds and we have, from (57), (58), and (73c),

$$u_{\rm III}^{-1}(t)=n[1+n^{-1}v(t)]\gamma^2=n(1+a^2t^2)+at(1+a^2t^2)^{1/2} =n+at+na^2t^2+\tfrac{1}{2}a^3t^3-\tfrac{1}{8}a^5t^5+\cdots. \quad (84)$$

The final result of putting (84) into (81) is

$$Y(y(t)-x)=-nx[1-\tfrac{1}{2}ax+\tfrac{1}{6}(1+2n^2)(ax)^2+\cdots] +t[1-ax+\tfrac{1}{2}(1+2n^2)(ax)^2+\cdots] -nx(at)^2[1-\tfrac{3}{2}ax+\cdots]+\cdots. \quad (85)$$

The ratio of the term quadratic in t to the term linear in t is of the order $n(ax)(at)$, which can be chosen to be small. In this case we may rewrite (80) in the form

$$B(x,t)=e^{i[k(x)x-\omega(x)t]}, \quad (86)$$

where

$$k(x)=n\omega[1-\tfrac{1}{2}ax+\tfrac{1}{6}(1+2n^2)(ax)^2], \quad (87a)$$

$$\omega(x)=\omega[1-ax+\tfrac{1}{2}(1+2n^2)(ax)^2]. \quad (87b)$$

Equation (87b) gives the frequency shift of the wave as a function of x correct to second order in ax. Observe that the first-order correction is just the usual gravitational frequency shift effect and is independent of the properties of the medium. Only the second-order correction term includes specific effects due to the dielectric properties of the medium.

The order of magnitude of the second-order frequency correction may be obtained by choosing representative values for a and x:

$$a=g\approx 10\ \text{m/sec}^2\approx 10^{-16}/\text{m},$$
$$x=10\ \text{m},$$
$$ax\approx 10^{-15}.$$

We see that the gravitational first-order effect is barely at the limits of present-day measuring techniques so that the second-order dielectric contribution is quite undetectable.

C. Uniform Rotation

Just as for the cases of constant linear acceleration, we calculate the explicit form of the constitutive relations for uniform rotation. The result of the calculation is presented in Table IV. Table V presents the same information calculated according to the PYT prescription. We note the same kind of differences between Tables IV and V as between Tables I and II.

The quantities appearing in Tables IV and V are listed here.

$$\mathbf{v}=(0,\Omega,0), \quad (88)$$

$$\gamma=(1-r^2\Omega^2)^{-1/2}, \quad (89)$$

$$\mathbf{S}=\mathrm{diag}(1,r^{-2},1). \quad (90)$$

In this case we are looking for azimuthal waves and so we seek solutions of the form $f(r)e^{i(\kappa\theta-\omega t)}$. When the contents of Table IV are introduced into the field equations (1) and (2) we obtain the following set of equations for azimuthal waves:

$$0=\partial_r B_r+i\kappa B_\theta\,, \tag{91a}$$

$$0=i\omega B_r+i\kappa E_z\,, \tag{91b}$$

$$0=i\omega B_\theta-\partial_r E_z\,, \tag{91c}$$

$$0=-i\omega B_z+\partial_r E_\theta-i\kappa E_r\,, \tag{91d}$$

$$0=\partial_r(\alpha r E_r+\lambda B_z)+i\kappa r^{-1}E_\theta\,, \tag{91e}$$

$$0=-i\omega(\alpha r E_r+\lambda B_z)-i\kappa(\beta r^{-1}B_z-\lambda E_r)\,, \tag{91f}$$

$$0=-i\omega(r^{-1}E_\theta)+\partial_r(\beta r^{-1}B_z-\lambda E_r)\,, \tag{91g}$$

$$0=-i\omega(\alpha r E_z-\lambda B_r)-\partial_r(rn^{-2}B_\theta)+i\kappa(\beta r^{-1}B_r+\lambda E_z)\,. \tag{91h}$$

The quantities α, β, and λ are identical to the quantities in Table III, provided we replace v by $r\Omega$.

Observe that these equations may be divided into two sets of four equations each. Equations (91a)–(91c) and (91h) form one set determining the quantities E_z, B_r and B_θ; Eqs. (91d)–(91g) form the other set, determining the quantities E_r, E_θ, and B_z. From the first set of equations we eliminate B_r and B_θ and are left with an equation for E_z:

$$[n^{-2}\partial_r r\partial_r+(\omega^2\alpha r-2\lambda\kappa\omega-\beta r^{-1}\kappa^2)]E_z=0\,. \tag{92}$$

From the second set of equations we eliminate E_r and E_θ and are left with an equation for B_z:

$$[n^{-2}\partial_r r\partial_r+(\omega^2\alpha r-2\lambda\kappa\omega-\beta r^{-1}\kappa^2)]\times[B_z/(\kappa\lambda-\omega r\alpha)]=0\,. \tag{93}$$

We see that the two modes of oscillation are governed by the same differential equation.

If we neglect all terms of second order and higher in the quantity $r\Omega$ then the equation for E_z reduces to a Bessel equation. We could attempt a perturbative solution using the appropriate Bessel functions as a basis for an expansion. Instead, we shall employ the geometrical optical approximation so that we can compare our results to those of PYT.

The geometrical optical approximation consists of considering an idealized beam traveling the path $r=r_0$ and of supposing the r dependence of the fields to vanish. Given these conditions the differential equations (92) and (93) immediately yield the result

$$\alpha(\omega r_0/\kappa)^2-2\lambda(\omega r_0/\kappa)-\beta=0\,. \tag{94}$$

The solution of the quadratic equation (94) is

$$\omega r_0/\kappa=[\lambda\pm(\lambda^2+\alpha\beta)^{1/2}]/\alpha \tag{95a}$$

or

$$\omega r_0/\kappa=(\lambda\pm n^{-1})/\alpha\,. \tag{95b}$$

Here (68) has been used to obtain (95b). We recognize the equivalence between this result and Eq. (72b). The ratio $\omega r_0/\kappa$ is just the azimuthal velocity of the wave. For each of these three cases the azimuthal velocity is

TABLE V. Constitutive relations for all three cases of *uniform rotation* calculated according to PYT.

Case I	$\mathbf{D}=r\epsilon\mathbf{S}\cdot\mathbf{E}+r(\epsilon-\mu_0^{-1})\mathbf{v}\times\mathbf{B}$ $\mathbf{H}=(r\mu\mathbf{S})^{-1}\cdot\mathbf{B}+r(\epsilon-\mu_0^{-1})[\mathbf{v}\times\mathbf{E}+\mathbf{v}\times(\mathbf{v}\times\mathbf{B})]$
Case II	$\mathbf{D}=r\epsilon\mathbf{S}\cdot\mathbf{E}-r\epsilon\mathbf{v}\times\mathbf{B}$ $\mathbf{H}=(r\mu\mathbf{S})^{-1}\cdot\mathbf{B}-r\epsilon\mathbf{v}\times\mathbf{E}+r\epsilon\mathbf{v}\times(\mathbf{v}\times\mathbf{B})$
Case III	$\mathbf{D}=r\epsilon\mathbf{S}\cdot\mathbf{E}-r\mu_0^{-1}\mathbf{v}\times\mathbf{B}$ $\mathbf{H}=(r\mu\mathbf{S})^{-1}\cdot\mathbf{B}-r\mu_0^{-1}\mathbf{v}\times\mathbf{E}+r\mu_0^{-1}\mathbf{v}\times(\mathbf{v}\times\mathbf{B})$

given explicitly by the following formulas:

$$\text{(Case I)}\quad \omega r_0/\kappa=(r_0\Omega\pm n^{-1})/(1\pm n^{-1}r_0\Omega)\,, \tag{96a}$$

$$\text{(Case II)}\quad \omega r_0/\kappa=-r_0\Omega\pm n^{-1}\,, \tag{96b}$$

$$\text{(Case III)}\quad \omega r_0/\kappa=\pm n^{-1}(1-r_0^2\Omega^2)/(1\pm n^{-1}r_0\Omega)\,. \tag{96c}$$

Again we see that Case I expresses the relativistic velocity addition law.

The speeds of two azimuthal waves traveling in opposite directions may be obtained directly from (95b):

$$u_1=(n^{-1}+\lambda)/\alpha\,, \tag{97a}$$

$$u_2=(n^{-1}-\lambda)/\alpha\,. \tag{97b}$$

The difference in speeds is then given by

$$\Delta u=2\lambda/\alpha\,. \tag{98}$$

Table VI gives the explicit form of (98) (in units of $2r_0\Omega$) for the three cases before and after terms of the order of $(r_0\Omega)^2$ are neglected.

In order to compare the contents of Table VI with the PYT results we must convert velocity difference into frequency difference. This is accomplished by the relation

$$\Delta\omega/\omega=n\Delta u\,. \tag{99}$$

Table VII provides an opportunity for comparing the present results (AR) with the PYT results. Two sets of expressions are given for the ratio $\Delta\omega/\omega$ (in units of $2r_0\Omega$) in the limit that $(r_0\Omega)^2$ may be neglected. A brief study of Table VII will reveal that the present method predicts that the frequency difference between clockwise and counterclockwise waves depends only upon the index of refraction. The PYT method, on the other hand, predicts an additional dependence of the frequency difference upon the relative permeability $\mu_r=\mu/\mu_0$. If there exist dielectrics with sufficiently large

TABLE VI. Velocity difference (Δu) between clockwise and counterclockwise beams in units of $2r_0\Omega$ before and after $(r_0\Omega)^2$ is neglected for all three cases.

	Case I	Case II	Case III
$\lvert r_0\Omega\rvert<1$	$\dfrac{n^2-1}{n^2-(r_0\Omega)^2}$	1	$\dfrac{1-(r_0\Omega)^2}{n^2-(r_0\Omega)^2}$
$(r_0\Omega)^2\ll1$	$1-n^{-2}$	1	n^{-2}

TABLE VII. Relative frequency difference ($\Delta\omega/\omega$) between clockwise and counterclockwise beams in units of $2r_0\Omega$, calculated according to AR and PYT in the limit $(r_0\Omega)^2$ may be neglected.

	Case I	Case II	Case III
AR	$n-n^{-1}$	n	n^{-1}
PYT	$n-\mu_r n^{-1}$	n	$\mu_r n^{-1}$

μ_r then it would be possible to determine which, if either, of the two methods is correct. Unfortunately, all the work in the past has been done with substances for which $\mu_r \approx 1$.

VI. DISCUSSION

The object of this paper is the development of a consistent and generally applicable method of treating electromagnetic radiation in noninertial systems. We have shown how to write a generally covariant set of field equations and constitutive relations which are valid in all reference frames connected by holonomic coordinate transformations. The explicit form of the constitutive relations is a result of one assumption: The relations between E_μ, $\mathfrak{D}^\mu$, $\mathfrak{B}^\mu$, and H_μ are linear. This assumption is absolutely minimal and implies no special properties to be possessed by the medium.

The PYT procedure, on the other hand, involves an explicit assumption about the dynamics of an accelerated medium. This assumption requires the medium portion of the constitutive tensor to have very special properties: the Cartesian components of $\chi_{(m)}{}^{\mu\nu\rho\sigma}$ in the co-moving frame are to be independent of the motion of the medium. Literally, this means that the state of motion of the medium has no physical consequences for electromagnetic radiation. The PYT assumption requires the dynamical equivalence of *all* reference frames in so far as they are used to describe electromagnetic radiation in a medium.

The medium portion of the constitutive tensor is obtained from a suitable averaging process which lumps the dynamical properties of the molecules constituting the medium into a simple addition to the free-space constitutive tensor $\chi_{(0)}{}^{\mu\nu\rho\sigma}$. Since the dynamical properties of the molecules are *not* invariant under all coordinate transformations, it would obviously require a very special kind of interaction to produce a medium for which its average dynamical properties were independent of its motion. For this reason it is necessary to consider the PYT procedure to be *ad hoc*. The procedure is also noncovariant because it requires $\chi_{(0)}{}^{\mu\nu\rho\sigma}$ and $\chi_{(m)}{}^{\mu\nu\rho\sigma}$ to be defined and measured in different reference frames in general. This poses the additional physical problem of measuring $\chi_{(m)}{}^{\mu\nu\rho\sigma}$ apart from $\chi_{(0)}{}^{\mu\nu\rho\sigma}$.

The consistency of the present formulation is implicit in two facts: First, for a medium stationary in an inertial frame the equations reduce to the familiar Maxwell-Minkowski form, which is a necessary condition. Second, for a moving medium viewed from an inertial frame we obtain the relativistic velocity addition law which is also a necessary condition. The PYT procedure fails to satisfy the second condition and must therefore be judged inconsistent. Of course, it must be emphasized that consistency is not synonomous with correctness. It is possible that not enough account has been taken of nonlinear effects or of dependence upon velocity gradients. Only observation can decide these issues.

The solution of the field equations for a plane wave in a dielectric undergoing constant linear acceleration yields two results. First is the gravitational frequency-shift effect, which is independent of the specific dielectric properties of the medium as expected. Second is the dielectric frequency-shift effect, which is of second order and effectively undetectable. The emergence of the gravitational frequency-shift effect is further evidence of the over-all consistency and correctness of the present method. The second-order dielectric effect does not present any problems in principle but may require an enormous amount of practical ingenuity to detect and measure.

The solution of the field equations for azimuthal waves in a uniformly rotating dielectric provides an opportunity for a comparison between the present formalism and the PYT procedure. We saw that frequency differences between clockwise and counterclockwise waves depend only upon the index of refraction if the present method is correct but upon both the index of refraction and the relative permeability if the PYT is correct. It should be possible to test the predictions of both methods by using dielectrics of sufficiently large permeability.

APPENDIX

The proof that $T^{\mu'\nu'}(\tau,\tau',\xi)$ affects the parallel transport of a vector is straightforward. Let the vector $V^\mu(\tau)$ be defined in an inertial frame so that its image in a noninertial frame is given by

$$V^{\mu'}(\tau)=A^{\mu'}{}_\nu(\tau)V^\nu(\tau). \tag{A1}$$

The parallel transport of the vector along a path labeled by the parameter τ is then expressed by the following equations:

$$dV^\mu(\tau)/d\tau=0, \tag{A2}$$

$$\frac{dV^{\mu'}(\tau)}{d\tau}=-\left\{\begin{matrix}\mu'\\ \rho'\sigma'\end{matrix}\right\}V^{\rho'}(\tau)\frac{dx^{\sigma'}}{d\tau}. \tag{A3}$$

The combination of Eqs. (A1)–(A3) yields

$$\frac{dV^{\mu'}(\tau)}{d\tau}=\frac{dA^{\mu'}{}_\nu(\tau)}{d\tau}V^\nu(\tau)$$

$$=-\left\{\begin{matrix}\mu'\\ \rho'\sigma'\end{matrix}\right\}A^{\rho'}{}_\nu(\tau)\frac{dx^{\sigma'}}{d\tau}V^\nu(\tau). \tag{A4}$$

Since $V^{\mu}(\tau)$ is arbitrary, Eq. (A4) is equivalent to the relation

$$\frac{dA^{\mu'}{}_{\nu}(\tau)}{d\tau}=-\left\{\begin{matrix}\mu'\\ \rho'\sigma'\end{matrix}\right\}A^{\rho'}{}_{\nu}(\tau)\frac{dx^{\sigma'}}{d\tau}, \qquad (A5)$$

which gives the parallel-transported components of $A^{\mu'}{}_{\nu}(\tau)$.

The transport tensor may be written in the form

$$T^{\mu'}{}_{\nu'}(\tau,\tau',\xi)=A^{\mu'}{}_{\rho}(\tau,\xi)A^{\rho}{}_{\nu'}(\tau',\xi). \qquad (A6)$$

We shall assume that the components of $V^{\mu'}(\tau)$ parallel transported from τ' to τ are given by

$$V^{\mu'}(\tau)=T^{\mu'}{}_{\nu'}(\tau,\tau')V^{\nu'}(\tau')=A^{\mu'}{}_{\rho}(\tau)A^{\rho}{}_{\nu'}(\tau')V^{\nu'}(\tau'). \qquad (A7)$$

Then by differentiation along the path, we obtain

$$\frac{dV^{\mu'}(\tau)}{d\tau}=\frac{dA^{\mu'}{}_{\rho}(\tau)}{d\tau}A^{\rho}{}_{\nu'}(\tau')V^{\nu'}(\tau'). \qquad (A8)$$

Substitute Eq. (A5) into Eq. (A8) and use Eq. (A7) to find the result:

$$\frac{dV^{\mu'}(\tau)}{d\tau}=-\left\{\begin{matrix}\mu'\\ \rho'\sigma'\end{matrix}\right\}\frac{dx^{\sigma'}}{d\tau}A^{\rho'}{}_{\lambda}(\tau)A^{\lambda}{}_{\nu'}(\tau')V^{\nu'}(\tau')$$

$$=-\left\{\begin{matrix}\mu'\\ \rho'\sigma'\end{matrix}\right\}\frac{dx^{\sigma'}}{d\tau}V^{\rho'}(\tau). \qquad (A9)$$

Thus $V^{\mu'}(\tau)$ as defined by Eq. (A7) satisfies the equation of parallel transport. We must conclude that the transport tensor provides us with the parallel-transported components of any vector if we have the components at some point on the path. This result is obvious if we look at Eq. (A7) in the inertial frame

$$V^{\mu}(\tau)=V^{\mu}(\tau'). \qquad (A7')$$

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Experimental Test of the Pion-Nucleon Forward Dispersion Relations at High Energies*

K. J. Foley, R. S. Jones,† S. J. Lindenbaum, W. A. Love, S. Ozaki, E. D. Platner, C. A. Quarles,‡ and E. H. Willen

Brookhaven National Laboratory, Upton, New York 11973

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The small-angle differential scattering cross sections of protons for pions have been measured to high precision at the Brookhaven AGS. The range of incident momenta was 8–20 GeV/c for π^+, and 8–26 GeV/c for π^-. The real part of the pion-nucleon forward scattering amplitude was determined by observing its interference with the known Coulomb amplitude. Combining these results with precision measurements of pion-proton total cross sections over this energy range provided a critical test of the predictions of the forward dispersion relations. The results demonstrate the validity of the dispersion relations up to at least 20 GeV/c laboratory momentum. The predictions of charge independence are also verified by comparing these experimental measurements with forward charge-exchange scattering cross sections. Furthermore, if microscopic causality is violated, this occurs at "distances" less than 10^{-15} cm.

I. INTRODUCTION

The purpose of this experiment was to test the pion-nucleon forward dispersion relations at high energy by measuring the real part of the pion-nucleon forward scattering amplitude. The real part was measured by observing the Coulomb-nuclear interference in pion-nucleon differential elastic scattering in the angular range 0–22 mrad at incident laboratory momenta from 8–26 GeV/c for π^--p and 8–20 GeV/c for π^+-p. These energies are sufficiently high that the dispersion-relation predictions are very insensitive to uncertainties in the low-energy parameters and the range of energies is large enough that we can perform sufficient subtractions to remove the dependence on the asymptotic behavior of the total cross sections. As a separate part of the experiment, we measured total cross sections in this energy region with an absolute precision of 0.3%.[1] This enabled us to evaluate the dispersion-relation integrals more precisely and also improved the determination of the real part of the scattering amplitude.

An earlier incomplete investigation[2] had established that there were sizable real parts of the pion-nucleon forward scattering amplitude at high energy. However,

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† Present address: University of Minnesota, Minneapolis, Minn.

‡ Present address: Texas Christian University, Fort Worth, Tex.

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