# Hartree-Fock Calculation of Helium Hypernuclear Binding Energies*

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The binding energies of the $s$-shell hypernuclei ($_{\Lambda}H^4$, $_{\Lambda}He^5$, and $_{\Lambda\Lambda}He^6$) have been calculated in the Hartree-Fock model, using Gaussian particle-particle potentials. Parameters of the $\Lambda$-$N$ interaction were determined from the experimental binding energies. These numerical results agree qualitatively with those of other calculations and also with the experimental $\Lambda$-$N$ scattering data, if short-range repulsions are included. Equality of the $\Lambda$-$N$ and $\Lambda$-$\Lambda$ potentials is consistent with these calculations.

## I. INTRODUCTION

ONE of the richer sources of information about the nature and strength of the $\Lambda$-$N$ interaction has been the study of the hypernuclei with $A \leq 6$. These isotopes which have thus far been identified and their respective separation energies are[1] $_{\Lambda}H^3$, $B_{\Lambda}=0.32\pm0.17$ MeV; $_{\Lambda}H^4$, $B_{\Lambda}=1.95\pm0.14$ MeV; $_{\Lambda}He^4$, $B_{\Lambda}=2.07\pm0.09$ MeV; $_{\Lambda}He^5$, $B_{\Lambda}=3.10\pm0.02$ MeV; $_{\Lambda}He^6$, $B_{n\Lambda}=4.09$ MeV; and the double hyperfragment $_{\Lambda\Lambda}He^6$, $B_{\Lambda\Lambda}=10.7\pm0.6$ MeV. The isotope $_{\Lambda}H^2$ has not been observed, but a large number of heavy hyperfragments, i.e., with $A>6$, have been seen. Until recently, these hypernuclei were the major reservoir of experimental data on the $\Lambda$-$N$ system. Within the past several years, however, scattering experiments have yielded directly the two-body $\Lambda$-$N$ scattering lengths and effective ranges[2]; the most recent determination is[3]

triplet state: $a_t=-2.07$ F, $r_t=4.50$ F,

singlet state: $a_s=-2.46$ F, $r_s=3.87$ F.

Several features of the $\Lambda$-$N$ potential have been deduced solely from the principle of charge symmetry.[4] Since the lightest exchange quantum giving rise to this interaction is either a $K$ meson or two (or more) pions or a combination, the range of this potential is necessarily much smaller than that in the $N$-$N$ potential which is governed by a single-pion exchange. Second, if the two- (or more) pion exchange is the dominant process, the interaction is an ordinary Wigner force, while if a single kaon exchange dominates, the resulting potential is of exchange character which changes sign when operating on two-particle states of negative relative parity, e.g., the relative $\Lambda$-$N$ $p$ state occurring in $_{\Lambda}He^6$. The resultant $\Lambda$-$N$ potential is then spin dependent, as is indicated by the low-energy scattering parameters, and also it may have the type of tensor component present in the $N$-$N$ potential.

One would hope to relate the hypernuclear binding energies listed above to the free $\Lambda$-$N$ interaction. This is not possible in every case, however. For example, Bodmer[5] has pointed out that in those hypernuclei in which the core nucleus has isospin $T=0$ (i.e., $_{\Lambda}He^5$, $_{\Lambda\Lambda}He^6$, and also $_{\Lambda}He^6$) certain terms in the free-particle interaction are suppressed. These correspond to channel couplings which require that the virtually excited core undergo an isospin change; for the case in which the core is $He^4$, the excited state has $T=1$. Presumably, such isotriplet amplitudes are present in the $_{\Lambda}He^5$ core, but these admixtures are small because of the large excitation energy of $He^4$. This suppression should be small in the other cases. Thus, one cannot hope to correctly calculate the binding energies of all the above hypernuclei in terms of a single two-body $\Lambda$-$N$ potential.

The hypernuclei $_{\Lambda}H^3$, $_{\Lambda}H^4$, $_{\Lambda}He^4$, and $_{\Lambda}He^5$ have been subjected to extensive and elaborate theoretical analyses, the first of these $_{\Lambda}H^3$ being investigated most thoroughly. The majority of these investigations have been variational calculations.[5–8] Dietrich, Mang, and Folk[9] and, later, Beck and Gutsch,[10] assumed all interparticle potentials to be square wells (allowing the wells to have a short-range hard core) and used an independent-pair approximation with two strongly correlated particles reacting under the influence of a background due to the other nucleons. In the purely variational methods it has been generally assumed that the hypernuclear wave functions are totally space symmetric with all particle pairs in relative $s$ states. (Bodmer,[5] in treating $_{\Lambda}H^3$, included the possibility of a

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[1] D. H. Davis and J. Sacton, *High-Energy Physics and Nuclear Structure* (North-Holland Publishing Co., Amsterdam, 1967), pp. 21–33; see also G. Bohm *et al.*, Nucl. Phys. **B4**, 511 (1968).

[2] S. Ali *et al.*, Phys. Letters **26B**, 453 (1967).

[3] See, e.g., G. Alexander, O. Benary, U. Karshon, A. Shapira, G. Yekutieli, R. Englemann, H. Filthuth, A. Fridman, and B. Schiby, Phys. Letters **19**, 715 (1966).

[4] R. H. Dalitz and B. W. Downs, Phys. Rev. **110**, 958 (1958).

[5] A. R. Bodmer, Phys. Rev. **141**, 1387 (1966).

[6] D. B. Lichtenberg, Nuovo Cimento **8**, 463 (1958); B. W. Downs and R. H. Dalitz, Phys. Rev. **114**, 593 (1959); B. W. Downs, D. R. Smith, and T. N. Truong, *ibid.* **129**, 2730 (1963); D. R. Smith and B. W. Downs, *ibid.* **133**, B461 (1964).

[7] R. C. Herndon, Y. C. Tang, and E. W. Schmid, Nuovo Cimento **23**, 259 (1964); Phys. Rev. **137**, B294 (1965); R. C. Herndon and Y. C. Tang, *ibid.* **153**, 1091 (1967); **159**, 853 (1967).

[8] R. H. Dalitz and B. W. Downs, Phys. Rev. **111**, 967 (1958).

[9] K. Dietrich, K. J. Mang, and R. Folk, Nucl. Phys. **50**, 177 (1964).

[10] F. Beck and U. Gutsch, Phys. Letters **14**, 133 (1965).

mixed-symmetry $S'$ state, known to be present in $H^3$ and $He^3$.[11]) In these calculations a $\Lambda$-$N$ potential well shape with different strengths in triplet and singlet states was assumed, and the energy was minimized with respect to trial functions of varying degrees of complexity in order to determine the well depths. Two conclusions result from these calculations: First, the singlet potential is larger than the triplet potential, a prediction apparently corroborated by the later analysis of the scattering data. Second, in order to account simultaneously for the binding of ${}_{\Lambda}H^3$ and the nonexistence of ${}_{\Lambda}H^2$, it has proven necessary to introduce a repulsive core term into the $\Lambda$-$N$ potential. The latest calculation reported by Herndon and Tang[7] gives a hard-core radius of 0.6 F (assuming an exponential well outside the core). Their resulting potential gives scattering parameters in fair agreement with the experimental values,[2] although the effective ranges are too low. A third aspect of the $\Lambda$-$N$ potential is that in these light nuclei, where all particle pairs are mainly $s$ states, the admixture of higher angular-momentum components (e.g., $d$ waves by the tensor force) produces no significant effect other than perhaps redefining the central part of the two-body potential.

The nuclei ${}_{\Lambda}He^6$ and ${}_{\Lambda\Lambda}He^6$ represent anomalous systems. For the first of these, the core nucleus $He^5$ is unstable against neutron decay, while the hypercore ${}_{\Lambda}He^5$ is stable. For this reason, the separation energy $B_{n\Lambda}$ given above is for the neutron-$\Lambda$ pair. Early variational calculations of this separation energy gave[12] inconclusive results, as did some latter attempts with nonvariational procedures, e.g., Lovitch and Rosati[13] and also Ananthanarayanan.[14] The difficulty associated with this system is that the last nucleon is in a relative $p$ state with respect to the other particles, thus requiring knowledge of the $p$-wave $N$-$\alpha$ and $N$-$\Lambda$ potentials. On the other hand, ${}_{\Lambda\Lambda}He^6$ is a system in which all particles can be in $s$-state wave functions, because the spin, isospin, and strangeness quantum numbers are all distinct. Moreover, in this case one can expect to obtain information about the nature of the $\Lambda$-$\Lambda$ potential because the effective $\Lambda$-$N$ potential is assumed to be identical with that in ${}_{\Lambda}He^5$. On the basis of their calculations for the binding energy of the heavier hypernucleus ${}_{\Lambda\Lambda}Be^{10}$, Tang and Herndon[15,16] correctly predicted the existence of ${}_{\Lambda\Lambda}He^6$, estimating the $\Lambda\Lambda$-separation energy to be 9.68 MeV, fairly close to the experimental value. Ananthanarayanan[14] has also attempted to evaluate this binding energy, using a Dawson-Walecka[17] procedure with attractive Yukawa wells; the resulting binding energies are too large. In part, this is due to the absence of a repulsive core in the two-body potentials. Since a core is present in the $N$-$N$ and $\Lambda$-$N$ systems, one would expect it to be present in the $\Lambda$-$\Lambda$ system as well, and, as has been argued,[14,18] it is very plausible that all three interactions exhibit the same core with the same parameters.

All of the above calculations have assumed that the interactions of a $\Lambda$ particle with a nucleus results from two-body potentials. Three-body interactions can also be present,[19] in particular, if the two-pion exchange dominates the $\Lambda$'s dynamics. Bodmer and Sampathar[20] have concluded, however, that in the ${}_{\Lambda}H^3$, ${}_{\Lambda}H^4$, and ${}_{\Lambda}He^4$ systems these three-body forces do not give significant effects, although this may not be true of the other hypernuclei.[21]

In this paper we present results for yet another calculation of these hypernuclear binding energies. This calculation uses a somewhat different method, albeit a variational one, namely, the Hartree-Fock model in configuration space. In the Hartree-Fock method, we determine the best (i.e., that giving the largest total binding energy) shell-model wave function, a Slater determinant of single-particle orbitals. The interactions that we use are of the Volkov form,[22] a sum of an attractive Gaussian well and a repulsive Gaussian core. When applied to $He^4$, this model gives a binding energy and a charge form factor which are in good agreement with the experimental data.[23] Its use in the hypernuclei calculation then allows a simple interpretation of the nuclear core distortion. In Sec. II we present the Hartree-Fock equations for the several hypernuclei. Section III contains the numerical results, while Sec. IV states our conclusions.

## II. HARTREE-FOCK EQUATIONS

The Hartree-Fock method minimizes the expectation value of the Hamiltonian with respect to variations in a Slater determinant $\Phi_0$ of single-particle wave functions $\psi_\lambda$. Here, $\lambda$ refers to the set of quantum numbers needed to specify a single-particle state: $n$ the principle quantum number, $l$ the orbital angular momentum, $m_l$ the projection of $l$, $m_s$ the projection of spin, $t$ the isospin third component, and $s$ the strangeness. The $\psi_\lambda$ are represented by

$$\psi_\lambda=[\phi_{nls}(r)/r]Y_{lm_l}(\hat{r})\Xi_{m_s ts}, \qquad (1)$$

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[22] A. B. Volkov, Nucl. Phys. **74**, 33 (1965).
[23] B. F. Gibson, A. Goldberg, and M. S. Weiss, Nucl. Phys. **118A**, 225 (1968).

with $\Xi$ a function of the three variables $m_s$, $t$, and $s$. In fact, for these hypersystems, all nucleons are in either $s$ or $p$ states with $l=0$, 1, and their reduced radial orbitals will be labelled briefly as $\phi_{100}(r)=\phi_s(r)$ $\phi_{110}(r)=\phi_p(r)$, and all $\Lambda$'s are in the lowest $s$ states labelled $\phi_{10,-1}(r)=\phi_\Lambda(r)$. The variations are then made with respect to the form of the $\phi$ functions.

The Hamiltonian is assumed to be the sum of the usual kinetic and potential energy operators

$$\mathcal{H}=\mathcal{T}+\mathcal{U}, \tag{2a}$$

and we minimize

$$\epsilon=\langle\Phi_0\mid\mathcal{H}\mid\Phi_0\rangle=-E_b, \tag{2b}$$

where $E_b$ is the total binding energy. $\mathcal{T}$ excludes the c.m. energy so that

$$\mathcal{T}=\sum_{i=1}^{A}\frac{\mathbf{p}_i^2}{2m_i}-\frac{1}{2M}\Big[\sum_{j=1}^{A}\mathbf{p}_j\Big]^2$$
$$=\sum_{i=1}^{A}\frac{1}{2}\frac{M-m_i}{Mm_i}\mathbf{p}_i^2-\frac{1}{2M}\sum_{i\neq j}\mathbf{p}_i\cdot\mathbf{p}_j, \tag{3}$$

where $M$ is the total hypernuclear mass. Since in these cases every particle pair has at least one particle in an $s$-state orbital, the second term does not contribute in Eq. (3), and we can take

$$\mathcal{T}=\sum_{i=1}^{A}\frac{\mathbf{p}_i^2}{2\mu_i}, \qquad (\mathbf{p}_i=-i\hbar\nabla_i). \tag{4}$$

Each particle of mass $m$ in a system ${}_nX^{N+n}$ with $N$ nucleons and $n$ hyperons has "reduced mass" $\mu_i$

$$\mu_i({}_nX^{N+n})=Mm_i/(M-m_i) \tag{5}$$

with $M=nm_\Lambda+Nm_p$, $m_p=$proton mass, and $m_\Lambda=\Lambda$ mass. In view of the comments in Sec. I, the potential $\mathcal{U}$ is a sum of central spin-dependent two-body potentials $V_{ij}$. Variation of $\epsilon$, Eq. (3), then yields the well-known Hartree-Fock equations. Because the nuclear Hartree-Fock equations are, in their full generality, rather complex, we quote below only the equations appropriate to the hypersystems in which we are interested.

We first state explicitly the assumptions and approximations which have gone into these equations. First, the Hartree-Fock method assumes that the only correlation between particles is statistical, arising from the Pauli principle. This is in contrast to the trial functions used, e.g., by Herndon and Tang,[7] where two-body rescatterings are emphasized. Moreover, in all of these hypernuclei (with the exception of ${}_\Lambda\mathrm{He}^6$), the $m_s$, $t$, and $s$ degrees of freedom serve to distinguish each of the particles, eliminating the statistical correlations also. Second, we ignore Coulomb effects. These corrections can be considered later as perturbations. Third, we assume that the radial functions $\phi$ are independent of $m_l$ and $m_s$. This is actually valid only for closed shell systems, e.g., ${}_{\Lambda\Lambda}\mathrm{He}^6$. However, one cannot avoid such an assumption in configuration space Hartree-Fock calculations without enormously increasing the numerical complexities of the problem. In any case, these calculations do generate the best product wave function within this restriction.

We present the Hartree-Fock equations and their numerical results in terms of "effective" two-body interactions, i.e., after the spin dependence has been removed. We shall then attempt to relate these effective potentials to the spin-dependent interactions $V_{ij}$. In the systems ${}_\Lambda\mathrm{He}^5$ and ${}_{\Lambda\Lambda}\mathrm{He}^6$ the core nucleus is $\mathrm{He}^4$. In this core each nucleon sees one nucleon with spin parallel (triplet) and two with spins opposed ($\frac{1}{2}$ triplet$+\frac{1}{2}$ singlet). Hence the effective potential between two nucleons a distance $r$ apart in the $\mathrm{He}^4$ nucleus is

$$U_{NN}(\mathrm{He}^4,r)=\tfrac{1}{2}\big[V_{NN}{}^{(S)}(r)+V_{NN}{}^{(T)}(r)\big], \tag{6}$$

where $V_{NN}{}^{(T)}$ and $V_{NN}{}^{(S)}$ are the triplet and singlet $N$-$N$ potentials, respectively. This is also the effective potential between nucleons in ${}_\Lambda\mathrm{He}^5$ and ${}_{\Lambda\Lambda}\mathrm{He}^6$. Similarly, the effective $\Lambda$-$N$ and $\Lambda$-$\Lambda$ potentials in these systems are

$$U_{N\Lambda}(\mathrm{He}^4,r)=\tfrac{1}{4}\big[V_{N\Lambda}{}^{(S)}(r)+3V_{N\Lambda}{}^{(T)}(r)\big], \tag{7}$$
$$U_{\Lambda\Lambda}(r)=V_{\Lambda\Lambda}{}^{(S)}(r). \tag{8}$$

The Hartree-Fock equations for these nuclei are then, for ${}_\Lambda\mathrm{He}^5$,

$$\left[\frac{-\hbar^2}{2\mu_p({}_\Lambda\mathrm{He}^5)}\frac{d^2}{dr^2}+3I_{NN}(r)+I_{N\Lambda}(r)\right]\phi_s(r)$$
$$=\epsilon_s({}_\Lambda\mathrm{He}^5)\phi_s(r), \tag{9}$$
$$\left[\frac{-\hbar^2}{2\mu_\Lambda({}_\Lambda\mathrm{He}^5)}\frac{d^2}{dr^2}+4I_{\Lambda N}(r)\right]\phi_\Lambda(r)=\epsilon_\Lambda({}_\Lambda\mathrm{He}^5)\phi_\Lambda(r),$$

and for ${}_{\Lambda\Lambda}\mathrm{He}^6$,

$$\left[\frac{-\hbar^2}{2\mu_p({}_{\Lambda\Lambda}\mathrm{He}^6)}\frac{d^2}{dr^2}+3I_{NN}(r)+2I_{N\Lambda}(r)\right]\phi_s(r)$$
$$=\epsilon_s({}_{\Lambda\Lambda}\mathrm{He}^6)\phi_s(r), \tag{10}$$
$$\left[\frac{-\hbar^2}{2\mu_\Lambda({}_{\Lambda\Lambda}\mathrm{He}^6)}\frac{d^2}{dr^2}+4I_{\Lambda N}(r)+I_{\Lambda\Lambda}(r)\right]\phi_\Lambda(r)$$
$$=\epsilon_\Lambda({}_{\Lambda\Lambda}\mathrm{He}^6)\phi_\Lambda(r).$$

In each set of equations the one-body potentials are given by

$$I_{NN}(r)=\int dr'\phi_s{}^2(r')u_{NN}(r,r'),$$
$$I_{N\Lambda}(r)=\int dr'\phi_\Lambda{}^2(r')u_{N\Lambda}(r,r'), \tag{11}$$
$$I_{\Lambda N}(r)=\int dr'\phi_s{}^2(r')u_{N\Lambda}(r,r'),$$
$$I_{\Lambda\Lambda}(r)=\int dr'\phi_\Lambda{}^2(r')u_{\Lambda\Lambda}(r,r'),$$

with $u_{NN}$ the spherical average of the effective two-body potential $U_{NN}$, etc.,

$$u_{NN}(r, r') = \frac{1}{2}\int_{-1}^{1} dx\, U_{NN}[\mathrm{He}^4, (r^2+r'^2-2rr'x)^{1/2}]. \tag{12}$$

The reduced masses $\mu_p$ and $\mu_\Lambda$ are given in Eq. (5). Evaluation of the separation energies requires also the solution of the Hartree-Fock equations for $He^4$

$$\left[\frac{-h^2}{2\mu_p(\mathrm{He}^4)}\frac{d^2}{dr^2}+3I_{NN}(r)\right]\phi_s(r)=\epsilon_s(\mathrm{He}^4)\phi_s(r). \tag{13}$$

Equations (9), (10), and (13) in conjunction with Eq. (11), form three *unrelated* sets of nonlinear equations, each of which is to be solved separately.

Several comments about these equations should be made. First, we are not interested here in the structure of $He^4$ *per se*, but rather in using knowledge of that structure to study the properties of ${}_\Lambda He^5$ and ${}_{\Lambda\Lambda}He^6$. Hence we choose an effective potential $U_{NN}(\mathrm{He}^4, r)$ empirically, i.e., that potential which when used in Eq. (13) gives a satisfactory model of $He^4$. We then use that potential in Eqs. (9) and (10). The fact that $U_{NN}(\mathrm{He}^4, r)$ can be related to the basic two-nucleon potential, Eq. (6) is convenient but irrevelant. Second, the nonlinearity of these equations requires that the method of solution be a self-consistent one; this will be discussed later. Third, the single-particle energies $\epsilon_s$ and $\epsilon_\Lambda$ are not the separation energies of the corresponding particles. The separation energies are computed as the differences in the binding energies. From the Hartree-Fock equations and from Eq. (2) these binding energies are

$$\mathrm{He}^4: \quad E_b=-4\epsilon_s(\mathrm{He}^4)+6\int dr\phi_s^2(r)I_{NN}(r), \tag{14a}$$

$${}_\Lambda\mathrm{He}^5: \quad E_b=-4\epsilon_s({}_\Lambda\mathrm{He}^5)-\epsilon_\Lambda({}_\Lambda\mathrm{He}^5)+6 \times\int dr\,\phi_s^2(r)I_{NN}(r)+4\int dr\,\phi_s^2(r)I_{N\Lambda}(r), \tag{14b}$$

$${}_{\Lambda\Lambda}\mathrm{He}^6: \quad E_b=-4\epsilon_s({}_{\Lambda\Lambda}\mathrm{He}^6)-2\epsilon_\Lambda({}_{\Lambda\Lambda}\mathrm{He}^6)+6 \times\int dr\,\phi_s^2(r)I_{NN}(r)+8\int dr\,\phi_s^2(r)I_{N\Lambda}(r) + \int dr\,\phi_\Lambda^2(r)I_{\Lambda\Lambda}(r), \tag{14c}$$

where the $\phi_s$, $\phi_\Lambda$ and the $I_{NN}$, etc. are generated from the equations for that system. Finally, one cannot learn from these three hypernuclei the relative strengths of the various triplet and singlet interactions, since the same combination of each appears in all three sets of equations.

In ${}_\Lambda H^4$ (and ${}_\Lambda He^4$) we cannot remove unambiguously the spin-dependence of the $N$-$N$ potential in the core $H^3$. Here again we adopt the philosophy that we are not interested in $H^3$ *per se*, and that we may define an effective $N$-$N$ potential which gives the correct binding energy for $H^3$ from the equation

$$\left[\frac{-h^2}{2\mu_p(\mathrm{H}^3)}\frac{d^2}{dr^2}+2I_{NN}(r)\right]\phi_s(r)=\epsilon_s(\mathrm{H}^3)\phi_s(r), \tag{15}$$

with the binding energy being

$$\mathrm{H}^3: \quad E_b=-3\epsilon_s(\mathrm{H}^3)+3\int dr\,\phi_s^2(r)I_{NN}(r). \tag{16}$$

The self-consistent potential $I_{NN}(r)$ has the same form as in Eq. (11), but using $U_{NN}(\mathrm{H}^3, r)$. Since the $\Lambda$-$N$ scattering data indicate the singlet potential to be stronger than the triplet potential, the effective $\Lambda$-$N$ potential is

$$U_{N\Lambda}(\mathrm{H}^3, r)=\tfrac{1}{3}[2V_{\Lambda N}{}^{(s)}(r)+V_{\Lambda N}{}^{(T)}(r)]. \tag{17}$$

The Hartree-Fock equations for ${}_\Lambda H^4$ are

$$\left[\frac{-h^2}{2\mu_p({}_\Lambda\mathrm{H}^4)}\frac{d^2}{dr^2}+2I_{NN}(r)+I_{N\Lambda}(r)\right]\phi_s(r)=\epsilon_s({}_\Lambda\mathrm{H}^4)\phi_s(r), \tag{18}$$

$$\left[\frac{-h^2}{2\mu_\Lambda({}_\Lambda\mathrm{H}^4)}\frac{d^2}{dr^2}+3I_{\Lambda N}(r)\right]\phi_\Lambda(r)=\epsilon_\Lambda({}_\Lambda\mathrm{H}^4)\phi_\Lambda(r),$$

with $I_{NN}(r)$, $I_{N\Lambda}(r)$, and $I_{\Lambda N}(r)$ defined in a way analogous to Eq. (11), using $U_{NN}(\mathrm{H}^3, r)$ and $U_{N\Lambda}(\mathrm{H}^3, r)$. The binding energy is

$$\mathrm{H}^4: \quad E_b=-3\epsilon_s({}_\Lambda\mathrm{H}^4)-\epsilon_\Lambda({}_\Lambda\mathrm{H}^4)+3 \times\int dr\,\phi_s^2(r)I_{NN}(r)+3\int dr\,\phi_s^2(r)I_{N\Lambda}(r). \tag{19}$$

We assume all effective two-body potentials to be of the Volkov form[22]

$$U(r)=-W_1\exp[-(r/a_1)^2]+W_2\exp[-(r/a_2)^2], \tag{20}$$

with $W_1$, $W_2$, $a_1$, and $a_2$ to be fixed as discussed above. Here both $W_1$ and $W_2$ are positive; the second term allows for a short-range repulsion ($a_1>a_2$). It is unlikely that the binding energies depend on the detailed shapes of these interactions, but rather on the presence or absence of the core, and on the over-all depths and ranges. Thus, this potential is convenient if only for its mathematical simplicity.

## III. RESULTS

The Hartree-Fock systems of equations in Sec. II were solved self-consistently. The computer code that was used is a fairly general one, applicable to a much wider class of Hartree-Fock problems than those considered here, and its details will be described elsewhere.[24] One chooses an initial set of wave functions

[24] R. L. Tarp, University of California Lawrence Radiation Laboratory Report No. UCRL-50430 (unpublished).

TABLE I. Λ-separation energies, no repulsive core.

(a) ${}_{\Lambda}He^{5}$

| Potential | $W_1^{(\Lambda N)}$ (MeV) | $a_1^{(\Lambda N)}$ (F) | $B_\Lambda$ (MeV) | $\epsilon_\Lambda$ (MeV) | $\Delta\epsilon_s$ (%) | $\Delta$ (F) | $U_4$ (MeV) |
|---|---|---|---|---|---|---|---|
| 1 | 149 | 0.60 | 1.9 | −6.8 | 17 | +0.05 | 717 |
| 2 | 35.9 | 1.05 | 1.6 | −5.9 | 12 | +0.01 | 926 |
| 3 | 158 | 0.60 | 3.07 | −8.3 | 22 | +0.07 | 760 |
| 4 | 39.4 | 1.05 | 3.12 | −7.6 | 18 | +0.03 | 1016 |

(b) ${}_{\Lambda}H^{4}$

| | | | |
|---|---|---|---|
| 4 | 39.4 | 1.05 | <0, unbound |
| 5 | 52.0 | 1.05 | 2.01 |

(c) ${}_{\Lambda\Lambda}He^{6}$, Λ−Λ potential parameters, no core

| Λ−N potential | $W_1^{(\Lambda\Lambda)}$ (MeV) | $a_1^{(\Lambda\Lambda)}$ (F) | $B_{\Lambda\Lambda}$ (MeV) |
|---|---|---|---|
| 3 | 39.4 | 1.05 | 12.9 |
| 4 | 39.4 | 1.05 | 9.0 |
| 3 | 30.0 | 1.05 | 10.8 |
| 4 | 49.0 | 1.05 | 10.7 |

$\phi_s(r)$ and $\phi_\Lambda(r)$, and evaluates the self-consistent one-body potentials $I_{NN}(r)$, $I_{N\Lambda}(r)$, etc., Eq. (11). The resulting Schrödinger equations, Eq. (9) or (10), etc., are integrated numerically to obtain a new set of wave functions and eigenvalues $\epsilon_s$ and $\epsilon_\Lambda$. This cycle is repeated using the new wave functions until the changes in the eigenvalues over successive iterations are less than one part in $10^4$. The final wave functions and potentials are then used to evaluate the binding energies, Eq. (14), etc. Trial cases have shown that the program produces accurate (1%) eigenvalues and wave functions unless the magnitude of the eigenvalue is less than 4 MeV on a nuclear scale of dimensions. In all the cases discussed here, the criterion for accuracy was satisfied.

One set of parameters given by Volkov for the $N$-$N$ potential in $He^4$ is

$$W_1^{(NN)}=83.34\text{ MeV}, \qquad a_1^{(NN)}=1.6\text{ F},$$
$$W_2^{(NN)}=145\text{ MeV}, \qquad a_2^{(NN)}=0.82\text{ F}.$$

Solution of the Hartree-Fock equations [(13) and (14a)] in which $U_{NN}(He^4, r)$ is taken to be a Volkov potential with these parameters, gives the correct binding energy, $E_b(He^4)=28.31$ MeV. In addition, the charge form factor computed from the resulting wave function agrees satisfactorily with the most recent data[25] for momentum transfers below $q^2\approx 7\text{ F}^{-2}$. Hence we can with some confidence use this $N$-$N$ potential in the hypernuclear calculations involving $U_{NN}(He^4, r)$.

The triton binding energy is $E_b(H^3)=8.48$ MeV. This value results from Eqs. (15) and (16) if we increase the depth of the attractive part of the well to $W_1^{(NN)}=85.6$ MeV, and we use this value and the other parameters above for those cases involving $U_{NN}(H^3, r)$.

We assume that the Λ-$N$ and Λ-Λ potentials in all cases are also of Volkov form, and we attempt to choose the corresponding well parameters to give the correct separation energies. These are evaluated throughout by solution of the appropriate Hartree-Fock equations and by use of the relations

$$B_\Lambda({}_{\Lambda}He^5)=E_b({}_{\Lambda}He^5)-E_b(He^4),$$
$$B_\Lambda({}_{\Lambda}H^4)=E_b({}_{\Lambda}H^4)-E_b(H^3),$$

and

$$B_{\Lambda\Lambda}({}_{\Lambda\Lambda}He^6)=E_b({}_{\Lambda\Lambda}He^6)-E_b(He^4).$$

We first ignore repulsive core effects in the hyperon interactions ($W_2^{(\Lambda N)}=W_2^{(\Lambda\Lambda)}=0$). Dalitz and Downs[8] employed a two-body model of ${}_{\Lambda}He^5$ in which the Λ interacts with the averaged field of the $He^4$ core. They fixed the depth of the Λ-$He^4$ potential by requiring that the experimental $B_\Lambda$ be secured under radial compression of the $He^4$ core. Their potentials were constructed from Gaussian Λ-$N$ interactions whose intrinsic range corresponds to Yukawa's for $2\pi$ or $K$ exchange.[26] From their results we obtain the following:

Two-pion exchange:

$$W_1^{(\Lambda N)}=35.9\text{ MeV}, \qquad a_1^{(\Lambda N)}=1.05\text{ F};$$

[25] R. Frosch, J. McCarthy, R. Rand, and M. R. Yearian, Phys. Rev. **160**, 874 (1967).

[26] This transformation is discussed by B. W. Downs, *Lectures in Theoretical Physics* (Wiley-Interscience, Inc., New York, 1960), Vol. II, and in Ref. 18.

Kaon exchange:

$$W_1^{(\Lambda N)}=149 \text{ MeV}, \qquad a_1^{(\Lambda N)}=0.60 \text{ F}.$$

The Hartree-Fock equations [(9) and (14)] give the results for these parameters shown in Table I(a) (potentials 1 and 2). Table I(a) also indicates the effects of the nuclear core distortion. $\Delta\epsilon_s$ is the percentage change in the single-particle energies associated with the nucleon orbitals, this change arising from the presence of the $\Lambda$. $\Delta$ is the core compression, defined by

$$\Delta=-[r_N({}_\Lambda\text{He}^5)-r_N(\text{He}^4)],$$

with $r_N$ the rms radius of the nucleon mass distribution. Finally, $U_4$ is the volume integral of the $\Lambda$-He$^4$ potential,

$$U_4=-4\pi\int_0^\infty r^2 dr 4I_{\Lambda N}(r).$$

$U_4$ is defined identically as that in other calculations.[8,27,28]

In order to clarify the nature of the nucleon orbital distortion, Fig. 1 illustrates the nucleon radial wave function $\phi_s(r)$ in both the He$^4$ and ${}_\Lambda$He$^5$ systems for potential 3. As would be expected, the orbital is pushed in slightly to smaller distances. As reflected in the rather small value of $\Delta$, this squeezing is not very great, although it has a marked effect on the single-particle energy eigenvalue.

The $B_\Lambda$'s calculated by the Hartree-Fock method are considerably too low. This appears puzzling in that the core compressions (5% and 1% for potentials 1 and 2, respectively) are of the same order as those of Dalitz and Downs[8] (8% and 3%). (Their nuclear compressibility was, however, quite high: 280 MeV.) In addition, the Dalitz-Downs values for $U_4$ are 910 and 695 MeV, slightly smaller than those in Table I(a).

In order to reproduce the experimental value of $B_\Lambda({}_\Lambda\text{He}^5)$ by the Hartree-Fock method, it is necessary to increase the $\Lambda$-$N$ well depths by about 10% to those given in Table I(a) (potentials 3 and 4).

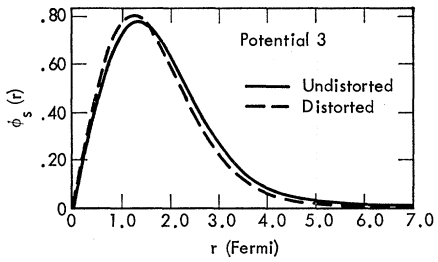


FIG. 1. Solid curve is the nuclear radial wave function in isolated He$^4$. The solid line with dots is the same quantity changed by the $\Lambda$ in ${}_\Lambda$He$^5$.

The Dalitz-Downs calculation, in fact, is also a variational calculation, minimizing the average of the same Hamiltonian as used above. The essential difference between the Dalitz-Downs and Hartree-Fock methods lies in the nature of the trial function. Use of a product trial function leads to the Hartree-Fock equations. Dalitz and Downs, on the other hand, chose a trial function of the form $\psi(r)\chi$, with $\chi$ involving only the internal coordinates of the He$^4$ core, and $\psi$ a function of the $\Lambda$-He$^4$ separation distance. Minimization of the energy then leads to the Dalitz-Downs results (see the Appendix). The product function permits no correlations (other than statistical) between particles, while the Dalitz-Downs function assumes a correlation of a specific form, between the $\Lambda$ and the He$^4$ core as a whole. On the other hand, the Hartree-Fock method allows for distortion effects by the $\Lambda$, whereas in the Dalitz-Downs model such distortion effects can in practice be included only in a simple way (e.g., an over-all compression). It is clear from the results in Table I(a) that the Dalitz-Downs model for ${}_\Lambda$He$^5$ is superior in that it produces a larger binding energy. This superiority is presumably a reflection of the large He$^4$ excitation threshold, which hinders those core distortions which form the virtue of the Hartree-Fock method. Presumably, also, these distortion effects are more important in other hypernuclei whose cores have much smaller excitation energies. Moreover, it is not clear how the Dalitz-Downs method, the replacement of the ${}_\Lambda$He$^5$ system by essentially a two-body system, could be applied to other cases, e.g., the limit of feasibility is ${}_{\Lambda\Lambda}$He$^6$, which would, by this method, be replaced by a three-body system.[18] Although, clearly, the Hartree-Fock method is not the best for ${}_\Lambda$He$^5$, its possible utility to other $s$-shell hypernuclei, where the two- or three-body approximations may not be superior, is plausible.

The potential parameters for ${}_\Lambda$H$^4$ are given in Table I(b). These potential parameters which are satisfactory in ${}_\Lambda$He$^5$ do not bind this lighter system, and one must increase the well depth to that indicated by potential 5.

Relating these well depths $W_1^{(\Lambda N)}$ for ${}_\Lambda$He$^5$ and ${}_\Lambda$H$^4$ (potentials 4 and 5) to the $\Lambda$-$N$ spin-singlet and triplet potentials, Eqs. (7) and (17), we find that the triplet well is very much shallower than that in the singlet state. However, the scattering parameters in Sec. I indicate that these wells should not be very different, and that the singlet potential is only slightly more attractive than the triplet potential. If one accepts the applicability of the Hartree-Fock model to these hypernuclear systems, one must conclude that, as suggested by Bodmer,[28] the triplet $\Lambda$-$N$ interaction in ${}_\Lambda$He$^5$ is greatly suppressed.

Having obtained an appropriate set of parameters for $U_{N\Lambda}(\text{He}^4, r)$, we proceed to the case of ${}_{\Lambda\Lambda}$He$^6$, choosing various values of the depth $W_1^{(\Lambda\Lambda)}$ of the well

[27] $U_4$'s and compressions for various calculations are discussed in recent review articles; see R. H. Dalitz, in *Interaction of High-Energy Particles with Nuclei*, edited by T. E. O. Ericson (Academic Press Inc., New York, 1967).

[28] A. R. Bodmer, *Hypernuclear Spectroscopy in High Energy Physics and Nuclear Structure* (North-Holland Publishing Co., Amsterdam, 1967).

TABLE II. Λ-separation energies, Volkov-type potentials (see Sec. III).

| | $W_1^{(\Lambda N)}$ (MeV) | $a_1^{(\Lambda N)}$ (F) | $B_\Lambda$ (MeV) | $U_4$ (MeV) |
|---|---|---|---|---|
| ${}_\Lambda H^4$ | 94.0 | 1.21 | 2.07 | ··· |
| ${}_\Lambda He^5$ | 85.8 | 1.21 | 3.10 | 1606 |

$U_{\Lambda\Lambda}(r)$. This interaction cannot arise from a kaon exchange, and we choose its range to correspond to that of the two-pion exchange, $a_1^{(\Lambda\Lambda)}=1.05$ F. The results are shown in Table I(c). As noted above, this Λ-Λ potential is the singlet interaction. It is interesting that the well depth giving the correct value of $B_{\Lambda\Lambda}$, $W_1^{(\Lambda\Lambda)}=$ 49 MeV, is rather close to the value of $W_1^{(\Lambda N)}$, 52 MeV, obtained from ${}_\Lambda H^4$. The relatively large uncertainty in the measured value of $B_{\Lambda\Lambda}({}_{\Lambda\Lambda}He^6)$ allows the possibility that these different well depths could be equal, in agreement with Ali and Bodmer.[18]

To include the effects of the short range repulsion, we accept the hypothesis that the "core" parameters should be approximately the same for the $N$-$N$, Λ-$N$, and Λ-Λ potentials. Accordingly, we set $W_2$ and $a_2$ equal to the Volkov values given above for all interactions. We drop here the restriction that the potential ranges correspond to any specific-exchange quanta, and we search for values of $W_1^{(\Lambda N)}$ and $a_1^{(\Lambda N)}$ by requiring that when used in the $U_{\Lambda N}$ corresponding to each nucleus, they give the correct separation energies for ${}_\Lambda H^4$ and ${}_\Lambda He^5$. A set of such parameters is given in Table II. It is to be noted that the value of $U_4$ for ${}_\Lambda He^5$ is rather large for this potential. Inclusion of this short range repulsion also results in there being no net core compression, $\Delta=0$, and, in fact, the nucleon orbital in this case is identical with the undistorted function. The Herndon-Tang calculation[7] of $B_\Lambda({}_\Lambda He^5)$ also utilized a Λ-$N$ potential containing repulsion, in this case an infinite short-range repulsion, and, in consequence, a trial function with explicit two-body correlations. They found also negligible core distortion. Their value of $U_4$ is 1021 MeV, considerably lower than that resulting from the Hartree-Fock calculation. Further comparison is impossible since the Hartree-Fock procedure cannot be used with infinite repulsions.

Again, the attractive well depth in ${}_\Lambda He^5$ is smaller than that in ${}_\Lambda H^4$, reflecting the possible isospin suppression. The Λ-$N$ potential in ${}_\Lambda H^4$ should be directly related to the free interaction. The scattering length and effective range of this potential are in fact −2.15 and 3.35 F, respectively, in reasonable agreement with the average of the experimental values. We have also recomputed $B_{\Lambda\Lambda}({}_{\Lambda\Lambda}He^6)$ assuming the Λ-Λ potential to be identical with the free Λ-$N$ interaction [retaining $U_{N\Lambda}({}_\Lambda He^5, r)$ as the Λ-$N$ interaction]. The result is $B_{\Lambda\Lambda}({}_{\Lambda\Lambda}He^6)=10.8$ MeV, in excellent agreement with the experimental value.

Finally, we must consider the difference in Λ separation energies for the isodoublet ${}_\Lambda H^4$ and ${}_\Lambda He^4$. Part of this difference arises from the proton Coulomb repulsion in ${}_\Lambda He^4$. Previous calculations of this Coulomb energy have given results of the wrong sign to account for this difference, and it has been proposed that the hyperon interactions are not charge symmetric but contain a symmetry breaking term.[29] We estimate the Coulomb energy $\Delta B_\Lambda^{(C)}$ by treating the Coulomb repulsion as a perturbation on the Hartree-Fock generated wave functions. If $E_C(He^3)$ and $E_C({}_\Lambda He^4)$ are the Coulomb energies for each nucleus

$$\Delta B_\Lambda^{(C)}=E_C(He^3)-E_C({}_\Lambda He^4).$$

We use in each case the wave functions computed with Volkov Potentials including the repulsive core terms. In first order, $E_C(He^3)=0.70$ MeV, compared to the experimental trinucleon Coulomb energy, 0.76 MeV. Proton finite size has not been included, but this should not affect the difference $\Delta B_\Lambda^{(C)}$. Again, in first order, $E_C({}_\Lambda He^4)=0.72$ MeV, so that $\Delta B_\Lambda^{(C)}=-0.02$ MeV. The measured difference is $\Delta B_\Lambda\approx 0.1$ MeV. Hence, in this calculation also, Coulombic effects do not account for this isodoublet splitting.

## IV. CONCLUSIONS

The above calculations suffer, of course, from the intrinsic flaw in all variational procedures: The method is guaranteed to give only the lowest upper bound to the binding energy within the class of allowed trial functions, in this case products of single-particle functions. We again emphasize that the Hartree-Fock approximation does not allow for two-body correlations in the trial wave function. Within this proviso the following conclusions can be drawn: (1) It is possible to fit the ${}_\Lambda H^4$, ${}_\Lambda He^5$, and ${}_{\Lambda\Lambda}He^6$ binding energies with a Hartree-Fock model by making quite reasonable choices for the two-body interaction parameters. (2) It is possible to choose a two-body Λ-$N$ potential with a repulsive "core" which gives the correct ${}_\Lambda H^4$ binding energy and which is consistent with the two-body scattering parameters. (3) This two-body potential does not give the correct ${}_\Lambda He^5$ binding energy. This hypernucleus requires a considerably smaller Λ-$N$ attractive potential, indicating that the type of suppression effects suggested by Bodmer are present and important. (4) Coulombic effects are of the wrong sign to account for the difference in binding energies of ${}_\Lambda H^4$ and ${}_\Lambda He^4$; these conclusions agree essentially with those of other variational calculations[7] assuming very different nuclear models. (5) In addition, we may con-

[29] R. H. Dalitz and F. von Hippel, Phys. Letters **10**, 153 (1964).

clude from these calculations that equality of the $\Lambda$-$N$ and $\Lambda$-$\Lambda$ interactions is consistent with the binding-energy data of the various hypernuclear systems.

We can, unfortunately, say nothing at this point about the exchange character of the $\Lambda$-$N$ potential, since our calculations here considered only $s$-shell hypernuclei, and effects of Majorana exchange do not enter. Also we can say nothing about the effect of three-body forces.

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## APPENDIX: RELATIONSHIP BETWEEN THE HARTREE-FOCK AND DALITZ-DOWNS METHODS

The calculation of the binding energy of ${}_{\Lambda}\mathrm{He}^5$ requires the evaluation and minimization of $\langle\Psi, H\Psi\rangle$, with $H$ the Hamiltonian excluding the total c.m. kinetic energy

$$H=\sum_{i=1}^{4}\frac{\mathbf{P}_i^2}{2m_p}+\frac{\mathbf{P}_\Lambda^2}{2m_\Lambda}-\frac{(\mathbf{P}_\alpha+\mathbf{P}_\Lambda)^2}{2(m_\alpha+m_\Lambda)}$$
$$+\tfrac{1}{2}\sum_{i\neq j,\,i,j=1}^{4}V_{NN}(\mathbf{r}_i-\mathbf{r}_j)+\sum_{i=1}^{4}V_{\Lambda N}(\mathbf{r}_\Lambda-\mathbf{r}_i). \quad \text{(A1)}$$

Here $\mathbf{P}_\alpha$ is the total momentum of the $\mathrm{He}^4$ core which has c.m. position $\mathbf{r}_\alpha$, $\mathbf{P}_\alpha=\sum_{i=1}^{4}P_i$, and $m_\alpha$ its mass, $m_\alpha=4m_p$. $V_{NN}$ and $V_{\Lambda N}$ are the nucleon-nucleon and $\Lambda$-nucleon potentials, respectively. Use of a trial function of a product of single-particle orbitals leads to the Hartree-Fock equations [Eqs. (9)]. To generate the Dalitz-Downs equations, we write $H$ as

$$H=H_\alpha+\frac{\mathbf{P}^2}{2\mu}+\sum_{i=1}^{4}V_{\Lambda N}(\mathbf{r}_\Lambda-\mathbf{r}_i), \quad \text{(A2)}$$

where $H_\alpha$ is the internal energy of the $\mathrm{He}^4$ core,

$$H_\alpha=\sum_{i=1}^{4}\frac{\mathbf{P}_i^2}{2m_p}-\frac{\mathbf{P}_\alpha^2}{2m_\alpha}+\tfrac{1}{2}\sum_{i\neq j;\,i,j=1}^{4}V_{NN}(\mathbf{r}_i-\mathbf{r}_j). \quad \text{(A3)}$$

$\mathbf{P}$ is the momentum conjugate to the $\Lambda$-$\mathrm{He}^4$ separation, $\mathbf{r}=\mathbf{r}_\Lambda-\mathbf{r}_\alpha$,

$$\mathbf{P}=\frac{m_\alpha}{m_\alpha+m_\Lambda}\mathbf{P}_\Lambda-\frac{m_\Lambda}{m_\alpha+m_\Lambda}\mathbf{P}_\alpha, \quad \text{(A4)}$$

and $\mu$ is the usual reduced mass

$$\mu=m_\alpha m_\Lambda/(m_\alpha+m_\Lambda).$$

Let the three $\mathrm{He}^4$ internal coordinates be labelled by $\mathbf{S}_1$, $\mathbf{S}_2$, and $\mathbf{S}_3$. The energy $H_\alpha$ thus depends only on the $\mathbf{S}$ variables. The $i$th nucleon position $\mathbf{r}_i$ is

$$\mathbf{r}_i=\mathbf{r}_\alpha+\mathbf{t}_i,$$

with $\mathbf{t}_i$ an appropriate linear combination of the $\mathbf{S}$ vectors. Thus the $\Lambda$-$\mathrm{He}^4$ potential is written

$$\sum_{i=1}^{4}V_{\Lambda N}(\mathbf{r}_\Lambda-\mathbf{r}_i)=\sum_{i=1}^{4}V_{\Lambda N}(\mathbf{r}-\mathbf{t}_i).$$

The Dalitz-Downs trial function is of the form

$$\Psi_{D-D}=\chi(\mathbf{S}_1,\mathbf{S}_2,\mathbf{S}_3)\psi(\mathbf{r}), \quad \text{(A5)}$$

and thus

$$\langle\Psi_{D-D},H\Psi_{D-D}\rangle=E_\alpha+\Big\langle\psi(\mathbf{r}),$$
$$\times\Big[\frac{\mathbf{P}^2}{2u}+\sum_{i=1}^{4}\int d\mathbf{S}_1d\mathbf{S}_2d\mathbf{S}_3\,|\,\chi(\mathbf{S}_1\mathbf{S}_2\mathbf{S}_3)\,|^2V_{\Lambda N}(r-\mathbf{t}_i)\Big]\psi(\mathbf{r})\Big\rangle, \quad \text{(A6)}$$

with $E_\alpha=\langle\chi, H_\alpha\chi\rangle$.

If $P(\mathbf{R})$ is the mass density of the $\mathrm{He}^4$ core with respect to its c.m. $\mathbf{r}_\alpha$, then

$$P(\mathbf{R})=\sum_{i=1}^{4}\int d\mathbf{S}_1d\mathbf{S}_2d\mathbf{S}_3d\mathbf{r}_\alpha\,|\,\chi(\mathbf{S}_1,\mathbf{S}_2,\mathbf{S}_3)\,|^2\delta(\mathbf{r}_\alpha)\delta(\mathbf{R}-\mathbf{r}_i) \quad \text{(A7)}$$
$$=\sum_{i=1}^{4}\int d\mathbf{S}_1d\mathbf{S}_2d\mathbf{S}_3\,|\,\chi(\mathbf{S}_1,\mathbf{S}_2,\mathbf{S}_3)\,|^2\delta(\mathbf{R}-\mathbf{t}_i),$$

and thus

$$\langle\Psi_{D-D},H\Psi_{D-D}\rangle$$
$$=E_\alpha+\Big\langle\psi(\mathbf{r}),\Big[\frac{\mathbf{P}^2}{2u}+\int d\mathbf{R}\,P(\mathbf{R})\,V_{\Lambda N}(r-\mathbf{R})\Big]\psi(\mathbf{r})\Big\rangle. \quad \text{(A8)}$$

Minimization of $\langle H\rangle$ from (A8) with respect to variations in $\psi(\mathbf{r})$ then leads to the Dalitz-Downs equations.