Coulomb Forces in the Three-Body Problem*

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The Faddeev-Lovelace equations, describing three particles interacting via short-range separable potentials, are modified to include the case in which two of the particles are charged. The three-body amplitudes are split into pure Coulomb and Coulomb-distorted amplitudes, and a set of integral equations similar to Lovelace's are obtained for the Coulomb-distorted amplitudes. The pure Coulomb contributions are taken to be zero for rearrangement channels and are approximated by two-body Coulomb amplitudes for the elastic scattering channel. Numerical results for deuteron-induced reactions on ^{16}O are compared with experimental data, with encouraging results.

I. INTRODUCTION

SEVERAL treatments[1–4] of the three-body problem which include the Coulomb force have recently appeared in the literature. In one of these, Nutt[1] considered the case in which three charged particles interact only via Coulomb forces. This requires a knowledge of the two-body Coulomb T matrix off the energy shell. Although in some special cases, such as that in which one particle is much heavier than the other two, it can be reduced to a manageable form, this is a complex mathematical entity and the general case seems to be intractable at present.

The Coulomb force is encountered in a simpler way, however, in the case in which the three particles are strongly interacting with only two of them charged. Although the Faddeev kernels still contain the two-body Coulomb T matrix in this case, there exists the possibility of dealing with it if the charge number is low by replacing it with the Coulomb potential.[4,5] Another possibility in this case is to redefine[2,6] the three-body scattering amplitudes in such a way that they do not contain pure Coulomb contributions and solve the three-body equations with the understanding that Rutherford-type terms are ignored.

The objective of the present paper is to begin with a standard definition of the scattering amplitudes and then systematically eliminate the pure Coulomb contributions from the final three-body equations. Three-body amplitudes obtained by solving these equations will then be added to the pure Coulomb amplitudes to obtain the final form. It is believed that this procedure is more instructive than defining directly the three-body scattering amplitudes which are free of the pure Coulomb contributions.

In the case considered the three-body breakup channel is ignored, and the nuclear forces are assumed to be separable. Form factors of the Hulthén form are used, and the bound pairs are assumed to have $l=0$ bound states only. The formalism is based on an approximate form for the Coulomb wave function in momentum space originally suggested by Guth and Mullin and used by Schulman[3] (hereafter the GMS approximation). A set of equations similar to that obtained by Lovelace[7] is developed with the resulting advantage of dealing directly with the scattering amplitudes.

A convention used by Noble[2] is adopted to avoid the difficulties encountered in using the Coulomb force in a Lippmann-Schwinger (LS) type of equation.[8] This treats the Coulomb potentials as though they were cut off at a distance much larger than the ranges of the other interactions involved in the problem. After obtaining the final equations, this cutoff radius is allowed to become arbitrarily large.

It is convenient to rewrite the two-body operators modified by the Coulomb force using the GMS approximation for the Coulomb wave function, and this is done in Sec. II. Section III deals with the Faddeev-Lovelace equations for this case and their reduction to a convenient form for computation. In Sec. IV some of the matrix elements are shown explicitly, and Sec. V deals with the numerical method. Application of the resulting formalism to the d-^{16}O system is given in Sec. VI.

II. TWO-BODY OPERATORS AND THE COULOMB GREEN'S FUNCTION

The system considered consists of two charged and one neutral particle in which one pair is always bound, so that the breakup channel is ignored. Let the charged particles be labeled 1 and 2 and the neutral 3. Denote the pair interactions by V_1, V_2, and V_3, where $V_\alpha = V_{\beta\gamma}$, $\alpha, \beta, \gamma = 1, 2, 3$, is the nuclear interaction between particles β and γ. Following Noble,[2] the total Hamiltonian H is

$$H = H_0 + U + V, \tag{1}$$

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[1] G. L. Nutt, J. Math. Phys. **9**, 796 (1968).
[2] J. V. Noble, Phys. Rev. **161**, 945 (1967).
[3] L. Schulman, Phys. Rev. **156**, 1129 (1967); E. Guth and C. J. Mullin, *ibid.* **83**, 667 (1951).
[4] E. O. Alt, P. Grassberger, and W. Sandhas, Nucl. Phys. **B2**, 167 (1967).
[5] V. A. Alessandrini, C. A. Garcia, and H. Fanchiotti, Phys. Rev. **170**, 935 (1968).
[6] S. Adya, Phys. Rev. **166**, 991 (1968).
[7] C. Lovelace, Phys. Rev. **135**, B1225 (1964).
[8] W. F. Ford, J. Math. Phys. **7**, 626 (1966).

where U is the Coulomb potential between particles 1 and 2, and

$$V=V_1+V_2+V_3. \tag{2}$$

Using the kinematic system of Lovelace,[7] the c.m. momentum of the (2, 3) subsystem is

$$\mathbf{p}_1=\{\sqrt{2}[m_2m_3(m_2+m_3)]^{1/2}\}^{-1}[m_3\mathbf{k}_2-m_2\mathbf{k}_3], \tag{3}$$

and the momentum of particle 1 relative to the (2, 3) subsystem is

$$\mathbf{q}_1=\{\sqrt{2}[m_1(m_2+m_3)(m_1+m_2+m_3)]^{1/2}\}^{-1}\times[m_1(\mathbf{k}_2+\mathbf{k}_3)-(m_2+m_3)\mathbf{k}_1], \tag{4}$$

where m_1, m_2, and m_3 are the masses and $\mathbf{k}_1$, $\mathbf{k}_2$, and $\mathbf{k}_3$ are the momenta of the three particles in the c.m. of the three-body system. Then the c.m. kinetic energy of the system is

$$H_0=p_1^2+q_1^2. \tag{5}$$

Other sets of momenta may be obtained by cyclic permutation of (3) and (4).

Define the operators

$$G(S)=(H-S)^{-1}, \tag{6}$$

$$G_0(S)=(H_0-S)^{-1}, \tag{7}$$

and

$$G_0^C(S)=(H_0+U-S)^{-1}. \tag{8}$$

Because particle 3 is neutral, its momentum is conserved in the pure Coulomb field, so that in the "third-particle coordinates" in momentum space

$$\langle\mathbf{p}_3,\mathbf{q}_3\mid G_0^C(S)\mid\mathbf{p}_3',\mathbf{q}_3'\rangle=\delta(\mathbf{q}_3'-\mathbf{q}_3)\langle\mathbf{p}_3\mid G_0^C(S-q_3^2)\mid\mathbf{p}_3'\rangle. \tag{9}$$

Using the spectral resolution of $G_0^C(S)$, Eq. (9) can be written

$$\langle\mathbf{p}_3,\mathbf{q}_3\mid G_0^C(S)\mid\mathbf{p}_3',\mathbf{q}_3'\rangle=\delta(\mathbf{q}_3'-\mathbf{q}_3)\times\int\frac{d^3k\langle\mathbf{p}_3\mid\phi_k^C\rangle\langle\phi_k^C\mid\mathbf{p}_3'\rangle}{q_3^2+k^2-S}, \tag{10}$$

where $\langle\mathbf{p}\mid\phi_k^C\rangle$ is the Coulomb wave function in momentum space. This simple δ-function factoring for $\mathbf{q}_3$ is true only in the $(\mathbf{p}_3,\mathbf{q}_3)$ set; consequently the matrix elements of the Coulomb Green's function should be taken in that set and then transformed into the desired set of coordinates by use of closure.

The GMS[3] approximation for the Coulomb wave function is that

$$\int\phi_k^C(\mathbf{p})f(\mathbf{p})\,d^3p\cong f(\mathbf{k})\int\phi_k^C(\mathbf{p})\,d^3p=f(\mathbf{k})[\psi_k^C(\mathbf{r})]_{r=0}, \tag{11}$$

where $\psi_k^C(\mathbf{r})$ is the Coulomb wave function in configuration space. At the origin

$$[\psi_k^C(\mathbf{r})]_{r=0}=\left(\frac{2\pi\mu e^2Z_1Z_2/k}{\exp(2\pi e^2Z_1Z_2\mu/k)-1}\right)^{1/2}\equiv C(k), \tag{12}$$

where μ is the reduced mass, Z_1 and Z_2 are the charge numbers of the (1, 2) pair, and $|C(k)|^2$ is the barrier penetration factor. Thus $\phi_k^C(\mathbf{p})$ can be replaced by $C(k)\delta(\mathbf{k}-\mathbf{p})$ whenever it occurs in the integrand:

$$\phi_k^C(\mathbf{p})\to C(k)\delta(\mathbf{k}-\mathbf{p}). \tag{13}$$

This approximation is valid only if $f(\mathbf{p})$ is square-integrable, but it will only be used with functions of the Hulthén type for which the criterion is always satisfied. Using it, the Coulomb Green's-function matrix element in (10) can be replaced by

$$\langle\mathbf{p}_3,\mathbf{q}_3\mid G_0^C(S)\mid\mathbf{p}_3',\mathbf{q}_3'\rangle\to\frac{\delta(\mathbf{q}_3'-\mathbf{q}_3)\delta(\mathbf{p}_3'-\mathbf{p}_3)\mid C(p_3)\mid^2}{p_3^2+q_3^2-S}. \tag{14}$$

In the presence of the Coulomb field, the two-particle amplitudes in the three-particle Hilbert space which will be needed are defined as

$$T_\alpha(S)\equiv V_\alpha+V_\alpha G_\alpha^C(S)V_\alpha, \tag{15}$$

which satisfies

$$T_\alpha(S)\equiv V_\alpha+V_\alpha G_0^C(S)T_\alpha(S), \tag{16}$$

where

$$G_\alpha^C(S)=(H_0+V_\alpha+U-S)^{-1}. \tag{17}$$

If the nuclear potential is separable, (16) can be solved exactly to give

$$T_\alpha(S)=\mid\alpha\rangle\hat{\tau}_\alpha(S)\langle\alpha\mid, \tag{18}$$

where the caret indicates a two-body Hilbert-space operator,

$$V_\alpha=\lambda_\alpha\mid\alpha\rangle\langle\alpha\mid, \tag{19}$$

and

$$\hat{\tau}_\alpha(S)=[\lambda_\alpha^{-1}+\hat{\tau}_\alpha(S)]^{-1}, \tag{20}$$

$$\hat{\tau}_\alpha(S)=\langle\alpha\mid G_0^C(S)\mid\alpha\rangle. \tag{21}$$

The form (19) is chosen on the assumption that the pair α possesses only one bound state.

The relation

$$\langle\mathbf{p}_\alpha,\mathbf{q}_\alpha\mid T_\alpha(S)\mid\mathbf{p}_\alpha',\mathbf{q}_\alpha'\rangle=\delta(\mathbf{q}_\alpha-\mathbf{q}_\alpha')\langle\mathbf{p}_\alpha\mid\hat{t}_\alpha(S-q_\alpha^2)\mid\mathbf{p}_\alpha'\rangle \tag{22}$$

is exact for $\alpha=3$, while for $\alpha=1$, 2 it is true only under the GMS approximation (13). Taking the form factor to be of the Hulthén type,[9]

$$\langle\alpha\mid\mathbf{p}_\alpha\rangle\equiv g_\alpha(p_\alpha)=N_\alpha/(p_\alpha^2+\mu_\alpha^2), \tag{23}$$

and the matrix element in (22) then can be evaluated explicitly. Thus only the matrix element of $\hat{\tau}_\alpha(S)$ is needed for (18) and (20).

For $\alpha=3$,

$$\langle\mathbf{q}_3\mid\hat{\tau}_3(S)\mid\mathbf{q}_3'\rangle\equiv\delta(\mathbf{q}_3'-\mathbf{q}_3)r_3(S;\mathbf{q}_3)=\delta(\mathbf{q}_3'-\mathbf{q}_3)\int\frac{\mid g_3^C(k)\mid^2}{k^2+q_3^2-S}\,d^3k, \tag{24}$$

[9] Y. Yamaguchi, Phys. Rev. 95, 1628 (1954).

where use has been made of (10). Using (23),

$$g_3^C(k) = \int g_3(p)\phi_k^C(\mathbf{p})\,d^3p, \tag{25}$$

and this can be evaluated explicitly, giving[10]

$$g_3^C(k) = g_3(k)C_0(\eta)\,\exp[2\eta\,\tan^{-1}(k\mu_3^{-1})], \tag{26}$$

with

$$\eta = e^2Z_1Z_2\mu/k. \tag{27}$$

For $\alpha=1, 2$ the integrations in the matrix elements of $\hat{\tau}_\alpha(S)$ can be carried out explicitly using the GMS approximation (13) to give

$$\langle \mathbf{q}_\alpha \mid \hat{\tau}_\alpha(S) \mid \mathbf{q}_\alpha' \rangle$$
$$\equiv \delta(\mathbf{q}_\alpha'-\mathbf{q}_\alpha) r_\alpha(S;\mathbf{q}_\alpha)$$
$$= \delta(\mathbf{q}_\alpha'-\mathbf{q}_\alpha)\int \frac{|\,g_\alpha(p_\alpha)\,|^2 C^2(|\,\gamma_1\mathbf{p}_\alpha+\gamma_2\mathbf{q}_\alpha|)}{p_\alpha^2+q_\alpha^2-S}\,d^3p_\alpha, \tag{28}$$

where

$$\gamma_1\equiv\gamma_1^{\alpha 3}=(m_\alpha m_3/M_\alpha M_3)^{1/2}, \quad \gamma_2\equiv\gamma_2^{\alpha 3}=(m_\alpha M/M_\alpha M_3)^{1/2}, \tag{29}$$

with

$$M=m_1+m_2+m_3, \qquad M_\gamma=M-m_\gamma, \tag{30}$$

where m_α is the mass of particle α. The parameter λ_α is chosen to be

$$\lambda_\alpha^{-1}=-r_\alpha(q_\alpha^2-E_\alpha;\mathbf{q}_\alpha). \tag{31}$$

Expressions (24), (28), and (31) show that the propagator (21) can be written

$$\langle \mathbf{q}_\alpha \mid \hat{\tau}_\alpha(S) \mid \mathbf{q}_\alpha' \rangle = \langle \mathbf{q}_\alpha \mid [\langle \alpha \mid G_0^C(S)G_0(q_\alpha^2-E_\alpha) \mid \alpha\rangle$$
$$\times(S-q_\alpha^2-E_\alpha)]^{-1} \mid \mathbf{q}_\alpha'\rangle, \qquad \alpha=1, 2, 3. \tag{32}$$

III. FADDEEV EQUATIONS

Denoting by γ the channel in which particle γ is free, Lovelace[7] defines the transition amplitude between channels α and β on the energy shell to be

$$\hat{X}_{\alpha\beta}(S) = \langle \chi_\alpha \mid \bar{U}_{\alpha\beta}^{\pm}(S) \mid \chi_\beta \rangle, \tag{33}$$

where

$$\bar{U}_{\alpha\beta}^{+}(S) = \bar{V}_\alpha - \bar{V}_\alpha G(S)\bar{V}_\beta, \tag{34}$$

$$\bar{U}_{\alpha\beta}^{-}(S) = \bar{V}_\beta - \bar{V}_\alpha G(S)\bar{V}_\beta, \tag{35}$$

and

$$\bar{V}_\gamma = V - V_\gamma, \tag{36}$$

with $\mid \chi_\gamma\rangle$ being the channel wave function, i.e.,

$$(H_0+V_\alpha) \mid \chi_\alpha\rangle = E \mid \chi_\alpha\rangle, \tag{37}$$

and

$$S=E+i\epsilon. \tag{38}$$

The limit $\epsilon\to 0$ is understood in (33). Though the operators $\bar{U}_{\alpha\beta}^{+}(S)$ and $\bar{U}_{\alpha\beta}^{-}(S)$ possess different off-shell extensions, they yield the same scattering amplitudes on the energy shell, and consequently only $\bar{U}_{\alpha\beta}^{+}(S)$ will be considered.

A generalization of Lovelace's equations to the case of particles 1 and 2 charged and particle 3 neutral can be accomplished by the replacement

$$V_3 \to V_3+U, \tag{39}$$

where U is the Coulomb potential and V_3 the nuclear potential. Using (39), Eq. (34) takes the form

$$\bar{U}_{\alpha\beta}^{+}(S) = (U_\alpha+\bar{V}_\alpha) - (U_\alpha+\bar{V}_\alpha)G(S)(\bar{V}_\beta+U_\beta), \tag{40}$$

where

$$U_\alpha = (1-\delta_{\gamma 3})U \tag{41}$$

and $G(S)$ is defined by (1) and (6). The channel wave functions become the solutions of

$$(H_0+V_\gamma+\delta_{\gamma 3}U) \mid \chi_\gamma\rangle = E \mid \chi_\gamma\rangle. \tag{42}$$

For separable nuclear potentials the solutions of (42) can be written explicitly. Using (19), Eq. (42) takes the form

$$(H_0+\delta_{\gamma 3}U-E) \mid \chi_\gamma\rangle = -\lambda_\gamma \mid \gamma\rangle\langle\gamma \mid \chi_\gamma\rangle. \tag{43}$$

For $\gamma\neq 3$ this possesses the solution[11]

$$\mid \chi_\gamma\rangle = -G_0(E) \mid \gamma\rangle, \tag{44}$$

and for $\gamma=3$

$$\mid \chi_3\rangle = -G_0^C(E) \mid 3\rangle, \tag{45}$$

with $G_0^C(E)$ defined by (8). Solutions (44) and (45) are valid provided that

$$\lambda_\gamma\langle\gamma \mid G_0(E) \mid \gamma\rangle = -1 \tag{46}$$

for $\gamma=1, 2$, and

$$\lambda_3\langle 3 \mid G_0^C(E) \mid 3\rangle = -1 \tag{47}$$

for $\gamma=3$. This can be demonstrated by direct substitution in (43).

Equation (40) can be transformed in a straightforward manner into a set of coupled integral equations similar to that obtained by Lovelace.[7] However, because of the presence of U_α and U_β in (40), the kernels of the integral equations will contain pure Coulomb two-body T matrices.[4] This causes no difficulty if one is interested only in the case in which all the three particles are bound, since in this case only one angular momentum state is required after partial-wave analysis. However, if one of the charged particles is free, as in the present case, it will no longer be possible to do partial-wave analysis on the integral equations, since it is known that the partial-wave series for the on-shell pure Coulomb scattering amplitudes do not converge. This causes the inconvenience of having to deal with multidimensional integrals.

These difficulties can be avoided if the explicit presence of U_α and U_β in (40) is eliminated by separating the pure Coulomb contributions. The techniques

[10] D. R. Harrington, Phys. Rev. **139**, B691 (1965).

[11] G. C. Chirardi and A. Rimini, J. Math. Phys. **5**, 722 (1964).

for carrying out a similar separation are developed in the literature[12] and will be adopted here for the case of the Coulomb potential with minor modifications. Defining

$$\Omega_\beta^+(S)\equiv 1-G(S)\,(\bar{V}_\beta+U_\beta), \tag{48}$$

Eq. (40) takes the form

$$\bar{U}_{\alpha\beta}^+(S)=(\bar{V}_\alpha+U_\alpha)\Omega_\beta^+(S). \tag{49}$$

Also defining

$$\omega_\alpha^{(-)\dagger}(S)\equiv 1-U_\alpha G_\alpha^C(S), \tag{50}$$

where

$$G_\alpha^C(S)=(H_0+V_\alpha+U-S)^{-1}, \tag{51}$$

the identity operator can be written

$$1=\omega_\alpha^{(-)\dagger}(S)+U_\alpha G_\alpha^C(S). \tag{52}$$

The Coulomb potential U_α can be eliminated from (49) by multiplying the first term on the right side by (52) and rearranging so that

$$\bar{U}_{\alpha\beta}^+(S)=\omega_\alpha^{(-)\dagger}(S)\,\bar{V}_\alpha\Omega_\beta^+ + U_\alpha[G_\alpha^C(S)\,\bar{V}_\alpha+1]\Omega_\beta^+(S). \tag{53}$$

Defining $A_{\alpha\beta}$ to be the second term on the right side in (53) and using (48), it can be written

$$A_{\alpha\beta}=U_\alpha G_\alpha^C(S)\,(H_0+V_\beta+U-U_\beta-E-i\epsilon), \tag{54}$$

where use has been made of the identity

$$(H_0+V_\alpha+U-S)^{-1}-(H_0+V+U-S)^{-1} =(H_0+V_\alpha+U-S)^{-1}\bar{V}_\alpha(H_0+V+U-S)^{-1} \tag{55}$$

and

$$\bar{V}_\alpha-\bar{V}_\beta=V_\beta-V_\alpha. \tag{56}$$

Making use of (41) and the fact that $A_{\alpha\beta}$ is defined inside the matrix element $\langle\chi_\alpha\mid A_{\alpha\beta}\mid\chi_\beta\rangle$, Eq. (54) becomes

$$A_{\alpha\beta}=-i\epsilon U_\alpha G_\alpha^C(S). \tag{57}$$

Using the "second" resolvent identity,[7]

$$G_\alpha^C(S)=G_\alpha(S)-G_\alpha^C(S)\,U_\alpha G_\alpha(S), \tag{58}$$

where $\alpha=1$ is assumed to be the incident channel and

$$G_\alpha(S)=(H_0+V_\alpha-S)^{-1}. \tag{59}$$

Then

$$\langle\chi_\alpha\mid A_{\alpha\beta}\mid\chi_\beta\rangle=-i\epsilon\langle\chi_\alpha\mid\omega_\alpha^{(-)\dagger}(S)\,U_\alpha G_\alpha(S)\mid\chi_\beta\rangle. \tag{60}$$

Consider the system d-^{16}O. If the incident channel is labeled α, then for $\beta=\alpha$

$$\langle\chi_\alpha\mid A_{\alpha\alpha}\mid\chi_\alpha\rangle=\langle\chi_\alpha\mid\omega_\alpha^{(-)\dagger}(S)\,U_\alpha\mid\chi_\alpha\rangle. \tag{61}$$

In this case U is the Coulomb potential of a proton relative to ^{16}O. If U_d is the Coulomb potential of a deuteron relative to ^{16}O, then $U-U_d$ vanishes asymptotically.[2] Replacing U by U_d in (61), one obtains the Rutherford scattering amplitude[12]

$$A_C=\langle\chi_d\mid\omega_d^{(-)\dagger}(S)\,U_d\mid\chi_d\rangle. \tag{62}$$

[12] K. Greider and L. Dodd, Phys. Rev. **146**, 671 (1966).

For $\alpha\neq\beta$ and assuming that U_α may not give rise to rearrangement, i.e., ignoring Coulomb stripping-type contributions, then

$$\langle\chi_\alpha\mid A_{\alpha\beta}\mid\chi_\beta\rangle=0, \qquad \alpha\neq\beta. \tag{63}$$

This is the same assumption used for the optical-model potential.[12] In the present case the Coulomb potential is treated essentially in the same way as the optical-model potential is treated in rearrangement collisions.

With (62) and (63) the matrix elements of (54) take the form

$$\langle\chi_\alpha\mid\bar{U}_{\alpha\beta}^+(S)\mid\chi_\beta\rangle =\langle\chi_\alpha\mid\omega_\alpha^{(-)\dagger}(S)\,\bar{V}_\alpha\Omega_\beta^+(S)\mid\chi_\beta\rangle+A_C\delta_{\alpha\beta}. \tag{64}$$

It remains to eliminate the explicit dependence of Ω_β^+ on U_β in (48). This is accomplished by using techniques similar to the ones used in deriving the two-body, two-potential scattering formula.[13] Writing

$$\Omega_\beta^+(S)=[1-G(S)\,\bar{V}_\beta]\omega_\beta^{(+)}(S), \tag{65}$$

where

$$\omega_\beta^{(+)}(S)=1-G_\beta^C(S)\,U_\beta \tag{66}$$

and

$$G_\beta^C(S)=(H_0+V_\beta+U-S)^{-1}, \tag{67}$$

and substituting this into (64), one obtains

$$\langle\chi_\alpha\mid\bar{U}_{\alpha\beta}^+(S)\mid\chi_\beta\rangle =\langle\chi_\alpha\mid\omega_\alpha^{(-)\dagger}(S)\,U_{\alpha\beta}^+(S)\omega_\beta^{(+)}(S)\mid\chi_\beta\rangle+A_C\delta_{\alpha\beta}, \tag{68}$$

where

$$U_{\alpha\beta}^+(S)=\bar{V}_\alpha-\bar{V}_\alpha G(S)\,\bar{V}_\beta. \tag{69}$$

This is similar to the operator defined by Lovelace's Eq. (3.2).[7]

With $\alpha=1$ as the fixed incident channel, the target is labeled as 1, the proton as 2, and the neutron as 3. Ignoring the breakup channel, then $\beta=1$, 2, or 3. Having defined the channels, one may look for more manageable forms for the transition amplitudes than the ones defined in (68). The "distorting" operators $\omega_\alpha^{(-)\dagger}$ and $\omega_\beta^{(+)}$ depend on both the nuclear and Coulomb potentials, while the channel wave function $\mid\chi_\beta\rangle$ is asymmetric in the sense that it is a solution of the nuclear plus Coulomb potentials in (42). For $\beta=3$, Eq. (41) gives $U_\beta=0$, and from Eq. (66), $\omega_\beta^{(+)}(S)=1$. For $\beta=1$, 2,

$$\omega_\beta^{(+)}(S)\mid\chi_\beta\rangle=-i\epsilon G_\beta^C(S)\mid\chi_\beta\rangle. \tag{70}$$

Since the $T_\beta(S)$ are solutions of (16), it follows that

$$G_\beta^C(S)=G_0^C(S)-G_0^C(S)\,T_\beta(S)\,G_0^C(S). \tag{71}$$

From (17), (32), and (44),

$$\omega_\beta^{(+)}(S)\mid\chi_\beta\rangle\mid\mathbf{q}_\beta\rangle=-i\epsilon G_0^C(S)\mid\chi_\beta\rangle\mid\mathbf{q}_\beta\rangle -G_0^C(S)\mid\beta\rangle\mid\mathbf{q}_\beta\rangle, \tag{72}$$

[13] M. L. Goldberger and K. M. Watson, *Collision Theory* (Wiley-Interscience, Inc., New York, 1964).

where the property (22) and the fact that the above operations are on the energy shell with

$$S=E+i\epsilon=q_\beta^2-E_\beta+i\epsilon \tag{73}$$

have been used. The first term on the right side of (72) can be dropped in the limit $\epsilon\to 0$ without causing any of the known difficulties,[14] because on the energy shell $G_0^C(S)$ is nonsingular. Similar arguments can be used for $\langle\chi_\alpha \mid \omega_\alpha^{(-)\dagger}(S)$. Using (45) and (72), Eq. (68) then takes the form

$$\langle\chi_\alpha \mid \bar{U}_{\alpha\beta}^+(S) \mid \chi_\beta\rangle = \langle\alpha \mid G_0^C(S)\,U_{\alpha\beta}^+(S)\,G_0^C(S) \mid \beta\rangle + A_C\delta_{\alpha\beta}. \tag{74}$$

Define the on-the-energy-shell amplitude

$$\hat{X}_{\alpha\beta}(S)=\langle\alpha \mid G_0^C(S)\,U_{\alpha\beta}^+(S)\,G_0^C(S) \mid \beta\rangle. \tag{75}$$

This effectively indicates that the channel wave function for channel γ is $G_0^C(S) \mid \gamma\rangle$. Since on the energy shell $G_0^C(S)$ is nonsingular, the Coulomb potential could have been included in (43) for all channels, or

$$(H_0+U-S) \mid \chi_\gamma\rangle=-\lambda_\gamma \mid \gamma\rangle\langle\gamma \mid \chi_\gamma\rangle-O(\epsilon). \tag{76}$$

Thus the channel wave functions, correct to terms of order ϵ, are given by

$$\mid \chi_\gamma\rangle=-G_0^C(S) \mid \gamma\rangle, \tag{77}$$

with the condition

$$\lambda_\gamma\langle\gamma \mid G_0^C(S) \mid \gamma\rangle=-1, \tag{78}$$

on the energy shell. This condition is the same as (31) and is necessary in order to make the T matrix of (16) possess a branch point on the energy shell (a simple pole for the two-body problem). This is not possible with condition (46), and thus it was necessary to remove the explicit dependence of $\bar{U}_{\alpha\beta}^+(S)$ on the Coulomb potentials, which then reappears in the equation for the channel wave function (76).

Since A_C in (74) is known, it remains to calculate $\hat{X}_{\alpha\beta}(S)$ in (75). The procedure used is similar to that of Lovelace,[7] except for the replacement of $G_0(S)$ by $G_0^C(S)$. Using the resolvent identity

$$G(S)=G_\beta^C(S)-\sum_{\delta\neq\beta} G(S)\,V_\delta G_\beta^C(S), \tag{79}$$

Eq. (69) takes the form

$$U_{\alpha\beta}^+(S)=\sum_{\gamma\neq\alpha} V_\gamma-\sum_{\delta\neq\beta} V_\gamma G(S)\,V_\delta = \sum_{\gamma\neq\alpha} V_\gamma-\sum_{\delta\neq\beta} U_{\alpha\delta}^+(S)\,G_\delta^C(S)\,V_\delta, \tag{80}$$

with $G_\delta^C(S)$ given by (67). Using the relation

$$G_0^C(S)\,T_\delta(S)=G_\delta^C(S)\,V_\delta, \tag{81}$$

Eq. (80) becomes

$$U_{\alpha\beta}^+(S)=\sum_{\gamma\neq\alpha} V_\gamma-\sum_{\delta\neq\beta} U_{\alpha\delta}^+(S)\,G_0^C(S)\,T_\delta(S). \tag{82}$$

Following Lovelace,[7] the "potential" is defined to be

$$\hat{Z}_{\alpha\beta}(S)=(1-\delta_{\alpha\beta})\,\langle\alpha \mid G_0^C(S) \mid \beta\rangle, \tag{83}$$

and the off-shell extension of $\hat{X}_{\alpha\beta}(S)$ in (75) to be

$$\hat{X}_{\alpha\beta}(S)=\langle\alpha \mid G_0^C(S)\,U_{\alpha\beta}^+(S)\,G_0^C(S) \mid \beta\rangle - \hat{Z}_{\alpha\beta}(S)[1+\lambda_\beta\hat{\tau}_\beta(S)]. \tag{84}$$

This reduces to (75) on the energy shell by virtue of (78) and (21). Finally, using (16)–(21), Eqs. (82) and (84) give

$$\hat{X}_{\alpha\beta}(S)=-\hat{Z}_{\alpha\beta}(S)-\sum_\delta \hat{X}_{\alpha\delta}(S)\hat{\tau}_\delta(S)\hat{Z}_{\delta\beta}(S). \tag{85}$$

The solutions to (85) are the three-body amplitudes to which the pure Coulomb contributions must be added. They can be found if all the $\hat{Z}_{\alpha\beta}(S)$ and all the $\hat{\tau}_\delta(S)$ are known. Again this equation is identical to that of Lovelace,[7] except that the "potential" $\hat{Z}_{\alpha\beta}(S)$ and the propagators $\hat{\tau}_\delta(S)$ now contain $G_0^C(S)$ instead of $G_0(S)$.

As mentioned earlier, the pure Coulomb contributions are approximated to be zero in all channels except in the elastic channel, and even in the elastic channel they are approximated by the two-body pure Coulomb scattering amplitude.[2] This approximation is not very well justified when the deuteron is close to the target nucleus. For the rearrangement channel, however, the particles must come close enough together for the nuclear force to cause a mass transfer reaction, and thus the nuclear interaction should dominate in that channel. On the other hand, the deuteron can scatter elastically without coming close enough to the target nucleus to feel the effect of the nuclear forces. Replacing the proton Coulomb potential by the deuteron Coulomb potential for deuteron scattering is justified if the Coulomb repulsive force is strong enough. The approximation can be expected to be reasonably good for systems like d-^{16}O or d-^{28}Si and not for systems like p-d.

IV. MATRIX ELEMENTS OF THE POTENTIAL OPERATORS AND THE SCATTERING AMPLITUDES

The matrix elements of the potential "operators" defined in (83) must be evaluated before (85) can be solved. The operations are elementary and the results will only be stated. For $\alpha, \beta\neq 3$,

[14] L. Foldy and W. Tobocman, Phys. Rev. **105**, 1099 (1957); S. Epstein, *ibid.* **106**, 598 (1957); B. Lippmann, *ibid.* **102**, 264 (1956).

$$\langle \mathbf{q}_\alpha \mid Z_{\alpha\beta}(S) \mid \mathbf{q}_\beta' \rangle = (1-\delta_{\alpha\beta})\gamma_{12}{}^{-3} \frac{g_\alpha[\gamma_{12}{}^{-1}(\mathbf{q}_\beta'+\gamma_{12}'\mathbf{q}_\alpha)]g_\beta[\gamma_{12}{}^{-1}(\gamma_{12}'\mathbf{q}_\beta'+\mathbf{q}_\alpha)]C^2(\gamma_{12}{}^{-1} \mid \gamma_1'\mathbf{q}_\alpha-\gamma_1\mathbf{q}_\beta' \mid)}{\gamma_{12}{}^{-2}(\gamma_{12}\mathbf{q}_\beta'+\mathbf{q}_\alpha)^2+q_\beta'^2-S}, \quad (86)$$

where

$$\gamma_1'=\gamma_1{}^{\beta 3}, \quad \gamma_2'=\gamma_2{}^{\beta 3}, \quad \gamma_1=\gamma_1{}^{\alpha 3}, \quad \gamma_2=\gamma_2{}^{\alpha 3}, \quad \gamma_{12}=\gamma_2{}^{\alpha\beta}, \quad \gamma_{12}'=\gamma_1{}^{\alpha\beta}, \quad (87)$$

and $\gamma_1{}^{\delta\gamma}$ and $\gamma_2{}^{\delta\gamma}$ are defined in (29). For $\alpha=3$,

$$\langle \mathbf{q}_3 \mid \hat{Z}_{3\beta}(S) \mid \mathbf{q}_\beta \rangle = (1-\delta_{3\beta})\gamma_2{}^{-3}g_3{}^C(\gamma_2{}^{-1}(\gamma_1\mathbf{q}_3+\mathbf{q}_\beta')) \frac{C(\mid \gamma_2{}^{-1}(\gamma_1\mathbf{q}_3+\mathbf{q}_\beta') \mid)g_\beta(\gamma_2{}^{-1}(\mathbf{q}_3+\gamma_1\mathbf{q}_\beta'))}{\gamma_2{}^{-2}(\gamma_1\mathbf{q}_3+\mathbf{q}_\beta')^2+q_3{}^2-S}. \quad (88)$$

A similar expression is obtained for $Z_{\alpha 3}(S)$ by replacing β by α in (88).

Using these expressions for the potentials and the propagators, (85) can be solved. Its matrix elements in the channel plane waves are

$$\langle \mathbf{q}_\alpha \mid \hat{X}_{\alpha\beta}(S) \mid \mathbf{q}_\beta' \rangle = -\langle \mathbf{q}_\alpha \mid \hat{Z}_{\alpha\beta}(S) \mid \mathbf{q}_\beta' \rangle - \sum_\delta \int \langle \mathbf{q}_\alpha \mid X_{\alpha\delta}(S) \mid \mathbf{q}_\delta'' \rangle \tau_\delta(S; \mathbf{q}_\delta'') \times \langle \mathbf{q}_\delta'' \mid \hat{Z}_{\alpha\beta}(S) \mid \mathbf{q}_\beta' \rangle d^3q_\delta'', \quad (89)$$

where

$$\langle \mathbf{q}_\delta'' \mid \hat{\tau}_\delta(S) \mid \mathbf{q}_\delta''' \rangle = \delta(\mathbf{q}_\delta''-\mathbf{q}_\delta''')\tau_\delta(S; \mathbf{q}_\delta''). \quad (90)$$

The expression in (89) is an integral equation with three-dimensional integrals. Two of the variables of integration can be eliminated by partial-wave analysis. Writing

$$\langle \mathbf{q} \mid \hat{X}_{\alpha\beta}(S) \mid \mathbf{q}' \rangle = \sum_{l=0}^{\infty} (2l+1)P_l(\cos\theta)X_{\alpha\beta}{}^l(q, q'; S), \quad (91)$$

where $\cos\theta=\hat{q}\cdot\hat{q}'$, and similar expressions for the $\hat{Z}_{\alpha\beta}(S)$ potentials, (89) can be reduced in partial waves to the form

$$X_{\alpha\beta}{}^l(q, q'; S) = -Z_{\alpha\beta}{}^l(q, q'; S) - 4\pi \sum_\delta \int X_{\alpha\delta}{}^l(q, q''; S)\tau_\delta(S; q'')Z_{\delta\beta}{}^l(q'', q'; S)q''{}^2dq'', \quad (92)$$

which is a one-dimensional integral equation. For the partial wave which contains bound states ($l=0$ in the present case), Eq. (92) cannot be solved by iteration, because the Born series does not converge for bound states and resonances.[15,16] Consequently, (89) is solved by matrix inversion techniques for all partial waves to avoid developing an unnecessarily lengthy computer code.

V. DEFORMED CONTOURS

The method of deformed contours used to solve (92) is by now standard,[16,17] and the present discussion will be limited to only a few comments. A detailed discussion is given elsewhere.[18] The procedure is to avoid the singularities which may develop on the path of integration of (92) by taking the limit $\epsilon\to 0$ in (86), (88), and (32) with $S=E+i\epsilon$. This is accomplished by rotating the momenta into the fourth quadrant using the mapping

$$\mathbf{q}_\delta\to\mathbf{q}_\delta \exp(-i\phi), \quad \mathbf{q}_\beta'\to\mathbf{q}_\beta' \exp(-i\phi). \quad (93)$$

The solution to (92) is then obtained and $\mathbf{q}_\beta'$ is rotated back to the real axis. By Cauchy's theorem this procedure is equivalent to solving (92) directly as long as no new singularities are crossed by the mapping (93).

For the uncharged particles and for the Hulthén form factors the partial-wave integrations over (86) and (88) can be carried out in a closed form. The singularities turn out to be the branch points of the $Q_l(Z)$ functions.[16] For the Coulomb case, however, integration over the angles in (86) and (88) must be carried out numerically because of $C(k)$. Consequently, the singularities were located as a function of the angle of integration and that nearest to the real axis was taken as $\phi_{\min}$, with $\phi=\frac{1}{2}\phi_{\min}$ in (93).

To illustrate this point consider the unmodified Hulthén form factors appearing in (86) and (88). Rotating only one of the momenta,

$$g_\alpha(k) = N_\alpha[(\gamma_5\mathbf{q}+\gamma_6\mathbf{q}' \exp(-i\phi))^2+\mu_\alpha{}^2]^{-1}, \quad (94)$$

gives the singularity

$$\phi_1 = \tan^{-1}[(\gamma_5{}^2q^2+\mu_\alpha{}^2-\gamma_5{}^2q^2x^2)^{1/2}/\gamma_5qx], \quad (95)$$

with $x=\hat{q}\cdot\hat{q}'$, and γ_5 and γ_6 are constants. As a function of x, ϕ_1 has a minimum at $x=1$, and

$$\phi_{1,\min} = \tan^{-1}(\mu_\alpha/\gamma_5 q). \quad (96)$$

The branch point in $C(k)$ causes no difficulty under (93), because only the square root with positive real

[15] S. Weinberg, Phys. Rev. **131**, 440 (1963).
[16] P. E. Shanley and R. Aaron, Ann. Phys. (N.Y.) **44**, 363 (1967), and refe rences therein.
[17] J. H. Hetherington and L. H. Shick, Phys. Rev. **137**, B935 (1965).
[18] K. Hamza, Ph.D. thesis, The Florida State University, 1968 (unpublished).

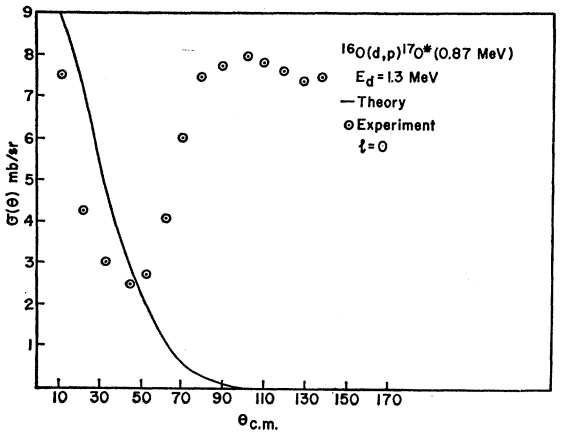

FIG. 1. Theoretical differential cross sections for $^{16}O(d, p)^{17}O^*$ (0.87 MeV) at deuteron incident energy of 1.3 MeV as compared with experimental data (Ref. 23).

value is taken. Also, the function $\exp[2\eta \tan^{-1}(k\beta^{-1})]$ is well behaved under (93). Writing $\eta=n/k$ in (27) and noting that for $|k|\to 0$ the function $\tan^{-1}(k\beta^{-1})\to k/\beta$, the factors multiplying $g_3(k)$ in (26) are well behaved under (93). This shows that the Coulomb force produces no new pathologies in (92).

VI. APPLICATIONS AND DISCUSSION

A relatively simple nuclear system to which the restricted problem treated here can be applied is the d-^{16}O system, if the deuteron bound state is taken to be the S state. For the outgoing channels there is an n-^{16}O bound state at 0.87 MeV relative to the ground state of ^{17}O with a binding energy of 3.275 MeV, and a p-^{16}O bound state at 0.500 MeV relative to the ground state of ^{17}F and with a binding energy of 0.096 MeV.[19]

The system is treated as three distinguishable particles. No attempt to include isospin is made, since it is known that the Coulomb force breaks isospin symmetry,[5] but the spin of the particles is included. For $l=0$ bound states spin and angular momenta uncouple in the LS representation,[7,16] and the total spin and its projection are conserved. Since ^{16}O has a spin of zero, the incident channel spin is that of the deuteron triplet state and has the value 1.

The scattering cross section is given by

$$d\sigma_{\alpha\beta}/d\Omega=[(2s_i+1)(2s_t+1)]^{-1}f_{\alpha\beta}\sum|\langle s_im_i, s_tm_t; \mathbf{q}| \times X_{\alpha\beta}|\mathbf{q}'; s_om_o, s_rm_r\rangle|^2, \quad (97)$$

where s and m are the spins and their projections for the incident particle i, the target t, the outgoing particle o, and the residual nucleus r, the sum is over all spin projections, and

$$f_{\alpha\beta}=(2\pi)^4(m_\alpha M_\alpha m_\beta M_\beta/M^2). \quad (98)$$

Substituting the values of the spins for the d-^{16}O system, this reduces to

$$d\sigma_{\alpha\beta}/d\Omega=f_{\alpha\beta}|X_{\alpha\beta}(\mathbf{q}, \mathbf{q}'; S)|^2. \quad (99)$$

The only parameters are those of the form factors defined in (23).

N_α is fixed by normalizing the wave functions defined in (44) and (45). The integral can be carried out in closed form[7] and one obtains for N_α

$$N_\alpha^2=\pi^{-2}\mu_\alpha\epsilon_\alpha(\epsilon_\alpha+\mu_\alpha)^3, \quad (100)$$

where

$$\epsilon_\alpha=\sqrt{E_\alpha} \quad (101)$$

and E_α is the binding energy of the α pair. The μ_α's are to be fixed from the two-body scattering data using the relation given by Yamaguchi[9]:

$$a_\alpha=2(\mu_\alpha+\epsilon_\alpha)^2/\mu_\alpha\epsilon_\alpha(\epsilon_\alpha+2\mu_\alpha), \quad (102)$$

where a_α is the scattering length and is related to the total scattering cross section at zero energy by

$$\sigma_T^\alpha(0)=4\pi a_\alpha^2. \quad (103)$$

The scattering length for the triplet p-n state has been measured to be 5.378 fm.[20] For neutron scattering from

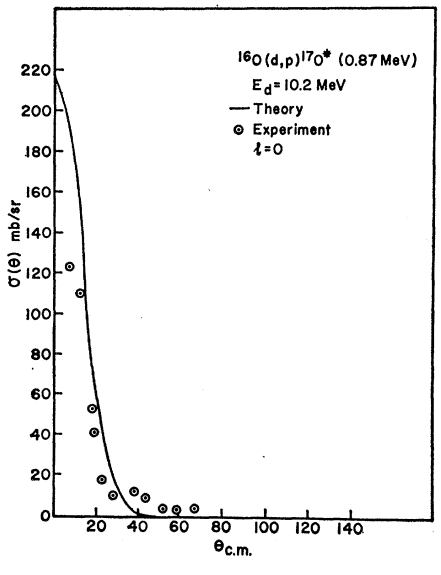

FIG. 2. Theoretical differential cross sections for $^{16}O(d, p)^{17}O^*$ (0.87 MeV) at deuteron incident energy of 10.2 MeV as compared with experimental data (Ref. 24).

[19] T. Lauritsen and F. Ajzenberg-Selov, *Energy Levels of Light Nuclei* (Printing and Publishing Office, National Academy of Sciences–National Research Council, Washington, D.C. 20025, 1962).

[20] M. Burgy, G. Ringo, and D. Hughes, Phys. Rev. **84**, 1160 (1950); E. Melkonian, *ibid.* **76**, 1744 (1949).

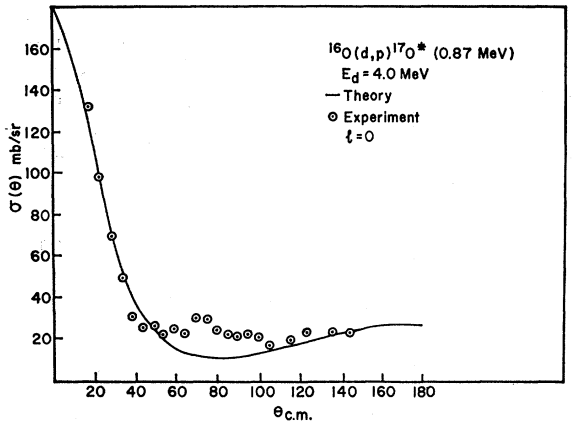

FIG. 3. Theoretical differential cross sections for $^{16}O(d, p)^{17}O^*$ (0.87 MeV) at deuteron incident energy of 4.0 MeV as compared with experimental data (Ref. 23).

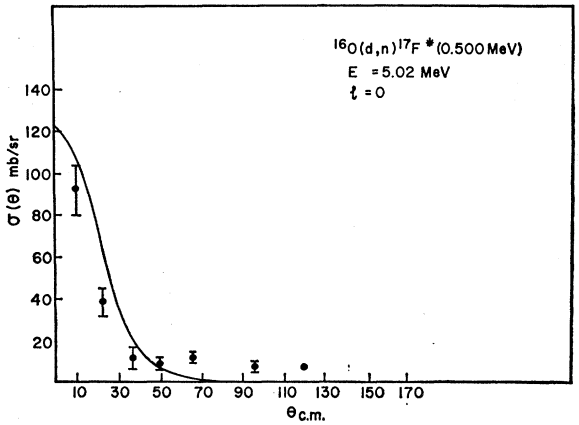

FIG. 5. Theoretical differential cross sections for $^{16}O(d, n)^{17}F^*$ (0.500 MeV) at deuteron incident energy of 5.02 MeV as compared with experimental data (Ref. 25).

^{16}O the total scattering cross section of thermal neutrons at 1 eV was measured to be 3.75 b.[21] This cross section is constant for the resonance-free range of energies 0.5–1000 eV, and it is used in (103) to extract the scattering length. The value of μ_α obtained in this way gave good results, and it was not further refined.

The same nuclear form factor $g_\alpha(k)$ is used for the n-^{16}O pair and p-^{16}O as was suggested by Harrington.[10] The presence of the Coulomb force modifies the nuclear form factor of p-^{16}O from $g_3(k)$ to $g_3^C(k)$ given in (26). The bound-state poles in the propagators are fixed through the λ_α's at the binding energy of each pair by numerical integration using expressions (31), (28), (24), and (73). Twelve-point Gaussian integration is used[22] for the partial-wave analysis and computing the λ_α's. To solve Eq. (92) using the method described in Sec. V, the integration range is subdivided into four sectors and the seven-point rule[22] is used on each of them.

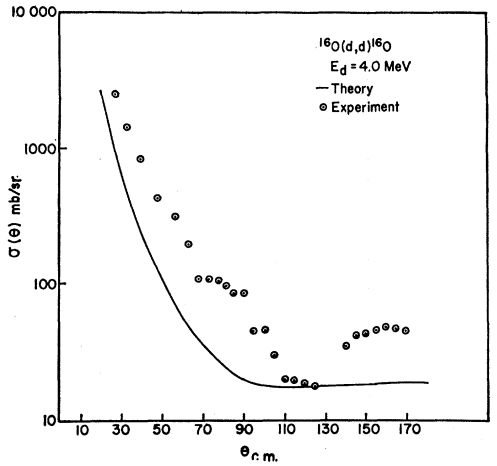

FIG. 4. Theoretical differential cross sections for $^{16}O(d, d)^{16}O$ at deuteron incident energy of 4.0 MeV as compared with experimental data (Ref. 23).

Some of the numerical results are compared with the data of Gallman *et al.*,[23] Hamburger,[24] and Yaramis[25] in Figs. 1–5. Remembering that only the simplest assumptions are made about the nuclear potential and only one bound state for each pair is considered, the results compare surprisingly well with the data. Since the three-body deuteron-induced reaction calculations reported in the literature[16,26] do not include the Coulomb force, it is not possible to make a direct comparison between them and the present results. A qualitative agreement is obtained, however, in the sense of a large forward peaking and small contributions at backward angles for the stripping channels.

The stripping differential cross sections above the 2.13-MeV (d, n) threshold shown in Figs. 2 and 3 are in good agreement with experiment, but the fit at an incident deuteron energy of 1.3 MeV in Fig. 1 is poor. This may stem from the fact that the higher angular momentum bound states are ignored.[16] Since both ^{17}O and ^{17}F have an $l=2$ ground state, it is expected that the present fit would be improved by including these two states. This remains to be done.

[21] D. Hughes and R. Schwartz, *Neutron Cross Sections*, compiled by D. J. Hughes and R. B. Schwartz (U.S. Government Printing Office, Washington, D.C., 1958).

[22] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun (U.S. Department of Commerce, National Bureau of Standards, Washington, D.C., 1966).

[23] A. Gallman, P. Fintz, and P. Hodgson, Nucl. Phys. **82**, 161 (1966).

[24] E. Hamburger, Phys. Rev. **123**, 619 (1961).

[25] B. Yaramis, Phys. Rev. **124**, 836 (1961).

[26] R. Aaron, R. D. Amado, and Y. Y. Yam, Phys. Rev. **136**, B650 (1964); R. Aaron and P. E. Shanley, *ibid.* **142**, 608 (1966).

Based on the fits at the energies available, it appears that agreement with experiment is good for the (d, n) channel and for the (d, p) channel above the (d, n) threshold. The elastic channel shown in Fig. 4 is dominated by pure Coulomb scattering and shows none of the structure effects observed experimentally. It is expected that better agreement will be obtained in future analyses by including the higher angular momentum bound states.

One problem with the method presented here is that the Hulthén form factor used in the separable approximation for the nuclear forces can only produce a $1s$ bound state, while the 0.87-MeV state of ^{17}O is a $2s$ state. The method has nevertheless been applied to this case, however, because it only requires that the two-body T matrix give the correct on-shell data and have the necessary analytic properties.[27] This is assured by determining the parameters from the experimental binding energy and phase shifts as discussed above. While a form factor obtained from a $2s$ bound-state wave function[16] might possess different off-shell extentions than the simple one used here, it was hoped that this would have little effect on the three-body amplitudes, and the results obtained are encouraging. An effort to modify the method to include various choices of form factors is being made for future calculations.

ACKNOWLEDGMENTS

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[27] T. Morgan, Phys. Rev. **175**, 1260 (1968).

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Antisymmetrization of the Theory of Stripping Reactions*

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The Butler-Hewitt-McKellar-May theory of stripping reactions is modified to take into account the blocking effect of particle-hole correlations in the core, which populate the final neutron state with probability $1-P$. The basic relation of the theory between the direct-reaction matrix element M_C and the continuum matrix element M_S is altered to $M_C=S^{1/2}(P-S)^{-1}M_S$.

1. INTRODUCTION

THIS paper discusses the effect of exclusion-principle blocking on the theory of stripping reactions proposed by Butler, Hewitt, McKellar, and May.[1] We consider the reaction $A(d, p)B$.

It is well known that, provided that the neutron is captured into a single-particle state which is unoccupied in the core (the ground state of A), antisymmetrization of the theory with respect to the neutrons is trivial.[2,3] Indeed, in this case the theory reduces to the form it takes when the captured neutron is regarded as distinguishable from the others.

When the state of the captured neutron is already partially occupied in the core with probability $1-P$, then the exclusion principle restricts the total single-particle strength in the state of P. In the distorted-wave Born approximation (DWBA) there is no formal modification in the theory,[4] but the sum of the spectroscopic factors is now P.

One would be surprised if the BHMM theory were not modified because of the emphasis in this theory on the many-body aspects of the wave function Ψ_B. In this paper we show that the direct-reaction matrix element M_C is related to the continuum matrix element M_S calculated by BHMM by

$$M_C=[A/(P-S)]M_S, \tag{1}$$

where A is the spectroscopic amplitude and $S=|A|^2$ the spectroscopic factor. Note that this reduces to the BHMM result

$$M_C=[A/(1-S)]M_S \tag{2}$$

in the absence of blocking ($P=1$).

The form of Eq. (1) could, in fact, be guessed. When the captured-neutron wave function contains no off-

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[1] S. T. Butler, R. G. L. Hewitt, B. H. J. McKellar, and R. M. May, Ann. Phys. (N.Y.) **43**, 282 (1967) (referred to as BHMM).

[2] M. L. Goldberger and K. M. Watson, *Collision Theory* (Wiley-Interscience, Inc., New York, 1964), §11.6.

[3] Antisymmetrization of the protons introduces a qualitatively new effect, exchange stripping, with which we will not be concerned in this paper.

[4] J. B. French, *The Analysis of Reduced Widths, in Nuclear Spectroscopy*, edited by F. Ajzenberg-Selove (Academic Press Inc., New York, 1960), p. 890.