Position Operators and Proper Time in Relativistic Quantum Mechanics*

JOSEPH E. JOHNSON
Physics Department, University of South Carolina, Columbia, South Carolina 29208

(Received 21 May 1968; revised manuscript received 7 January 1969)

Covariant four-vector position operators X^μ are proposed, which form a natural operator generalization of the four-position in relativistic classical mechanics. These X^μ are defined by specifying commutation relations of the X^μ with the Poincaré generators P^μ and $M^{\mu\nu}$, and thereby extending the Poincaré algebra to a larger algebra whose representations are subsequently found. The X^μ are shown to be acceptable relativistic position operators within a proper-time dynamical framework. A single Hamiltonian is used for all spins, with a covariant proper-time description. The dynamics is capable of describing the time evolution of states which are not mass eigenstates. An automatic Foldy-Wouthuysen-type diagonalization is achieved for all spin representations, with spin and orbital angular momentum being separately conserved. The connection with the standard theory is made via the specific field equations. In making this connection to the standard theory of half-integral spins, the origin of *Zitterbewegung* and the nonseparate conservation of spin and angular momentum are clarified. The connection to Maxwell's equations provides an interesting statement of those equations and of gauge invariance. The unphysical representations of negative and imaginary mass and continuous spin are not present in this formalism. Other features are discussed.

I. INTRODUCTION

THERE is a very extensive literature[1] concerned with three-vector position operators X^i defined upon irreducible representations of the Poincaré algebra, but the Lorentz transformation properties of these operators and the associated states are not transparent. Several authors[2] have studied four-vector position operators which are closely related to the $g^\mu = M^{\mu\nu}P_\nu$ operators defined by Shirokov.[3] But such operators have, as a classical analog, the perpendicular four-vector from the origin to the world line of the particle. As such, they are not directly analogous to the classical four-position. Fleming[4] has shown how some of the more natural definitions of four-position may be recast in a manifestly covariant framework in which the c-number time is less objectionable to covariance and in which the position operators have a dynamical behavior more closely related to four-position in classical relativistic mechanics. But it is not clear how to construct the localized states or fields associated with these operators, especially in view of the dependence of these position operators on the spacelike surfaces. Nor is it clear which operator best represents four-position. Also, one can question the use of a c-number time in a relativistic theory when three-position must be represented by an operator.

If the fundamental particles are to be represented by fields which are in turn representations of the Poincaré algebra as is generally accepted, then it follows that the position operators X^μ for the particles must be defined upon the Poincaré representations. The X^μ operators would then have well-defined commutation rules with each other and with the P^μ and $M^{\mu\nu}$ Poincaré generators. Being antisymmetrical and satisfying the Jacobi identity, these commutators would be expected to form some algebra which would, in general, be an extension of the Poincaré algebra. The (generally reducible) Poincaré representations would thus form a representation space for the extended algebra. Conversely, if one knew the extended algebra, then by finding its representations, one would find the forms of the position operators which would be operative on the various Poincaré representations. If one were *initially* to postulate such an extended algebra, then one could build in certain desirable features for the X^μ from the beginning, and these features would be present in all representations of the algebra.

The purpose of this paper is to propose a four-vector position operator X^μ, where the time X^0 is treated as an operator on an equal footing with X^i. This X^μ will be defined in Sec. II by proposing for the abstract operators X^μ, P^μ, and $M^{\mu\nu}$ an algebraic structure which incorporates certain desirable features for the X^μ. The Dirac bra and ket notation will be used for the representation space, thus making the connection between the amplitudes or fields and the abstract representation space more transparent. In order that the time not be singled out from the X^i by the dynamics, a proper-time dynamical formulation will be used with a Poincaré-invariant Hamiltonian. We will treat only a single particle in this paper. Two particles with a model interaction will be treated elsewhere. In Sec. III, the representations of the XPM algebra will be found and the Lorentz and Poincaré algebra representation content will be studied. The physical interpretation of the various representations and of the dynamics will be studied in Sec. IV. A number of important advantages of the proposed position operator and dynamical formula-

* Portions of this work were contained in the author's Ph.D. thesis (State University of New York at Stony Brook, 1967) and were presented at the November 1967 New York APS meeting.

[1] T. D. Newton and E. P. Wigner, Rev. Mod. Phys. 21, 400 (1949); L. L. Foldy and S. C. Wouthuysen, Phys. Rev. 78, 29 (1950); A. S. Wightman, Rev. Mod. Phys. 34, 845 (1962).

[2] S. Sankaranarayanan and R. H. Good, Phys. Rev. 140, B510 (1965); R. A. Berg, J. Math. Phys. 6, 34 (1965); H. Bacry, *ibid.* 5, 109 (1964); Phys. Letters 5, 37 (1963).

[3] I. U. Shirokov, Zh. Eksperim. i Teor. Fiz. 33, 861 (1957) [English transl.: Soviet Phys.—JETP 6, 664 (1958)].

[4] G. N. Fleming, Phys. Rev. 137, B188 (1965).

tion are shown. In Sec. V, the connection is made to the standard wave equations for systems which approximate mass eigenstates.

II. BASIC POSTULATES

The commutation rules for the X^μ, P^μ, and $M^{\mu\nu}$ must now be chosen. We take[5] the Poincaré-algebra structure constants for the P^μ and $M^{\mu\nu}$

$$[P^\mu,P^\nu]=0\,, \tag{1}$$

$$[M^{\mu\nu},P^\lambda]=i(g^{\lambda\nu}P^\mu-g^{\lambda\mu}P^\nu)\,, \tag{2}$$

$$[M^{\mu\nu},M^{\rho\sigma}]=-i(g^{\mu\rho}M^{\nu\sigma}+g^{\nu\sigma}M^{\mu\rho}-g^{\nu\rho}M^{\mu\sigma}-g^{\mu\sigma}M^{\nu\rho})\,. \tag{3}$$

We can ensure that X^μ is a four-vector under the Lorentz transformations by demanding that

$$[M^{\mu\nu},X^\lambda]=i(g^{\lambda\nu}X^\mu-g^{\lambda\mu}X^\nu)\,. \tag{4}$$

The main question arises with the definition of the remaining commutators $[X^\mu,X^\nu]$ and $[P^\mu,X^\nu]$. We will assume, primarily on the basis of simplicity, that

$$[X^\mu,X^\nu]=0\,. \tag{5}$$

Physically, this implies that x, y, z, and t measurements do not mutually interfere for an irreducible system. While such commutatively is physically necessary macroscopically, it is not at all obvious when sufficiently small regions are probed. However, if not zero, one would expect this commutator to be small (perhaps of the order of $\hbar$ spin/mass^2), in which case (5) is a good approximation. A vanishing commutator also facilitates a study of the representations.

The commutator $[P^\mu,X^\nu]$ must be a covariant generalization of $[P^i,X^j]=-i\delta^{ij}$. The commutator

$$[P^\mu,X^\nu]=i(g^{\mu\nu}-P^\mu P^\nu/m^2)\,, \tag{6}$$

or one similar in form, is suggested for position operators when X^0 is to be treated as a c number and P^0 as the Hamiltonian. This is evident as (6) gives $[P^\mu,X^0]=0$ in the rest frame and $[P^0,X^i]=-iP^0P^i/m^2$ in an arbitrary frame. However, as mentioned in Sec. I, the method of constructing local fields for these operators is not obvious. Furthermore, we would not expect this commutator to be useful when X^0 is treated on an equal footing with X^i as an operator in all frames and when a manifestly covariant Hamiltonian is used instead of P^0. We will postulate the commutator

$$[P^\mu,X^\nu]=ig^{\mu\nu} \tag{7}$$

for the physical four-momentum and the physical four-position. This appears to be the simplest covariant generalization of the nonrelativistic commutator. Furthermore, it treats all X^μ and P^μ on an equal footing as operators. Finally, (7) has the property that the physical four-momentum is the generator for translations in the physical space-time. The most important features of (7) will appear when we use this algebra as a basis for four-position.

The commutator (7) has not been used in the past primarily because of the following argument: If the mass operator is defined as $m=\sqrt{(P_\mu P^\mu)}$, it follows that

$$[m,X^\mu]=iP^\mu/m\,, \tag{8}$$

which has been interpreted to mean that a mass eigenstate cannot be localized. In Sec. IV we will argue that (7) is not objectionable in a proper-time dynamical formulation where the X^μ operator is interpreted as the operator for four-position at a given instant of proper time. Furthermore, we will show that with the present treatment of proper time, one may achieve the accepted degree of localization at a fixed time X^0 even for a mass eigenstate. The noncommutativity of mass and four-position will be interpreted to mean that a mass eigenstate cannot be localized in four-space at a given instant of proper time.

The observables P^μ, X^μ, $M^{\mu\nu}$, and I can be shown to form a 15-parameter Lie algebra which we refer to as the *XPM* algebra (taking I to commute with all elements). We now wish to develop a relativistic quantum mechanics based upon this algebra, utilizing the theory of representations of Lie algebras and the standard Dirac bra-ket notation in order to achieve a basis-free description. This formalism is to be based on the following three postulates:

(i) We take the fundamental algebra of relativistic quantum mechanics to be the Lie algebra generated by X^μ, P^μ, $M^{\mu\nu}$, and I with the commutation rules as above. These elements are to be associated with ideal experimental measurements of the physical four-position, four-momentum, and four-tensor of total angular momentum.

(ii) In order to connect this abstract algebra with physics, we assume the principle of linear superposition and seek representations of the *XPM* algebra as linear operators on a linear vector space with the standard quantum-mechanical interpretation.

(iii) The dynamical development of a system is given by the operator $U(\tau)=e^{-i\tau H}$, where H is a Poincaré-invariant Hamiltonian (for a closed system), and τ is a real c number parametrizing the evolution. τ will later be shown to have an interpretation as the proper time of the system. In particular, for a single free particle, the Hamiltonian $H=m-m_0$, where m is the mass operator $\pm\sqrt{(P_\mu P^\mu)}$ and m_0 the c-number mass, is suggested by classical mechanics. Since $H=m-m_0$ and $H=m$ have the same commutation rules with all operators, we may take $H=m$ in the Heisenberg picture.

In the Schrödinger picture, the constant m_0 gives rise to a constant-phase transformation $e^{i\tau m_0}$ for a

[5] We use the metric $g^{\mu\nu}$, where μ, ν, $\lambda\cdots=0$, 1, 2, and 3 and i, j, k, $\cdots=1$, 2, and 3. $g^{00}=-g^{ii}=1$ and $g^{\mu\nu}=0$ for $\mu\neq\nu$. $P^0\equiv E/c$ and $x^0\equiv ct$. We set $\hbar=c=1$.

state with mass m_0. However, this phase will vanish when probabilities are computed and thus will not influence any observable. Thus we will omit it and use $H=m$. This Hamiltonian is spin-independent in form. We later show the connection to the standard Hamiltonians for various spins. The Poincaré (PM) algebra is still the symmetry algebra of the system. It is also a subalgebra of the XPM algebra. Thus any XPM representation will also be a Poincaré representation (in general, reducible). Thus we will find that the theory of free particles in the momentum representation is essentially the same here as with a formalism based upon the Poincaré algebra alone. The free-particle Hamiltonian m used here differs from the $P_\mu P^\mu/2m_0$ used in other proper-time formulations.[6] The advantages of using m (as will be seen) are the following: First, since m contains no reference to a particular mass, the resulting dynamics will hold not only for a particular mass but also for states which are not mass eigenstates. Secondly, the use of m will be instrumental in eliminating certain unphysical representations. Finally, m reduces to the standard Hamiltonian P^0 in the rest frame whereas $P_\mu P^\mu/2m_0$ does not.

III. LINEAR REPRESENTATIONS OF THE *XPM* ALGEBRA

Assume that there exists a representation[7] of the XPM algebra as linear operators on some vector space. Since successive operators upon a vector with a series of operators are well defined, we may use these operations to introduce a product of operators. Collectively, all such products and their linear combinations define the universal enveloping algebra. In what follows, we will work with the enveloping algebra using the same symbol for an element of the abstract algebra and for its realization as a linear operator.

We define the physical four-tensor of orbital angular momentum as

$$L^{\mu\nu}\equiv X^\mu P^\nu - X^\nu P^\mu\,, \tag{9}$$

and the physical four-tensor of intrinsic angular momentum as

$$S^{\mu\nu}\equiv M^{\mu\nu}-L^{\mu\nu}. \tag{10}$$

The following commutation rules are easily verified:

$$[L^{\mu\nu},P^\lambda]=i(g^{\lambda\nu}P^\mu - g^{\lambda\mu}P^\nu)\,, \tag{11}$$

$$[L^{\mu\nu},X^\lambda]=i(g^{\lambda\nu}X^\mu - g^{\lambda\mu}X^\nu)\,, \tag{12}$$

$$[M^{\mu\nu},L^{\rho\sigma}]=[L^{\mu\nu},L^{\rho\sigma}] = -i(g^{\mu\rho}L^{\nu\sigma}+g^{\nu\sigma}L^{\mu\rho}-g^{\nu\rho}L^{\mu\sigma}-g^{\mu\sigma}L^{\nu\rho})\,, \tag{13}$$

$$[S^{\mu\nu},S^{\rho\sigma}]=-i(g^{\mu\rho}S^{\nu\sigma}+g^{\nu\sigma}S^{\mu\rho}-g^{\nu\rho}S^{\mu\sigma}-g^{\mu\sigma}S^{\nu\rho}) =[M^{\mu\nu},S^{\rho\sigma}], \tag{14}$$

$$[S^{\mu\nu},X^\lambda]=[S^{\mu\nu},P^\lambda]=[S^{\mu\nu},L^{\rho\sigma}]=0. \tag{15}$$

Now, since

$$M^{\mu\nu}=S^{\mu\nu}+X^\mu P^\nu - X^\nu P^\nu\,, \tag{16}$$

we may use the algebra defined by X^μ, P^μ, $S^{\mu\nu}$, and I as a basis for the XPM enveloping algebra, i.e., the XPM and XPS enveloping algebras are the same. Thus we may find all representations of the XPM algebra by finding all representations of the XPS algebra. In other words, every XPM representation defines an XPS representation and conversely.

We now seek the representations of the XPS algebra. From (15) one sees that the XPS algebra reduces to the direct product of two of its subalgebras: the algebra of the X^μ and P^μ (and I), which is the nine-dimensional nilpotent Heisenberg algebra; and the algebra of the $S^{\mu\nu}$, which is the same as the six-dimensional noncompact homogeneous Lorentz algebra. The representations of the cross-product algebra may be obtained by taking the direct product of the separate representations. This separation is very fortunate, since all representations of both the XP algebra and the S algebra are known. We briefly review these representations in the Dirac notation and then take their direct product to obtain the representations of the XPM algebra.

Stone and von Neumann[8] have found all representations of the $(2n+1)$-dimensional Heisenberg algebra in which the X's and P's are realized as Hermitian operators. This Hermiticity entails no loss of generality, since we have demanded that these operators be physical and hence Hermitian. The most general basis state can be uniquely labeled by the Casimir, or center, operators and a maximum Cartan subalgebra. The only Casimir operator is I, which mathematically may have any real eigenvalue. For physical reasons, however, we consider only that irreducible representation for which $I=1$, i.e., we set $\hbar=1$. There are two natural Cartan subalgebras, the set P^μ or the set X^μ, either of which may be used to form a basis for the representation space.

The momentum basis $|k\rangle$ is defined by

$$P^\lambda|k\rangle = k^\lambda|k\rangle\,, \tag{17}$$

and the position basis $|y\rangle$ is defined by

$$X^\lambda|y\rangle = y^\lambda|y\rangle\,, \tag{18}$$

where k and y are each sets of real numbers. The orthogonality of eigenvalues of Hermitian operators requires

$$\langle k|k'\rangle = \delta^4(k^\mu - k'^\mu) \tag{19}$$

[6] J. H. Cooke, Phys. Rev. **166**, 1293 (1968); W. C. Davidson, *ibid.* **97**, 1131 (1955); P. M. Pearle, *ibid.* **168**, 1429 (1968).

[7] After completion of this work, it was pointed out to us that the representations of this algebra had also been found by J. S. Zmuidizinas, J. P. L. Technical Report No. 32–797, 1965 (unpublished). The algebra was studied there for a possible connection to internal symmetries but the X^μ were rejected as position operators for physical particles. That rejection is not applicable here, because we use a proper-time dynamics and do not admit the transformations $\exp(ia_\mu X^\mu)$ which take a physical state into an unphysical state.

[8] J. von Neumann, Ann. Math. Pure Appl. **104**, 570 (1931); M. H. Stone, Proc. Natl. Acad. Sci. U. S. **16**, 172 (1930).

and

$$\langle y|y'\rangle=\delta^4(y^\mu-y'^\mu)\,, \tag{20}$$

with unit normalization. It follows from these equations and the commutation rules, by familiar arguments, that

$$P^\mu|y\rangle=-ig^{\mu\nu}(\partial/\partial y^\nu)|y\rangle\,, \tag{21}$$

$$X^\mu|k\rangle=ig^{\mu\nu}(\partial/\partial k^\nu)|k\rangle\,, \tag{22}$$

$$\langle k|y\rangle=(2\pi)^{-2}e^{ik_\mu y^\mu}\,, \tag{23}$$

and

$$I=\int d^4k\;|k\rangle\langle k|=\int d^4y\;|y\rangle\langle y|\,, \tag{24}$$

where I is a unit operator on the space. The resolution of an arbitrary state $|\psi\rangle$ or operator Q on the momentum basis, for example, is

$$|\psi\rangle=I|\psi\rangle=\int d^4k\;|k\rangle\langle k|\psi\rangle=\int d^4k\,\psi(k)|k\rangle \tag{25}$$

and

$$Q=IQI=\int d^4k_1d^4k_2\;|k_1\rangle\langle k_1|Q|k_2\rangle\langle k_2|=\int d^4k_1d^4k_2Q_{k_1k_2}|k_1\rangle\langle k_2|\,. \tag{26}$$

These integrations are to be performed over the entire k (or y) space.

Gel'fand, Naimark,[9] and others have found all representations of the $S^{\mu\nu}$ (homogeneous Lorentz) algebra. We list these results for completeness.[10] One defines a new basis in the $S^{\mu\nu}$ algebra by

$$S^i\equiv\tfrac{1}{2}\epsilon_{ijk}S^{jk}\,, \tag{27a}$$

$$R^i\equiv S^{0i}\,, \tag{27b}$$

$$(S)^2\equiv(S^1)^2+(S^2)^2+(S^3)^2\,, \tag{27c}$$

$$S_\pm\equiv S^1\pm iS^2\,, \tag{27d}$$

$$R_\pm=R^1\pm iR^2\,. \tag{27e}$$

It then follows that

$$[S^i,S^j]=iS^k\,, \tag{28a}$$

$$[R^i,R^j]=-iS^k\,, \tag{28b}$$

$$[R^i,S^j]=iR^k\,, \tag{28c}$$

$$[S^3,S_\pm]=\pm S_\pm\,, \tag{28d}$$

$$[S_+,S_-]=2S^3\,, \tag{28e}$$

where i, j, and k are cyclic permutations of 1, 2, and 3. There are two Casimir operators, b_0 and b_1, which are defined by

$$b_0^2+b_1^2-1=\tfrac{1}{2}S^{\mu\nu}S_{\mu\nu}=(\mathbf{S})^2-(\mathbf{R})^2 \tag{29a}$$

and

$$b_0b_1=-\tfrac{1}{4}\epsilon^{\mu\nu\rho\sigma}S_{\mu\nu}S_{\rho\sigma}=\mathbf{S}\cdot\mathbf{R}\,. \tag{29b}$$

We define S^i to be the physical spin (vector) operator. It will play the same role as the w^μ $(\equiv\epsilon^{\mu\nu\rho\sigma}M_{\nu\rho}P_\sigma)$ operator of the Poincaré algebra. The Cartan subalgebra formed from $(S)^2$ and S^3 will be used because of its natural physical interpretation. An irreducible complete basis for the representation space is thus labeled by eigenvalues of b_0, b_1, $(S)^2$, and S^3, as $|b_0,b_1,s,\sigma\rangle$, where

$$(S)^2|b_0,b_1,s,\sigma\rangle=s(s+1)|b_0,b_1,s,\sigma\rangle \tag{30a}$$

and

$$S^3|b_0,b_1,s,\sigma\rangle=\sigma|b_0,b_1,s,\sigma\rangle\,. \tag{30b}$$

The action of the operators on this basis is given by

$$S^3|s,\sigma\rangle=\sigma|s,\sigma\rangle\,, \tag{31a}$$

$$S_\pm|s,\sigma\rangle=\surd(s\mp\sigma)\surd(s\pm\sigma\pm1)|s,\sigma\pm1\rangle\,, \tag{31b}$$

$$R^3|s,\sigma\rangle=C_s\surd(s^2-\sigma^2)|s-1,\sigma\rangle-A_s\sigma|s,\sigma\rangle-C_{s+1}\surd[(s+1)^2-\sigma^2]|s+1,\sigma\rangle\,, \tag{31c}$$

$$R_\pm|s,\sigma\rangle=\pm C_s\surd(s\mp\sigma)\surd(s\mp\sigma-1)|s-1,\,\sigma\pm1\rangle-A_s\surd(s\mp\sigma)\surd(s\pm\sigma+1)|s,\,\sigma\pm1\rangle\pm C_{s+1}\surd(s\pm\sigma+1)\surd(s\pm\sigma+2)|s+1,\,\sigma\pm1\rangle\,, \tag{31d}$$

where

$$s=b_0,\;b_0+1,\;b_0+2,\;\cdots,\;|b_1|-1\,, \tag{31e}$$

$$\sigma=-s,\;-s+1,\;\cdots,\;s-1,\;s\,, \tag{31f}$$

$$A_s=ib_0b_1/[s(s+1)]\,, \tag{31g}$$

$$C_s=(i/s)\surd[(s^2-b_0^2)(s^2-b_1^2)/(4s^2-1)]\,, \tag{31h}$$

and where $|s,\sigma\rangle$ is used to abbreviate $|b_0,b_1,s,\sigma\rangle$.

The eigenvalue spectrum of the Casimir operators is given by $b_0=\frac{1}{2}n$, where n is any integer and b_1 is any complex number. Since S ranges from b_0 to $|b_1|-1$, we may interpret b_0 physically as the lowest spin in the representation and $|b_1|-1$ as the highest spin (in those cases where the series b_0, b_0+1, $\cdots$, terminates in $|b_1|-1$).

Irreducible representations consisting of a single spin value, which are special cases of these general representations, are defined by $b_0=s$ and $b_1=s+1$. There is one other irreducible representation, called the "conjugate representation," which has the same value of the spin and which is defined by $b_0=s$ and $b_1=-(s+1)$. We use b to stand for the pair b_0, b_1. By a suitable choice of basis, the most general invariant bilinear form (scalar product) can be cast (for bounded spin representations) into the form

$$\langle b',s',\sigma'|b,s,\sigma'\rangle=(-1)^s\delta_{b'\bar{b}}-\delta_{s's}\delta_{\sigma'\sigma}\,, \tag{32}$$

[9] I. M. Gel'fand and A. M. Naimark, J. Phys. (USSR) **10**, 93 (1946); M. A. Naimark, Usp. Mat. Nauk **9**, 19 (1954).

[10] We simply summarize the results of I. M. Gel'fand, R. A. Minlos and Z. Ya. Shapiro [*Representations of the Rotation and Lorentz Groups and Their Applications* (The Macmillan Co., New York, 1963)], using Dirac notation.

where $\bar{b}$ is defined in terms of b in a way which depends both upon the type of representation and whether the representation is to be of the proper or improper (i.e., $S^{\mu\nu}$ with space and time reflections adjoined) algebra. For example, for a unique spin representation of the improper algebra, $\bar{b}$ is the conjugate representation. This situation is exemplified by the Dirac theory of spin-$\frac{1}{2}$ particles, where, it is recalled, in order to form the bilinear invariant $(\int d\sigma^\mu \bar{\psi}\gamma_\mu\psi)$, one must take the ψ space to be the direct sum of a representation space and its conjugate representation. The unit operator is given by

$$I=\sum_{b,s,\sigma}(-1)^s|\bar{b},s,\sigma\rangle\langle b,s,\sigma|\,, \tag{33}$$

where the sum is to extend over all representations b (i.e., admissible pairs b_0 and b_1), while s is summed from b_0 to $|b_\rho|-1$ and σ is summed from $-s$ to s. The symbol $\bar{b}$ has the same meaning as before. There are three types of representations: unique spin, finite-dimensional mixed spin, and infinite-dimensional mixed spin. The unitary representations are all infinite-dimensional except the trivial one-dimensional representation of spin zero. We will not discuss particular representations further, but take the information from the literature as it is needed.

The product representations may now be constructed as follows: We write $|k^\mu\rangle\otimes|b,s,\sigma\rangle$ as $|k,b,s,\sigma\rangle$ and $|y\rangle\otimes|b,s,\sigma\rangle$ as $|y,b,s,\sigma\rangle$. Either of these bases is complete in the irreducible product representation. The action of the various operators on these product states follows in a trivial manner from their action on the constituent spaces. The scalar product and completeness relation are given by

$$\langle k',b',s',\sigma'|k,b,s,\sigma\rangle=a_{b'b}(k,s,\sigma)\delta^4(k'^\mu-k^\mu)\delta_{s's}\delta_{\sigma'\sigma} \tag{34}$$

and

$$I=\int d^4k\sum_{b,s,\sigma}a^{-1}{}_{b'b}(k,s,\sigma)|k,b',s,\sigma\rangle\langle k,b,s,\sigma|\,. \tag{35}$$

We now must interpret the various operators and states justifying our particular definitions of physical operators. The dynamical development of systems will now be discussed, and this formalism will be connected to the standard theory using the nonunitary representations (except for spin zero) defined by taking the scalar product for the entire space to be the product of the scalar products for the separate XP and S spaces, i.e.,

$$a_{b'b}(k,s,\sigma)=\delta_{b'\bar{b}}(-1)^s\,, \tag{36}$$

where $\bar{b}$ is the representation conjugate to b. For unique spin representations, $(-1)^s$ may be omitted. By a nonsingular change of basis this scalar product may be brought to diagonal (indefinite) form. In this paper we will not investigate the very detailed question of finding all invariant bilinear forms (positive definite and indefinite) of which the above form is a special case.[11]

IV. PHYSICAL INTERPRETATION AND DISCUSSION

For a free particle, the invariant Hamiltonian was postulated to be $H=m$. This operator is well defined by the equation

$$m=\int d^4k\sqrt{(k_\rho k^\rho)}\,|k\rangle\langle k|\,. \tag{37}$$

This H has the correct dimensions and is the same as the energy (ordinary Hamiltonian) in the rest frame of the system (apart, possibly, from a sign). Since we adopt the normal probability interpretation of quantum mechanics, the time-development operator must be unitary, and thus the Hamiltonian must be Hermitian as τ is real. The Hamiltonian must also be positive definite on states which represent physical particles. Otherwise the system would not possess a ground state, since $-m$ has no lower bound. These conditions applied to a free particle, $H=m$, require m to be Hermitian positive definite. But within an irreducible representation, all real k^μ are mathematically permissible, and thus $\pm\sqrt{(P_\mu P^\mu)}$ can assume imaginary and negative values. It must be concluded that although these states are admissible by postulate I, they are excluded by postulates II and III. Thus, in order to maintain a Hermitian positive-definite Hamiltonian, one must restrict physical states to linear combinations of $|k,b,s,\sigma\rangle$ such that $k^2\geq 0$, and m must be defined with that sign for $\sqrt{(P_\mu P^\mu)}$ which gives $m\geq 0$ on the bilinear form being used. Thus, only positive timelike (or null) masses are consistent with all three postulates. This is in agreement with the fact that spacelike particles have not been observed and with the fact that, in a gravitational sense, mass appears to be positive definite even for antiparticles.

If a state is a linear combination of positive timelike masses at one time, it will remain so for a free particle in the course of time τ. It will be shown in a later paper that, even in the second-quantized version with interactions, invariant Hamiltonians may be constructed that produce no transitions to spacelike particles. The matrix elements of the position operator between spacelike states $|k_s\rangle$ and timelike states $|k_t\rangle$ vanish, i.e.,

$$\langle k_s|X^\mu|k_t\rangle=\delta'(k_t{}^\mu-k_s{}^\mu)=0\,, \tag{38}$$

and the commutator $[P^\mu,X^\nu]=ig^\mu$, is still realized on the physical subspace. That the operator exp $(ia_\mu X^\mu)$ can translate a timelike state into a spacelike state does not appear to cause any difficulty, because such a trans-

[11] S. Malin and A. O. Barut [Rev. Mod. Phys. 40, 632 (1968)] have discussed invariant forms for irreducible Poincaré representations.

formation does not normally arise in the theory. Thus, no difficulty seems to be encountered in using only a portion of the mathematically admissible irreducible representation space to describe physical states, although, as will be seen, exact localization is not possible for a physical state. The energy P^0 mathematically may assume negative as well as positive values. These negative-energy states, in contrast to the standard theory, are completely admissible and may be placed on a completely equal footing with the positive-energy states. The negative-energy states cause no difficulty[12] here because our Hamiltonian is not taken as P^0. Thus the treatment of mass and energy eigenvalues in the present formalism appears to give a much closer correspondence between admissible representations and the observed properties of particles associated with representations than the standard theory, which *a priori* admits negative and imaginary masses and in which admission of the negative-energy states is mathematically awkward.

The present formalism also admits a treatment of spin and angular momentum which has several advantages over the standard theory. One recalls that in the Dirac theory, spin and angular momentum are not separately conserved. One must make a (nonlocal) Foldy-Wouthuysen transformation to new variables, called the "mean observables," in order to get separate conservation. In the group-theoretical approach, one defines the spin, via the Poincaré algebra, by the operator W^μ with $(S)^2=w^2/m^2$. Three of the four W^μ obey the commutation rules of angular momentum only in the rest frame, and thus the W^μ operator is not intuitively conducive to interpretation of physical spin in a moving frame. Also, because of the transformation properties of W^μ, the irreducible representations of the Poincaré algebra are not connected in a simple fashion to the field operators in a second-quantized theory. Finally, there are continuous spin representations of the Poincaré algebra which do not seem to be realized in nature. The above-mentioned difficulties are not met in the present formalism. We defined the physical spin S^i as a three-component object which maintains the commutation rules of angular momentum [Eq. (28a)] in all frames of reference. Since $[S^i,m]=[S^i,L^{\mu\nu}]=[L^{\mu\nu},m]=0$, for a free particle the spin and angular momentum will be separately conserved. It is important to note that this result is representation- (spin-) independent. As a consequence, we will find S^i and $L^{\mu\nu}$ separately conserved for all particles or fields. Furthermore, the connection between the representations of the XPM algebra and the field operators is direct, since such operators have the correct transformation properties under $M^{\mu\nu}$ even in moving frames. Finally, one notes that the unphysical continuous spin representations present in the Poincaré algebra are not present in the present formalism when S^i is interpreted as the spin. The space part of the spin operators of the Poincaré algebra and the S^i operators of the XPM algebra are identical in the rest frame of the system. For $m\neq 0$ in the rest frame, $k^i=0$, one has $w^2/m^2=\frac{1}{2}(\epsilon^{\mu\nu\rho 0}M_{\nu\rho}P_0)^2/m^2=(S)^2$ as L^{ij} vanishes in the rest frame. Also, $w^3/m=\frac{1}{2}\epsilon^{3\mu\nu 0}P_0M_{\mu\nu}/m=S^3$. For the $m=0$ case in the frame where $k^0=k^3$ and $k^1=k^2=0$ we have the helicity $W^0/P^0=\frac{1}{2}\epsilon^{0\mu\nu 3}P_3M_{\mu\nu}/P_0=S^3$. The equivalence of S^i and the Poincaré spin in the rest frame allows one to establish the connection between the representations in that frame, e.g., $|k^\mu,b,s,\sigma\rangle_{XPM}=|m,l^i,s's_3'\rangle_{PM}$, where $m=\sqrt{(k_\mu k^\mu)}$, $k^i=l^i=0$, $s=s'$, and $\sigma=s^3$. The relationship between the representations is easily found for other frames by Lorentz transformation.

Considering now the position operators, we see that by construction the X^μ is a Hermitian four-vector in all representations. As P^μ, $M^{\mu\nu}$, $S^{\mu\nu}$, and $L^{\mu\nu}$ all commute with $H=m$, all of their eigenvalues are separately conserved for all representations of a closed system. The X^μ, however, obey

$$X^\mu\to U(\tau)X^\mu U^{-1}(\tau)=X^\mu(\tau)=X^\mu(0)+\tau P^\mu/m. \quad (39)$$

Upon taking expectation values we see that this becomes identical to the classical equation for a free system where $\langle P^\mu/m\rangle$ is the expectation value of the four-velocity. Consequently, the limit to classical relativistic mechanics is obtained. Taking the expectation value of this equation in the momentum representation in the rest frame of the particle, $|k\rangle=|k^0=m,\ k^i=0\rangle$, we get $\langle X^0(\tau)-X^0(0)\rangle=\tau$. Thus τ may be interpreted as the expectation value of the time interval on a clock at rest with respect to the system. Thus, the X^μ operators appear to form satisfactory physical position operators. It should be realized, however, that the form of the present theory is somewhat different from the standard theory because of the use of a proper-time dynamics. While the X^μ operators are well defined on the present states and bilinear forms, they naturally will not, in general, be valid on the standard states and inner products without further qualification.

From

$$|y\rangle=\int d^4k\,|k\rangle\langle k|y\rangle=\frac{1}{(2\pi)^2}\int d^4k\; e^{ik_\mu y^\mu}|k\rangle, \quad (40)$$

we see that a localized state consists of not only all momentum k^μ but also all masses, even imaginary ones. Thus a position eigenstate cannot be formed from physical states ($m\geq 0$). Even the maximum localization consistent with $m\geq 0$ will be a state of infinite energy and mass. This, however, presents neither a physical nor mathematical impass, since one never achieves *exact* localization in nature. The physical states which

[12] That is, there is no difficulty with the single-particle theory being studied here, since the Hamiltonian is positive-definite Hermitian even for negative-energy states. Whether it is possible to treat the negative-energy states on an equal footing with n interacting particles remains to be investigated.

enter the various equations will, if they are true representations of nature, be nonsingular distributions, and thus no transition can occur, in a system of finite energy, to a localized state in the course of time. The physical observables are always described in terms of densities. For example, if $|\psi\rangle$ is a physical state, then $|\psi\rangle=\int d^4k\,\psi(k)|k\rangle$, where $\psi(k)=0$ for $k^2<0$, $m<0$. Then $\langle y|\psi\rangle=\psi(y)$ is still a well-defined density.

The wave function in this formalism has a slightly different form than in the standard theory, since it is a function of five variables k^μ and τ or y^μ and τ (in addition to the spin-space variables). For example, in the Schrödinger picture and momentum representation for a free state we have $|\psi,\tau\rangle=U(\tau)|\psi\rangle$ or

$$\langle kbs\sigma|\psi,\tau\rangle=\psi_{bs\sigma}(k^\mu,\tau)=e^{-i\tau\surd(k_\mu k^\mu)}\psi_{bs\sigma}(k^\mu,0)\,. \quad (41)$$

The amplitude $|\psi,\tau\rangle$ obeys

$$i(\partial/\partial\tau)|\psi,\tau\rangle=m|\psi,\tau\rangle\,. \quad (42)$$

If $|\psi\rangle$ is normalized to unity at one proper time, as it must be to represent a single particle, then, as $U(\tau)$ is unitary, the unit normalization will hold at all τ. By expanding $\langle\psi|\psi\rangle=1$ in either the position or the momentum representation, e.g.,

$$\int d^4y\,(-1)^s\sum_{b,s,\sigma}\psi_{b's\sigma}{}^*(y,\tau)\psi_{bs\sigma}(y,\tau)=1\,,$$

we see that $\psi_{b,s,\sigma}(y,\tau)$ is a four-dimensional distribution in space-time for each value of τ. Physically, one may think of $\psi_\tau(y^\mu)$ in the following way: If we imagine a clock which moves along in the rest frame of the particle, then this clock gives us the proper time τ. Then $\psi_\tau(y^\mu)$ in the amplitude, at an instant τ, for finding the particle at the four-position y^μ, and thus $\psi_\tau{}^*(y^\mu)\psi_\tau(y^\mu)$ is the probability of finding the particle at the event y^μ at the proper time τ. If the different instants of proper time τ are indistinguishable (as for a stable system), then τ is not an observable, and the observable probability of finding the particle at y^μ (irrespective of the time τ) is obtained by integrating over all τ. Thus, the primary function of the proper time τ is just to give a covariant parametization to the dynamical development. If we wish to form a wave packet at a *particular* proper time, then in order to partially localize the distribution $|\psi\rangle$ in space-time, one must superimpose several neighboring mass eigenstates. This was to be anticipated from the position-mass noncommutativity. If a measurement were to be carried out at a given event τ, then the probability of finding the particle in a space-time volume V would be

$$P_V(\tau)=\int_V d^4y\,\psi_\tau{}^*(y^\mu)\psi_\tau(y^\mu)\,.$$

For example, the event τ could be characterized by the instant of decay of an unstable particle. Any system in a dynamical state of internal development such that the stage of the development is observable in principle from the outside will distinguish, by its stages, an internal time τ. Partial knowledge of τ in turn implies an indeterminancy in the invariant mass m of the system of $\Delta m\Delta\tau\geq\frac{1}{2}h$, which follows (even though τ is a c number) from the fact that m generates τ translations. This is the same as the $\Delta E\Delta t$ uncertainty in the standard theory. Thus, an unstable particle with lifetime $\Delta\tau$ must be described by superimposing invariant-mass states of width Δm. (In the standard theory, one cannot represent a particle as a superposition of mass states, since the dynamics is formulated only for mass eigenstates.) Conversely, it also follows that if a state is known to be a mass eigenstate, then τ is completely indeterminate, i.e., the system does not change internally in time τ and must thus be a stable (internally stationary) state. Thus it follows in this formalism that a mass eigenstate is stable. In particular, it follows that $m=0$ states must be stable particles.

As an example of the present formalism, consider a free wave packet $|\psi\rangle$ localized about the origin $y^\mu=0$ with $\langle\psi|p^i|\psi\rangle=0$. Any other similar packet can be obtained by performing a translation and a Lorentz transformation on this packet. For simplicity we take

$$\begin{aligned}\langle k|\psi,0\rangle=(\sigma_x/\surd\pi)^{3/2}(2\sigma m)^{-1/2}[(k^0)^2/k_\mu k^\mu]^{1/4}e^{-\sigma_x{}^2\mathbf{k}^2/2}\\ \times\{\Theta[\surd(k_\mu k^\mu)-m_0+\sigma_m]\\ -\Theta[\surd(k_\mu k^\mu)-m_0-\sigma_m]\}\,. \quad (43)\end{aligned}$$

Then

$$\langle k|\psi,\tau\rangle=e^{+i\tau\surd(k_\mu k^\mu)}\langle k|\psi,0\rangle \quad (44)$$

gives the proper-time evolution. Thus the probability of finding the particle at the event y^μ regardless of the time τ would be given by

$$\psi^*(y)\psi(y)=\int d\tau\left|\int d^4k\langle y|k\rangle\langle k|\psi,\tau\rangle\right|^2. \quad (45)$$

In the limit of a mass eigenstate ($\sigma_m\to0$) one gets

$$\psi^*(y)\psi(y)=(\pi)^{-3/2}\left(\sigma_x{}^2+\frac{4(y^0)^2}{\sigma_x{}^2m_0{}^2}\right)^{3/2}\times\exp\left(-\frac{\mathbf{y}^2}{2[\sigma_x{}^2+4(y^0)^2/\sigma_x{}^2m_0{}^2]}\right). \quad (46)$$

We note that the integration over $d\tau$ is to be performed after computing probabilities.[13] We see that the time development of this packet is the same as a Gaussian packet in the standard theory (as it must be if the proper-time dynamics is correct). That mass eigenstates may be sharply localized in the conventional sense may be seen as follows: If we assume

$$\psi(k^\mu)=\phi(k^i,m_0)[m_0\surd(2\sigma_m)]^{-1}\times[\theta(m-m_0+\sigma_m)-\theta(m-m_0-\sigma_m)],$$

[13] There are two natural ways to eliminate τ: First, one can compute amplitudes and then integrate over τ. Second, one can compute probabilities and then integrate over τ. The first method does not work, while the second appears to give reasonable results.

and evaluate $\psi^*(y)\psi(y)$ as above, we get in the limit as $\sigma_m \to 0$

$$\psi^*(y)\psi(y)=\frac{1}{(2\pi)^3}\int \frac{d^3k_1}{k_1{}^0}e^{ik_{\mu}y^{\mu}} \times\phi(k_1{}^i)\int \frac{d^3k_2}{k_2{}^0}e^{-ik_{2\mu}y^{\mu}}\phi(k_2{}^i), \quad (47)$$

where $k^0=(m_0{}^2+k^2)^{1/2}$, which gives the same localization (and time development) as the standard theory.

The dynamics of mass-zero particles cannot be formulated in terms of proper time as $U(\tau)|m=0\rangle = |m=0\rangle$, i.e., the state is unaltered in τ. Physically, this is because mass-zero particles have no rest frame, and thus the time τ of a clock in the rest frame has no meaning. However, since these states obey $P_\mu P^\mu|\psi\rangle=0$, it follows that

$$P^0|\psi\rangle=\sqrt{(\mathbf{P}^2)}|\psi\rangle, \quad (48)$$

which can be used to formulate the dynamics in the standard manner.

The free-particle Hamiltonian $m=\sqrt{(P_\mu P^\mu)}$ has the same form for all spins and does not have off-diagonal spin matrix elements. As a consequence of this, one automatically gets a Foldy-Wouthuysen–type separation for *all* spins, as can be seen from $ih(\partial/\partial\tau)\psi_{bs\sigma}(k,\tau) = e^{-i\tau\sqrt{(k_\mu k^\mu)}}\psi_{bs\sigma}(k,\tau)$. In the spin case, for example, the position variables y^i appear to be closely related to the mean variables of Foldy and Wouthuysen.

The continuous transformations generated by the Poincaré subalgebra form a Lie group, the Poincaré group. The most general element is given by

$$U(a,\eta)=\exp[-i(a_\mu P^\mu+\tfrac{1}{2}\eta_{\mu\nu}M^{\mu\nu})/h]. \quad (49)$$

As mentioned above these transformations commute with the mass m and thus form the symmetry group for a free system. The effect of these transformations on both the operators and the states may be found in a straightforward manner by expanding the exponential and using the commutation rules given previously. The connection between the unitary representations of the Poincaré algebra and the nonunitary representations of the Lorentz algebra (studied here) has been discussed to some extent in the literature.[14] We will not discuss the unitary representations in this paper, since we wish mainly to emphasize the correspondence of the *XPM* formalism to the fields and to the standard wave equations, which is most easily done via the nonunitary representations.

V. CONNECTION BETWEEN *XPM* FORMALISM AND STANDARD WAVE EQUATIONS

The manner in which the present formalism reduces to the standard wave equations of Klein, Gordon, Dirac, and Maxwell is now shown. Such a reduction may be obtained for states which are not highly unstable and which may be treated, consequently, as a mass eigenstate to very good approximation. The massive spinless representation, defined by $b_0=0$, $b_1=1$, and $s=\sigma=0$, is the only finite-dimensional unitary representation. In the case that the state $|\psi\rangle$ to be described is approximately a mass eigenstate, the relation

$$P_\mu P^\mu|\psi\rangle\approx m_0{}^2|\psi\rangle \quad (50)$$

leads to the Klein-Gordon equation

$$(\partial_\mu\partial^\mu+m_0{}^2)\psi(y)=0 \quad (51)$$

in the position representation. Generally, however, $|\psi\rangle$ will be a superposition of mass states, in which case one must use the proper-time dynamics, e.g.,

$$i(\partial/\partial\tau)|\psi,\tau\rangle=H|\psi,\tau\rangle. \quad (52)$$

The mass is to be determined for such a state by specifying the distribution $|\psi\rangle$ at one time. The scalar product is

$$\langle\psi|\phi\rangle=\int d^4k\,\psi^*(k)\phi(k), \quad (53)$$

which reduces to the conventional form

$$\int \frac{d^3k}{k^0}\psi^*(k)\phi(k),$$

when $\psi(k)$ and $\phi(k)$ are of the form $f(k^\mu)\delta(k^2-m_0{}^2)$. Since

$$\langle\psi|\psi\rangle=\int d^4k\,\psi^*(k)\psi(k), \quad (54)$$

we see that this general scalar product is positive definite and thus a probabilistic interpretation is possible on the first-quantized level. The normalization of this scalar product is invariant in proper time, i.e.,

$$(\partial/\partial\tau)\langle\psi,\tau|\psi,\tau\rangle=0. \quad (55)$$

We define a current operator

$$j^\mu\equiv eP^\mu/m \quad (56)$$

in analogy to classical mechanics. The expectation value of j^μ for a field $|\psi\rangle$ is

$$j_{\psi\tau}{}^\mu=\langle\psi,\tau|j^\mu|\psi,\tau\rangle=\int d^4y\,\tfrac{1}{2}(\langle\psi,\tau|y\rangle\langle y|j^\mu|\psi,\tau\rangle +\langle\psi,\tau|j^\mu|y\rangle\langle y|\psi,\tau\rangle)=\int d^\mu y j_{\psi,\tau}{}^\mu(y), \quad (57)$$

where $j_{\psi,\tau}{}^\mu(y)$ is the local current density at time τ. The observable density is obtained by integrating over all τ. When $|\psi\rangle$ is an approximate eigenstate, this $j_\psi{}^\mu(y)$ reduces to the standard expression in the

[14] S. Weinberg, Phys. Rev. **133**, B1318 (1963).

Klein-Gordon theory. However,

$$\partial_\mu j_\psi^\mu(y)=0 \tag{58}$$

for any $|\psi\rangle$, even when $|\psi\rangle$ is a superposition of mass eigenstates, and thus does not obey a Klein-Gordon equation.

A massive spin-$\frac{1}{2}$ particle is described by the direct sum of the representation $b_0=\frac{1}{2}$, $b_1=\frac{3}{2}$ and its conjugate representation $b_0=\frac{1}{2}$, $b_1=-\frac{3}{2}$. This sum of representations of the proper Lorentz algebra can be shown to be an irreducible representation of the improper Lorentz algebra. It consists of the four states $|k^\mu,\frac{1}{2},+\frac{1}{2},+\rangle$, $|k^\mu,\frac{1}{2},-\frac{1}{2},+\rangle$, $|k^\mu,\frac{1}{2},+\frac{1}{2},-\rangle$, and $|k^\mu,\frac{1}{2},-\frac{1}{2},-\rangle$. The free-particle Hamiltonian $\sqrt{(P_\mu P^\mu)}$ is diagonal and positive definite on physical states. This automatic diagonalization is reminiscent of the Foldy-Wouthuysen transformed Dirac theory. It is not obvious at this point how this formalism is connected to the standard Dirac theory even when one has approximate mass eigenstates. In order to show this connection, we ask if it is possible to extract the square root in the Hamiltonian $\sqrt{(P_\mu P^\mu)}$ in order to express it as an operator linear in P^μ, i.e., $\gamma_\mu P^\mu$, where γ_μ are operators which do not depend upon P^μ. We set $\sqrt{(P_\mu P^\mu)}=\gamma_\mu P^\mu$ to see under what conditions the γ_μ exist. From this requirement it follows immediately that $\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu=2g_{\mu\nu}$ by familiar arguments. Since $\sqrt{(P_\mu P^\mu)}$ is a Poincaré scalar, γ_μ must transform as a Lorentz four-vector and must be independent of X^μ (to maintain translational invariance). Since γ_μ is to be independent of P^μ and X^μ, it must be nondiagonal in the spin variables to be nontrivial.

From the study of the γ_μ (Clifford) algebra, it is well known that nontrivial representations exist in all half-integral representations. It thus appears that one could use either the Hamiltonian $\sqrt{(P_\mu P^\mu)}$ or the Hamiltonian $\gamma_\mu P^\mu$. Both have the same commutator (zero) with the Poincaré basis P^μ and $M^{\mu\nu}$. But on taking $U(\tau)=\exp(i\tau\gamma_\mu P^\mu)$, we find that $S^{\mu\nu}(\tau)$ and $L^{\mu\nu}(\tau)$ are not separately conserved, since they are τ-dependent. The position operators obey $X^\mu(\tau)=X^\mu(0)+\gamma^\mu\tau$, a familiar result in the Dirac theory for the space components which implies that γ^μ is the four-velocity. But these four-velocity operators do not mutually commute and they imply that the velocity of all particles is that of light as $\langle\gamma^\mu\rangle=1$. These difficulties are well known in the Dirac theory, but it appears surprising at first that we find them in the present formalism, when the Hamiltonian $\gamma_\mu P^\mu$ is used instead of $\sqrt{(P_\mu P^\mu)}$. The answer lies in the fact that, while $\sqrt{(P_\mu P^\mu)}$ and $\gamma_\mu P^\mu$ are equivalent with respect to the Poincaré algebra, they are not equivalent with respect to the other physical observables X^μ, $S^{\mu\nu}$, and $L^{\mu\nu}$ (as may be easily checked).

Thus $\gamma_\mu P^\mu$ may not be interpreted as the Hamiltonian. The situation is further complicated by the fact that an m eigenstate is not necessarily a $\gamma\cdot P$ eigenstate (although $\gamma\cdot P$ eigenstates are eigenstates of m). One easily shows that a mass eigenstate consists of two disjoint Poincaré invariant subspaces with $\gamma\cdot P=\pm m$ which coincide only when $m=0$. Instead of choosing the basis $|m, \mathbf{k}, Sg_{p_0}, Sg_{b_1}, s, \sigma\rangle$ one may choose the complete basis $|m, \mathbf{k}, Sg_{p_0}, \gamma\cdot P, s, w^3\rangle$. The helicity w^3 is used, as σ does not commute with $\gamma\cdot P$. The two signs of $\gamma\cdot P$ replace Sg_{b_1}. The two invariant subspaces then satisfy $(\gamma\cdot P\pm m)|\Psi\rangle=0$ depending upon the sign of $\gamma\cdot P$. Thus the Dirac equation holds for mass eigenstates when the Cartan subalgebra is chosen to include the sign of $\gamma\cdot P$. The position eigenstates cannot be formed from only one subspace as then one only has P_μ available with a given projection onto γ^μ. The transformation $\langle m, \mathbf{k}, Sg_{p_0}, Sg_{b_1}, s, \sigma|m, \mathbf{k}, Sg_{p_0}, \gamma\cdot P, s, w^3\rangle$ appears to be the proper-time equivalent of the Foldy-Wouthuysen transformation.

As a final example, we discuss a connection between two massless representations of the XPM algebra and Maxwell's equation for the electromagnetic field. Maxwell's equations may be stated either in terms of the fields

$$F^{\mu\nu}(y)=-F^{\nu\mu}(y) \quad \text{as} \quad \partial_\mu F^{\mu\nu}(y)=0$$

and

$$\epsilon_{\mu\nu\rho\sigma}\partial^\nu F^{\rho\sigma}(y)=0$$

or in terms of the four-potentials $A^\mu(y)$ as $\Box A^\mu(y)=0$, $\partial_\mu A^\mu(y)=0$, with the unobservable gauge transformation $A^\mu(y)\to A^\mu(y)+\partial^\mu\phi(y)$, where $\Box\phi(y)=0$. First, consider the mixed-spin representation $b_0=0$, $b_1=2$, which has the basis $|0,0\rangle$, $|1,1\rangle$, $|1,0\rangle$, and $|1,-1\rangle$ for $|s,\sigma\rangle$ with k, $b_0=0$, $b_1=2$ being understood in the labeling. By constructing a new basis $|\mu\rangle$ as $|0\rangle\equiv|0,0\rangle$,

$$|1\rangle\equiv-(i/\sqrt{2})(|1,1\rangle-|1,-1\rangle),$$
$$|2\rangle\equiv-(1/\sqrt{2})(|1,1\rangle+|1,-1\rangle),$$

and $|3\rangle\equiv i|1,0\rangle$, one easily verifies that the ket $|\mu\rangle$ transforms as a contravariant vector under Lorentz transformations. We may thus use either $|k,\mu\rangle$ or $|y,\mu\rangle$ as a basis for this mixed-spin representation. The (indefinite) scalar product is $\langle k',\lambda'|k,\lambda\rangle=g^{\lambda'\lambda}\delta^4(k'^\mu-k^\mu)$. A general state vector $|A\rangle$ in this representation has components $\langle y,\lambda|A\rangle\equiv A^\lambda(y^\mu)$ which we identify with the vector potential (apart, possibly, from constants of proportionality). From the fact that the mass is zero, it follows that $P_\mu P^\mu|k,\lambda\rangle=0$. As previously mentioned, one cannot form localized states from a linear superposition of these states, but the densities $A^\lambda(y)$ are still well defined and satisfy $\Box A^\lambda(y)=0$ as a consequence of being massless.

There is another representation, distinct from this mixed-spin representation, which is the unique spin-1 representation $b_0=1$, $b_1=2$ taken with its conjugate representation $b_0=1$, $b_1=-2$. Here there are six basis vectors, given by $|1,2,1\rangle$, $|1,2,0\rangle$, $|1,2,-1\rangle$, $|1,-2,1\rangle$, $|1,-2,0\rangle$, and $|1,-2,-1\rangle$, in which k (or y) and $s=1$

are understood. One may take a linear combination of these to form a new basis $|k,\rho\sigma\rangle$, which transforms as a contravariant second-rank antisymetric tensor. We consider the $m=0$ subspace and define the components of a state vector $|F\rangle$ on this basis as the electromagnetic field tensor $\langle k,\rho\sigma|F\rangle = F^{\rho\sigma}(k)$. It is to be noted that the field tensor $F^{\mu\nu}(k)$ and the vector potential are essentially different irreducible representations of the XPM (and hence Lorentz) algebra.

Further considerations are necessary to connect these representations to Maxwell's equations. Consider a ket vector $|k,\lambda_1\lambda_2,\cdots,\lambda_n\rangle$ which transforms as an n-rank irreducible tensor representation of the XPM algebra. Tensor operators of various ranks may be constructed from the elements of the XPM algebra and multiplied into the tensor kets with either contracted or uncontracted indices to form new representation states which transform according to a higher- or lower-order representation. For example, $P_\lambda|k,\lambda\rangle$ (sum over λ understood) transforms as a scalar representation, while $P^\mu|k\rangle$ transforms as a vector ket. We call these representations "derived." If a ket $|\psi\rangle$ is to describe particles belonging to only one representation, then we require its scalar product with other representations, including derived ones, to be zero (or unobservable). Such a condition has meaning only if the elimination of the derived representations is Poincaré-frame-independent. Imposing this condition on the $|F\rangle$ field, we get $\epsilon_{\mu\nu\rho\sigma}\langle y\rho\sigma|P^\nu|F\rangle = 0 = \langle y,\rho,\sigma|P^\sigma|F\rangle$ or $\epsilon_{\mu\nu\rho\sigma}\partial^\nu F^{\rho\sigma}(y) = 0 = \partial_\mu F^{\rho\sigma}(y)$, which are Maxwell's equation stated in terms of the fields. In terms of the potentials one has $\langle y_\mu|P^\mu|A\rangle = 0$ or $\partial_\mu A^\mu(y) = 0$. Requiring that the derived field $P^\mu|k\rangle$ be unobservable means that the replacement $|k,\mu\rangle \rightarrow |k,\mu\rangle + P^\mu|k\rangle$ is unobservable. In terms of $|A\rangle$, this implies that $A^\mu(y) \rightarrow A^\mu(y) + \partial^\mu\phi(y)$, the gauge transformation, is an unobservable replacement. In a first-quantized theory, the $A^\mu(y)$ and $F^{\mu\nu}(y)$ are connected by $F^{\mu\nu}(y) = \partial^\mu A^\nu(y) - \partial^\nu A^\nu(y)$. This may be written in our notation as $|y,\mu,\nu\rangle = P^\mu|y,\nu\rangle - P^\nu|y,\mu\rangle$. However, the descriptions of the electromagnetic field in terms of the four-vector and four-tensor representations are not completely equivalent from the point of view of representations. This derivation of Maxwell's equations is somewhat artificial, but it does exhibit the equations from a different point of view. The extra requirement employed here regarding derived fields replaces Gel'fand's requirement that a Lagrangian exist.

VI. SUMMARY

Since the observables of a system must be well-defined operators on the space $|\psi\rangle$ of physical states, $|\psi\rangle$ should form a representation space of the algebra of observables. Thus it appears that a useful method of approach to the study of new observables is to postulate a certain algebra for the observables of interest and then to find representations of this algebra. These representations are then possible states of the system. We have applied this philosophy in order to obtain a well-defined four-vector position operator for relativistic quantum mechanics. The basic algebra for quantum mechanics then becomes the XPM algebra instead of its subalgebra, the Poincaré algebra. The XPM algebra was defined by postulating certain commutation rules suggested by analogy to classical mechanics and nonrelativistic quantum mechanics. One could, in principle, carry through this program for some other algebra. Generally, this program is equivalent to studying properties of operators which may be defined on reducible representations of the Poincaré algebra. The XPM algebra and the proper-time dynamics postulated here give a more satisfactory formulation, in many respects, of the single-particle observables, although a more detailed investigation of the complete set of bilinear forms and the detailed properties of the particular representations is needed.

ACKNOWLEDGMENTS

I am deeply indebted to my advisor, Professor Max Dresden, for counsel and encouragement throughout this work. For useful conversations in the earlier part of this work, I would like to thank Professor P. A. M. Dirac and Professor. V. Bargmann.