The values of the inertial parameter using the five formulas are presented in Table V for Ne^{20} and Si^{28}. All the formulas, except the Levinson moment, were evaluated with H without any modifying $\alpha \mathbf{J}^2$ term. We find less spread in the values of the inertial parameter for Si^{28} than for Ne^{20} since (3.3a) is better satsified for the former.

Kelson[13] proposed a criterion to determine the validity of various formulas for the inertial parameter. The criterion is that, if $A(H)$ is a correct formula for the inertial parameter, then the following relation should hold

$$A(H)=A(H-\alpha \mathbf{J}^2)+\alpha.$$

Thus to test a particular formula, say the one proposed by Das Gupta and Van Ginneken, one would calculate $A(H)$ with Eq. (5.10) and calculate $A(H-\alpha \mathbf{J}^2)$ with the same equation replacing H and $H-\alpha \mathbf{J}^2$ and $|\Phi\rangle$ with $|\Phi_\alpha\rangle$. The Inglis formula, which is positive definite, clearly cannot satisfy this criterion. For the remaining formulas, this criterion really tests the dependence of $|\Phi_\alpha\rangle$ on α. If the rotational criterion that this dependence should be weak is satisfied, all the formulas give nearly the same values and satisfy Kelson's criterion reasonably well. To illustrate this point, the values of $A(H-\alpha \mathbf{J}^2)+\alpha$ as a function of α for the various formulas are presented for Ne^{20} in Fig. 3 and for Si^{28} in Fig. 4. If the Kelson criterion is satisfied, these quantities should be constant. We see that this condition is better satisfied for Si^{28}, where the relevant rotational criterion is better satisfied.

Since all of these formulas are determined self-consistently and with the use of the HF wave functions, the inertial parameter will reflect a weighted average of the excitation energies of the members of the ground band and will give the best results when applied to the spacing of levels with $I\sim[\langle\Phi|\mathbf{J}^2|\Phi\rangle]^{1/2}$. Inasmuch as purely rotational spectra are rarely observed in nature and a more adequate description of the spectrum is of the form

$$E_I=E_0+AI(I+1)+BI^2(I+1)^2,$$

where A is positive and B is negative, these formulas will generally underestimate the spacing E_2-E_0 of the ground- and first-excited states.

[13] I. Kelson, Phys. Rev. **160**, 775 (1967).

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Fluctuations in Nuclear Elastic Scattering Cross Sections*

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A method of fluctuation analysis is developed for elastic scattering of spin-zero particles from spin-zero nuclei. The principal difference between this model and the one proposed by Ericson is that some unitarity is included. It is shown that all features of conventional fluctuation analysis are modified to a small extent. Analysis of the phase shift may show entirely different behavior from that predicted by an Ericson model. Because of the simplicity of the elastic-scattering reaction, it is possible to obtain not only the total width but also the level spacing and partial width (as defined within the context of the model). These quantities are obtained as a function of angular momentum. This method is only applicable if a phase-shift analysis can be done on the data.

I. INTRODUCTION

THE representation of compound-nucleus reaction cross sections by stochastic processes as first proposed by Ericson[1] and Brink and Stephen[2] has had a great deal of success in the interpretation of excitation functions.

The method commonly employed to obtain such a representation is as follows. A general form is given expressing the amplitude in terms of a large number of resonance parameters. The resonance parameters are then assumed to be random variables with appropriate properties. In order to make the resulting process as simple as possible it is desirable to have all of the random variables independent. The choice of form has a strong bearing on the allowability of the independence assumption, as we shall see.

The success achieved by fluctuation theory has been in spite of the fact that the form used, together with the independence assumption, allows unitarity to be

* Work performed under the auspices of the U.S. Atomic Energy Commission.
[1] T. Ericson, Ann. Phys. (N.Y.) **23**, 390 (1963); Phys. Letters **4**, 258 (1963).
[2] D. M. Brink and R. O. Stephen, Phys. Letters **5**, 77 (1963).

violated. However, fluctuation analysis has been applied primarily to small, off-diagonal, elements of the S matrix whose contribution to the unitarity sums is very small. If the simple theory were applied to elastic scattering one has every reason to believe that this violation of unitarity would have observable effects since these elements (at least for the most important partial waves to fluctuation cross sections) are of order one.

For the same reasons one would expect that a theory which does not allow diagonal elements to violate unitarity of themselves will take into account most of the effects of the unitarity condition. In fact the further effects of unitarity should be of the same order as the first-order effects in off-diagonal elements. The object of this paper is to present a simple model for diagonal S-matrix elements which do not violate unitarity of themselves.

Section II reviews the Mittag-Leffler method and the R-matrix method of representing the amplitude as a stochastic process. Section III presents the mathematical form of the model and gives the reasons for its selection. The average value of S is computed in Sec. IV.

Section V deals with an approximate form valid in the limit, partial width/total width $\ll 1$. The average elastic fluctuation cross section is obtained for this (slightly) restricted case. It is also shown that the average partial width, the average total width, and the average level spacing can be measured as a function of angular momentum J.

In Sec. VI the properties of the phase shift are examined and Sec. VII contains an investigation of the properties of the residues predicted by this model.

Throughout it is assumed that we are dealing with spinless particles and that the symbol S denotes a diagonal S-matrix element in a single partial wave. The Appendix contains results pertaining to differential cross sections.

II. REVIEW OF BASIC METHODS

The first method of representing the amplitude as a stochastic process[1–3] is based on the Mittag-Leffler theorem[4,5] which states that any function (F) which is analytic everywhere except at isolated poles $(P_1, P_2, \cdots, P_N)$ and bounded at infinity can be written

$$f(z)=\text{const}+\sum_{i=1}^{N}\frac{b_i}{z-p_i}. \tag{1}$$

If S is assumed to be such a function of E it can be written as

$$S(E)=\bar{S}+\sum_{\lambda}\frac{b_\lambda}{E-E_\lambda+i\Gamma_\lambda{}^T/2}. \tag{2}$$

Of course, S is not bounded at infinity, nor is it meromorphic (because of the threshold branch cuts), but the error made due to these two incorrect assumptions is likely to be small if we are interested only in the rapid energy dependence of S.

Since the b_λ are known (from reaction theory) to be related to overlap integrals of (compound nucleus) wave functions, it is assumed that they are random variables with average values zero.

Thus Eq. (2) is an expression with parameters corresponding to each pole. The assumptions commonly made about the parameters in Eq. (2) are

$$\Gamma_\lambda{}^T \text{ is independent of } \lambda; \tag{3a}$$

$$\bar{b}_\lambda=0; \tag{3b}$$

$$\text{the } b_\lambda \text{ are statistically independent.} \tag{3c}$$

The first assumption is made because the total width is the sum of a large number of (assumed) independent partial widths and as such will have a χ^2 type of distribution with a large number of degrees of freedom. The second assumption has already been discussed. The third cannot be true because unitarity imposes the condition[6]

$$|S|^2\leq 1, \tag{4}$$

which implies a relationship among the b_λ. As was mentioned in the Introduction for off-diagonal elements, this is not a serious defect, while for diagonal elements this assumption may cause serious error. This is especially true if one asks for properties which assume unitarity in advance, such as the behavior of the phase shift.

The second method considered here is that of representation in terms of the R matrix.[7] Only the briefest mention will be made here, since the subject has been treated in detail by Moldauer.[8]

For only one channel open there is only one element of the S matrix, and the expressions are

$$S=(1+iPR)/(1-iPR) \tag{5}$$

with

$$R=\sum_n[\gamma_n{}^2/(E_n-E)]. \tag{6}$$

Since P and R are real, S is unitary regardless of the choice of γ_n (the basic random variables in this case).

[3] W. R. Gibbs, Los Alamos Scientific Laboratory Report No. LA 3266, 1965 (unpublished).

[4] G. Mittag-Leffler, Acta Soc. Sci. Fennicae **XI**, 273 (1880); Acta Math. **IX**, 1 (1884).

[5] Konrad Knopp, *Theory of Functions* (Dover Publications, Inc., New York, 1947), Part II.

[6] Of course, unitarity imposes many more conditions than this, but this is sufficient for our purposes.

[7] E. P. Wigner and L. Eisenbud, Phys. Rev. **72**, 29 (1947); A. M. Lane and R. G. Thomas, Rev. Mod. Phys. **30**, 257 (1958).

[8] P. A. Moldauer, Phys. Rev. Letters **18**, 249 (1967); **19**, 1047 (1967); Phys. Rev. **157**, 907 (1967); **171**, 1164 (1968).

In fact, S is always unitary even if more than one channel is open. Moldauer has shown that

$$T\equiv 1-|\tilde{S}|^2=1-\exp(-2\pi\Gamma_\alpha/D), \tag{7}$$

where Γ_α is the partial width.

This is to be compared with the relationship which is commonly used and is valid for small T:

$$T=2\pi\Gamma_\alpha/D. \tag{8}$$

We see that Eq. (8) is just the first-order expression of Eq. (7).

Let us digress for the moment to consider the original physical derivation of Eq. (8). It was argued that if the period of motion is $\tau=2\pi\hbar/D$ and the probability of getting out per period is T then the decay rate must be

$$T/\tau=\Gamma_\alpha/\hbar=TD/2\pi\hbar$$

or

$$T=2\pi\Gamma_\alpha/D.$$

This, of course, neglects depletion due to loss, which requires that T be small so that depletion is small. However, there is no need to make this approximation.

The probability of finding the nucleus still intact from decay through the channel in question after the nth period is

$$Q_n=(1-T)Q_{n-1}=(1-T)^n, \tag{9}$$

since $Q_0=1$.

Writing Q as a function of time

$$\begin{aligned} Q(t)&=(1-T)^{t/\tau}=(1-T)^{tD/2\pi\hbar}\\ &=\exp(-(tD/2\pi\hbar)\ln(1-T)^{-1})\\ &\equiv\exp(-\Gamma_\alpha t/\hbar) \end{aligned} \tag{10}$$

from which we identify

$$\Gamma_\alpha=(D/2\pi)\ln(1-T)^{-1}, \tag{11}$$

which is equivalent to Eq. (7).

Of course, much more general cases can be treated by the R matrix than just presented but in order to get useful results it is necessary to go to numerical calculations. Moldauer has shown numerically that the relationship (7) holds generally in the absence of direct reactions. Similar approaches[9] may yield useful results of an analytic nature.

III. PRODUCT FORM

The product representation may be obtained by imposing unitarity on the Mittag-Leffler form, as an approximation to the R matrix, as the result of a semiclassical argument, or directly by means of analyticity arguments. We will consider all of these possibilities to obtain a better understanding of the model to be adopted.

[9] Nazakat Ullah, Phys. Rev. **154**, 891 (1967); **164**, 1316 (1967); P. A. Moldauer, *ibid.* **135**, B642 (1964).

A. Modified Mittag-Leffler

If we consider Eq. (2) to contain a finite number of poles we may write

$$S=\prod_\lambda \frac{E-a_\lambda}{E-E_\lambda+\frac{1}{2}(i\Gamma_\lambda{}^T)}. \tag{12}$$

The condition (4) can be satisfied without requiring any correlation between parameters by demanding

$$\left|\frac{E-a_\lambda}{E-E_\lambda+\frac{1}{2}(i\Gamma_\lambda{}^T)}\right|^2\leq 1 \tag{13}$$

be satisfied for each factor individually. Since we wish to take each set of pole parameters to be independent of all other sets of pole parameters, a_λ must be the same as it would be if the level were isolated. Thus,

$$a_\lambda=E_\lambda-\tfrac{1}{2}(i\Gamma_\lambda{}^T)+i\Gamma_\lambda. \tag{14}$$

If we write

$$\Gamma_\lambda{}^T=\beta_\lambda+\Gamma_\lambda, \tag{15}$$

where β_λ will be called the inelastic width and Γ_λ the elastic width, S can be written as

$$S=\prod_\lambda \frac{E-E_\lambda+\frac{1}{2}i(\beta_\lambda-\Gamma_\lambda)}{E-E_\lambda+\frac{1}{2}i(\beta_\lambda+\Gamma_\lambda)}. \tag{16}$$

This is the basic form that will be used. It is to be noted that an undetermined, slowly varying phase should multiply this form to give the physical S.

B. R Matrix

To obtain this form from the R matrix we observe that, aside from the hard sphere phase shift, which becomes the slowly varying phase for this case, we can write [from the Mittag-Leffler theorem for $(dS/dE)/S$]

$$S=\prod_\lambda \frac{E-E_\lambda-\epsilon_\lambda+\frac{1}{2}i(\beta_\lambda-\Gamma_\lambda)}{E-E_\lambda+\frac{1}{2}i(\beta_\lambda+\Gamma_\lambda)} \tag{17}$$

if the penetrabilities are taken to be independent of energy. In order to satisfy the condition (4) for all energies and still have all parameters independent it is necessary to take

$$\epsilon_\lambda\equiv 0. \tag{18}$$

This is an approximation made to obtain this model and it may be necessary for some applications to relax this condition, but for the present we shall assume that Eq. (18) holds exactly.

C. Semiclassical Argument

Since $1-|S|^2$ is the probability that a particle incident upon a nucleus is removed from the incident beam (which is the same as being captured if we neglect direct coupling to other channels) then $|S|^2$ is the probability that it *not* be captured. If we now assume that there exist a number of independently acting "physical" states, the probability that the particle is

not captured is the product of the probabilities that it is not captured into each state individually. If these probabilities, $|S_\lambda|^2$, are assumed to be unmodified by the presence of other states then

$$|S|^2=\left|\prod_\lambda \frac{E-E_\lambda+\frac{1}{2}i(\beta_\lambda-\Gamma_\lambda)}{E-E_\lambda+\frac{1}{2}i(\beta_\lambda+\Gamma_\lambda)}\right|^2. \tag{19}$$

This is the same as Eq. (16) except for a phase.

D. Analyticity Arguments

To obtain the desired form directly from analyticity arguments, consider potential scattering. We need three following facts:[10] (1) S is analytic in the k plane with the exception of isolated poles and an essential singularity at infinity. (2) All poles occur below the real axis or on the imaginary axis. (3) $S(k)S(-k)=1$.

Let k_λ denote poles in the right half-plane, k_λ' denote zeros in the left half-plane, q_λ denote zeros in the right half-plane, q_λ' denote poles in the left half-plane, p_n denote poles on the imaginary axis.

Noting that $(d/dk)\ln S$ has residue -1 at every pole and residue $+1$ at every zero, we see that

$$\frac{d}{dk}\ln S=c+\sum\frac{-1}{k-p_n}+\sum\frac{1}{k+p_n}+\sum\frac{-1}{k-k_\lambda}+\sum\frac{1}{k-k_\lambda'}+\sum\frac{1}{k-q_\lambda}+\sum\frac{-1}{k-q_\lambda'}. \tag{20}$$

Using the third condition in the form $k_\lambda'=-k_\lambda$, $q_\lambda'=-q_\lambda$,

$$\frac{d}{dk}\ln S=c-2\left[\sum\frac{p_n}{k^2-p_n^2}+\sum\frac{k_\lambda}{k^2-k_\lambda^2}-\sum\frac{q_\lambda}{k^2-q_\lambda^2}\right] \tag{21}$$

or

$$\ln S=ck-2\int_0^k dk\times\left[\sum\frac{p_n}{k^2-p_n^2}+\sum\frac{k_\lambda}{k^2-k_\lambda^2}-\sum\frac{q_\lambda}{k^2-q_\lambda^2}\right]. \tag{22}$$

If we integrate this as it stands we arrive at the usual product representation.[11] Let us add and subtract k from the numerator of each term and integrate. By properly grouping the terms we see that

$$S=e^{ck}\left[\prod_n(k+p_n)\prod_\lambda\frac{k+k_\lambda}{k+q_\lambda}\right]^2\times\left(\prod_n\frac{1}{p_n^2-E}\prod_\lambda\frac{E-q_\lambda^2}{E-k_\lambda^2}\right). \tag{23}$$

Note that all poles in the product involving k are distant from the region of interest, as is the case with the first product involving E. Thus we come again to the form expressed in Eq. (17). As before we neglect the ϵ_λ to make the form more manageable.

For the more general case (not just potential scattering) everything would be the same except that the branch cuts due to thresholds spoil the analyticity. However, it is to be expected that any energy dependence which comes from these branch cuts would have a single-particle character.

For the plausibility arguments given above Eq. (16) will be adopted as a model for S.

IV. AVERAGE VALUE OF S

In this section we shall compute the average value of S under the approximation that all β_λ are equal and large compared to the average level spacing. This will be done at a fixed energy, so we may write

$$S=\prod\frac{x_\lambda+i(1-y_\lambda)}{x_\lambda+i(1+y_\lambda)} \tag{24}$$

with

$$x_\lambda=2(E-E_\lambda)/\beta,\qquad y_\lambda=\Gamma_\lambda/\beta.$$

Since all y_λ are to be considered to be independent, the ensemble average of S is given by

$$\bar{S}=\prod c_\lambda, \tag{25}$$

where

$$c_\lambda=\int_0^\infty\frac{x_\lambda+i(1-y)}{x_\lambda+i(1+y)}f(y)\,dy=\int_0^\infty\frac{1-ix_\lambda-y}{1-ix_\lambda+y}f(y)\,dy, \tag{26}$$

where $f(y)$ is the probability density function of y. This could well be taken to be of a Porter-Thomas form, but its specification is (almost) irrelevant for this calculation. Then we may write

$$\ln(\bar{S})=\sum_\lambda\ln(c_\lambda), \tag{27}$$

which we will take over to an integral (assuming a constant pole spacing, D)

$$\ln(\bar{S})=d^{-1}\int_{-\infty}^\infty\ln[c(z)]dz, \tag{28}$$

where

$$d=2D/\beta.$$

Since

$$c(z)\equiv\int_0^\infty\frac{1-iz-y}{1-iz+y}f(y)\,dy \tag{29}$$

has no singularities in the upper half-plane, let us com-

[10] See, e.g., R. E. Peierls, Proc. Roy. Soc. (London) **A253**, 16 (1959).

[11] Ning Hu, Phys. Rev. **74**, 131 (1948); N. G. van Kampen, *ibid.* **91**, 1267 (1953).

plete the contour there.[12] Thus we find

$$\ln(\bar{S}) = -d^{-1} \lim_{R\to\infty} iR\int_0^{\pi} e^{i\theta} d\theta \ln[c(Re^{i\theta})]. \quad (30)$$

Now we may observe that (to first order in $1/R$)

$$c(Re^{i\theta}) \to 1 - 2ie^{-i\theta}\bar{y}/R \quad (31)$$

so

$$\ln(\bar{S}) = -d^{-1} \lim_{R\to\infty} iR\int_0^{\pi} e^{i\theta} d\theta \ln\left[1-\left(\frac{2ie^{-i\theta}\bar{y}}{R}\right)\right]$$

$$= -\frac{2\pi\bar{y}}{d} = -\frac{\pi\bar{\Gamma}}{D} \quad (32)$$

or

$$\bar{S} = \exp(-\pi\bar{\Gamma}/D). \quad (33)$$

Thus we find again Eq. (7).

Note that this expression is general under the assumption of the form (16), the assumptions of equally spaced poles and $\beta/D \gg 1$, so that the transition from Eq. (27) to Eq. (28) is correct. Eq. (7) will be derived again in Sec. V under a slightly different set of assumptions.

V. STATISTICAL PROPERTIES OF AN APPROXIMATE FORM OF S

If the excitation energy is very high, then many channels are open and the width in the entrance channel will be very small compared to the total width from other channels or

$$\bar{y} \ll 1. \quad (34)$$

Since $f(y)$ is of an exponentially decreasing form $[\exp(-\bar{y}/y)]$, the probability that y exceeds $\bar{y}$ by a large factor is very small. Thus we may study the statistical properties at high energies by expanding in powers of y_λ. A word of caution before going further: There are certainly regions in every nuclear-scattering problem where this approximation does not hold and certainly regions where it does, and one must be very careful to be high enough in energy before using this approximation.

In order to carry out the program outlined above we first expand each factor in Eq. (21) to first order in y_λ

$$\frac{x_\lambda + i(1-y_\lambda)}{x_\lambda + i(1+y_\lambda)} = 1 - \frac{2y_\lambda}{1-ix_\lambda}. \quad (35)$$

Now define

$$p+iq \equiv \ln S \approx \sum_\lambda \ln\{1-[2y_\lambda/(1-ix_\lambda)]\}, \quad (36)$$

so to first order in y_λ

$$p = -2\sum_\lambda [y_\lambda/(1+x_\lambda^2)], \quad (37a)$$

$$q = -2\sum_\lambda [x_\lambda y_\lambda/(1+x_\lambda^2)]. \quad (37b)$$

Let us investigate the statistical properties of p and q. The first thing we may note is that there are many significant terms in each sum (since $\beta/D \gg 1$) so that p and q are normally distributed. This means that we need only the first and second moment to describe their distributions.

$$\bar{p} = -2\bar{y}\sum_\lambda \{1/(1+x_\lambda^2)\}_{\text{av}}$$

$$= \frac{-2\bar{y}}{d}\int_{-\infty}^{\infty} \frac{dx}{1+x^2},$$

where $d = 2D/\beta$ or

$$\bar{p} = -2\pi\bar{y}/d = -\pi\bar{y}\beta/D = -\pi\bar{\Gamma}/D \quad (38)$$

and

$$\bar{q} = -2\bar{y}\int_{-\infty}^{\infty} \frac{x\,dx}{1+x^2} = 0. \quad (39)$$

By the same simple arguments it may be shown that

$$\langle p^2\rangle_{\text{av}} = \bar{p}^2 + 2\pi\sigma_y^2/d, \quad (40a)$$

$$\langle q^2\rangle_{\text{av}} = 2\pi\sigma_y^2/d, \quad (40b)$$

where

$$\sigma_y^2 \equiv \text{Var}(y); \quad (41)$$

thus,

$$\sigma_p^2 \equiv \text{Var}(p) = 2\pi\sigma_y^2/d, \quad (42a)$$

$$\sigma_q^2 \equiv \text{Var}(q) = 2\pi\sigma_y^2/d. \quad (42b)$$

It may also be shown that p and q are independent. To summarize, the complex random variable S may be expressed in terms of the independent random variables p and q as follows:

$$S = \exp(p+iq), \quad (43)$$

where p and q are normally distributed

$$\bar{p} = -\pi\Gamma/D, \qquad \bar{q} = 0, \quad (44)$$

$$\sigma^2 = \sigma_p^2 = \sigma_q^2 = 2\pi\sigma_y^2/d. \quad (45)$$

For such a statistical system it is easy to show[13]

$$\bar{S} = e^{\bar{p}}. \quad (46)$$

Thus once again we arrive at Eq. (7).

We have the equipment now to derive all moments of the amplitude

$$A = 1 - e^{2i\varphi}S,$$

$$\bar{A} = 1 - e^{2i\varphi}\bar{S},$$

$$A = \bar{A} + \hat{A}$$

$$= \bar{A} + e^{2i\varphi}(\bar{S} - S).$$

[12] Although it is true that $C(z)$ has no singularities in the upper half-plane, it can have a zero there. The position of the zero depends on the distribution function chosen. For a Porter-Thomas function the zero remains in the lower half-plane provided $\bar{\Gamma} \lesssim 2.6\beta$. For two degrees of freedom the condition is $\bar{\Gamma} \lesssim 1.65\beta$.

[13] If $z = e^{x+iy}$ with x and y independent with zero means and equal variances (normally distributed) then $E(z) = E(e^{iy})E(e^x) = e^{-\sigma^2/2}e^{\sigma^2/2} = 1$.

The slowly varying phase factor $e^{2i\varphi}$ has been inserted explicitly. Let us investigate the statistical properties of $\hat{A}$

$$\hat{A}=(\bar{S}-S)e^{2i\varphi}$$
$$=\exp(\bar{p}+2i\varphi)[1-\exp(p-\bar{p}+iq)], \quad (47)$$
$$\langle\hat{A}\rangle_{\rm av}=0,$$
$$\sigma_{fl}=|\hat{A}|^2=\exp(2\bar{p})\{1+\exp[2(p-\bar{p})]$$
$$-\exp(p-\bar{p}+iq)-\exp(p-\bar{p}-iq)\}, \quad (48)$$
$$\bar{\sigma}_{fl}=\exp(2\bar{p})[\exp(2\sigma^2)-1],$$
$$\sigma_{fl}{}^2=\exp(4\bar{p})\{1+\exp[4(p-\bar{p})]+\exp[2(p-\bar{p})+2iq]$$
$$+\exp[2(p-\bar{p})-2iq]+2\exp[2(p-\bar{p})]$$
$$-2\exp[p-\bar{p}+iq]-2\exp[p-\bar{p}-iq]$$
$$-2\exp[3(p-\bar{p})+iq]-2\exp[3(p-\bar{p})-iq]$$
$$+2\exp[2(p-\bar{p})]\}, \quad (49)$$
$$\langle\sigma_{fl}{}^2\rangle_{\rm av}$$
$$=\exp(4\bar{p})[\exp(8\sigma^2)-4\exp(4\sigma^2)+4\exp(2\sigma^2)-1]. \quad (50)$$

As σ^2 goes to zero

$$\langle\sigma_{fl}{}^2\rangle_{\rm av}\to\exp(4\bar{p})8\sigma^4,$$
$$\bar{\sigma}_{fl}\to\exp(2\bar{p})2\sigma^2,$$

so that

$$(\langle\sigma_{fl}{}^2\rangle_{\rm av}-\bar{\sigma}_{fl}{}^2)/\bar{\sigma}_{fl}{}^2\to 1 \quad (51)$$

as expected.

For a Porter-Thomas distribution

$$\sigma_y{}^2=2\bar{y}^2, \quad (52)$$

so that

$$\sigma^2=2\pi\bar{\Gamma}^2/\beta D \quad (53)$$
$$=-[\ln(1-T_\alpha)]^2/\sum_{c\neq\alpha}\ln(1-T_c). \quad (54)$$

The last equality follows from Eq. (7) and the assumption that β is the sum of all other partial widths.

We may note that the magnitude of σ^2 depends on two factors, $\bar{\Gamma}/\beta$ and $\bar{\Gamma}/D$. The first of these has been assumed to be small, but the second has been allowed to have an arbitrarily large value. Thus it cannot be said, without some calculation, that σ^2 is small compared to one. However, one may always compute the partial wave cross section by

$$\bar{\sigma}_{fl}=(1-T_\alpha)[\exp(2\sigma^2)-1], \quad (55)$$

where σ^2 is given by Eq. (54). The reader may verify that Eq. (55) reduces to the usual Hauser Feshbach expression when all transmission coefficients are small compared to one.

The equations obtained in this section allow the determination of $\bar{\Gamma}$, D, and β as a function of J. If a partial-wave analysis is performed on the data (elastic scattering of two spin-zero particles) and $p(E)$ $\{\equiv\mathrm{Re}[\ln S(E)]\}$ is thus obtained over some energy range three quantities may be computed with reasonable accuracy. These are $\bar{p}$, $R(0)$, and the correlation length of $p(E)$. The latter numbers are obtained from the expression

$$R(\epsilon)\equiv\frac{\langle p(E)p(E+\epsilon)\rangle_{\rm av}-\bar{p}^2}{\bar{p}^2}=\frac{R(0)}{(1+\epsilon^2/\beta^2)}. \quad (56)$$

Since β is the correlation length one may calculate

$$D=\tfrac{1}{2}\pi R(0)\beta \quad (57)$$

and in turn

$$\bar{\Gamma}=-\bar{p}D/\pi. \quad (58)$$

These equations are subject to uncertainties from two causes (aside from the basic assumptions of the model). The first of these is the possibility of direct reactions removing a large amount of flux from the beam. Note that pure elastic potential scattering has no effect since this involves only the real part of the phase shift. Such a direct coupling to other channels is expected to affect Eq. (56) not at all, Eq. (57) in second order, and Eq. (58) in first order.

The second uncertainty comes from the assumption of a Porter-Thomas distribution for the partial widths. For a function with two degrees of freedom the right-hand side of Eq. (57) will be divided by 2.

VI. BEHAVIOR OF PHASE SHIFT

Let us return to the full form of S given by Eq. (16), that is to say without the condition expressed in (34). If we write

$$S=\eta e^{2i\delta}, \quad (59)$$

then

$$\delta=\tfrac{1}{2}\sum_\lambda\left[\tan^{-1}\frac{1-y_\lambda}{2(E-E_\lambda)}-\tan^{-1}\frac{1+y_\lambda}{2(E-E_\lambda)}\right] \quad (60)$$

and

$$\eta=\prod\left[\frac{4(E-E_\lambda)^2+(1-y_\lambda)^2}{4(E-E_\lambda)^2+(1+y_\lambda)^2}\right]^{1/2}. \quad (61)$$

Consider a single term in the sum for δ.

$$\delta_\lambda=\frac{1}{2}\left[\tan^{-1}\frac{1-y_\lambda}{2(E-E_\lambda)}-\tan^{-1}\frac{1+y_\lambda}{2(E-E_\lambda)}\right]. \quad (62)$$

As E passes through E_λ one of two things occurs. If y_λ is less than one then δ_λ increases from zero up to some maximum less than $\frac{1}{4}\pi$ then decreases to zero as $E=E_\lambda$. It then continues to decrease to a minimum which is greater than $-\frac{1}{4}\pi$ and then increases again to zero.

If y_λ is greater than one δ_λ simply increases by π as E passes through E_λ. These two behaviors are shown in Fig. 1.

Figure 2 shows some curves of δ calculated using Eq. (67) with equally spaced levels and y_λ generated as random numbers with a χ^2 distribution of two degrees

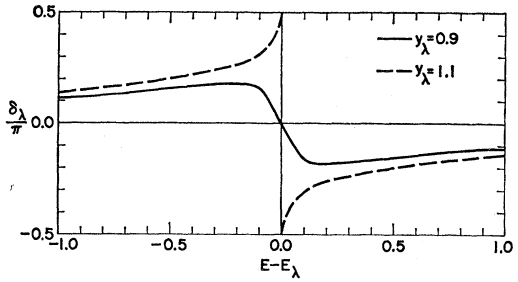

FIG. 1. Shown are two calculations to demonstrate the behavior of the phase shift as a function of the ratio $\Gamma_\lambda/\beta_\lambda \equiv y_\lambda$.

of freedom. The random numbers are the same in each curve but the average value of y has been changed to demonstrate the sharpness of the effect.

Since δ is the sum of the δ_λ and since the average contribution of all levels with $y_\lambda<1$ is zero, we may say that the number of times the phase shift increases by π is equal to the number of levels with $y_\lambda>1$.

For a distribution of y as used above we have

$$N=T\exp(-\beta/\bar{\Gamma})/D, \tag{63}$$

where N is the number of times the phase shift passes through π in an energy range T. Since the exponential factor is very important we may say that the most probable time to observe the effect is when the ratio $\beta/\bar{\Gamma}$ is minimized. This can be expected for high J values in the elastic scattering of α particles since the α-particle widths are weighted to higher J values than the total width (due largely to mass 1 particles). This effect has been observed by Singh *et al.*[14] One may easily develop the more realistic expressions and attempt to get quantitative agreement with the experiment but the great sensitivity of the results to optical-model parameters and the great uncertainty in the value of D make this comparison rather meaningless. Calculations of this type by the author have upheld the general features discussed above, for example, the proper partial waves are predicted to have larger effects.[15]

Note also that since the value of y_λ is unlikely to be much above one the corresponding value of η should show a dip at the same energy that the phase shift increases by π. This is also observed in Singh's experiment.

An interesting feature of this picture is the approximate prediction of the width of these structures. Since $\Gamma_\lambda \geq \beta$ for the effect to occur and since the distribution function of Γ_λ is decreasing rapidly we have the result that the total width is expected to be $\approx 2\beta$. This feature is also in reasonable agreement with the experiment.

One may note that effects such as these could never be reproduced by a sum representation (with statistical assumptions) and that the unitarity restriction apparently plays an important role in interpreting these data.

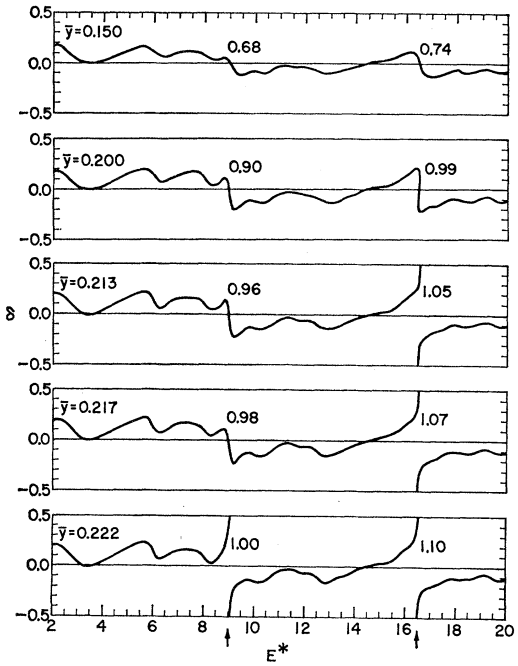

FIG. 2. A set of computer-generated excitation functions to demonstrate the sharpness of the phase-shift effect. The curves were generated by using random numbers from a Porter-Thomas distribution for the y_λ and equally spaced levels (strongly overlapping). The basic random numbers are the same for each curve, only the average value being changed as marked.

Perhaps some comments are in order concerning the fact that these states are simple states in the elastic channel. This is certainly true since they have large widths in that channel. However, this does not mean that their partial width is much larger than the others since we have seen that a few percent change in the partial width produces a complete change in the behavior of δ as a function of energy. In fact if these widths were much larger than the other widths, η would not show a dip. Thus the fact that the phase shift has a resonant behavior does *not* mean that the state has a very special character.

VII. RESIDUES

The form, Eq. (16), specifies the way in which the residues of the Mittag-Leffler expansion for S [Eq. (2)] are correlated in order that unitarity not be violated. The explicit relationship between the residues and the resonance parameters is easily seen to be

$$b_\mu = id_\mu,$$

$$d_\mu = \Gamma_\mu \prod_{\lambda \neq \mu} \frac{E_\mu - E_\lambda + \frac{1}{2}i(\beta_\lambda - \beta_\mu - \Gamma_\lambda - \Gamma_\mu)}{E_\mu - E_\lambda + \frac{1}{2}i(\beta_\lambda - \beta_\mu + \Gamma_\lambda - \Gamma_\mu)}. \tag{64}$$

[14] P. P. Singh, B. A. Watson, J. J. Kroepfl, and T. P. Marvin, Phys. Rev. Letters **18**, 31 (1966).

[15] For these calculations β was also treated as a random variable with few ($\sim$10) degrees of freedom (Ref. 7). This can be an important effect.

There are a number of algebraic identities relating sums of residues to sums of resonances parameters. Consider S to have a finite number of poles and zeros, i.e., the product in Eq. (16) is finite. Denote them as follows:

$$p_\lambda = E_\lambda - \tfrac{1}{2}i(\beta_\lambda + \Gamma_\lambda),$$
$$z_\lambda = E_\lambda - \tfrac{1}{2}i(\beta_\lambda - \Gamma_\lambda). \quad (65)$$
$$S = \prod_\lambda (E - z_\lambda)/(E - p_\lambda). \quad (66)$$

Now consider the integral of $E^m S$, $(m=0, 1, 2, \cdots)$ around a contour C, where C is a circle (of radius R) with center at the origin enclosing all poles and zeros of S. Then,

$$\int_C E^m S dE = 2\pi i \sum (\text{residues}) = 2\pi \sum d_\mu p_\mu{}^m. \quad (67)$$

On the circle we may express S as

$$S = \prod_\lambda \frac{1-(z_\lambda/E)}{1-(p_\lambda/E)}$$
$$= \exp\left[\sum_\lambda \ln\left(1-\frac{z_\lambda}{E}\right) - \sum_\lambda \ln\left(1-\frac{p_\lambda}{E}\right)\right] \quad (68)$$
$$= \exp\left[-\sum_{n=1}^{\alpha} \frac{g_n}{nE^n}\right],$$

where

$$g_n \equiv \sum_\lambda z_\lambda{}^n - \sum_\lambda p_\lambda{}^n. \quad (69)$$

We may now expand the exponential and collect terms with the same power of E.

$$S = 1 - \sum_{n=1}^{\alpha} \frac{C_n}{E^n}, \quad (70)$$

where

$$C_1 = g_1,$$
$$C_2 = \tfrac{1}{2}(g_2 - g_1{}^2), \quad (71)$$
$$C_3 = 1/18(6g_3 - 9g_1g_2 + g_1{}^3), \quad \text{etc.}$$

Note that the sum in Eq. (70) is convergent with the conditions stated on the contour C. Thus we may write

$$\int_C E^m S(E) dE = iR \int_0^{2\pi} R^m e^{im\theta} S(Re^{i\theta}) e^{i\theta} d\theta$$
$$= iR^{m+1}$$
$$\times \int_0^{2\pi} \left[\exp[i(m+1)\theta] - \sum_{n=1}^{\infty} \frac{C_n \exp[i(m-n+1)]}{R^n}\right] d\theta$$
$$= -i2\pi C_{m+1}. \quad (72)$$

Combining this result with Eq. (29) we have

$$\sum d_\mu p_\mu{}^m = -iC_{m+1}, \quad (73)$$

which is the desired relationship.

For $m=0$ the equation

$$\sum_\mu d_\mu = \sum_\mu \Gamma_\mu \quad (74)$$

is obtained.

It is interesting to note that, even for the case of purely elastic scattering with well-isolated levels, Eq. (63) displays a correlation between levels. To exhibit this feature, we note that in this limit Eq. (63) can be written

$$d_\mu = \Gamma_\mu \exp(i\varphi_\mu), \quad (75)$$

where

$$\varphi_\mu = -2\sum_{\lambda \neq \mu} [\Gamma_\lambda/(E_\mu - E_\lambda)]. \quad (76)$$

Consider the case in which there exists some finite number of levels clustered about an energy E_0, with all other levels very distant. For E_μ to the far left of E_0, φ_μ must be positive. As E_μ increases, φ_μ decreases until on the far right of the collection of levels it is negative. Thus we see that *for any isolated collection of elastic levels a correlation of the phases of their residues must exist.* For this reason a correlation of residues for an isolated group of levels does not provide any further evidence of intermediate structure.

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APPENDIX

Here are the formulas for differential cross sections which follow from the results given in main body of the paper. These expressions may be simplified by use of the quantity

$$x_J \equiv \exp\{-2[\ln(1-T_J)]^2 / \sum_{c \neq \text{elastic}} \ln(1-T_c)\}. \quad (A1)$$

The average differential fluctuation cross section is given by

$$\bar{\sigma}(\theta) = \tfrac{1}{4}\lambda^2 \sum_J (2J+1)^2 (1-T_J)(x_J - 1) P_J{}^2(\theta),$$

and

$$R \equiv (\langle \sigma^2 \rangle_{\text{av}} - \bar{\sigma}^2)/\bar{\sigma}^2 \quad (A2)$$

is given by

$$R = 1 + [\bar{\sigma}(\theta)]^{-2} \sum_J (x_J + 3)(x_J - 1) \bar{\sigma}_J{}^2 P_J{}^4(\theta). \quad (A3)$$

Because of the form of Eq. (A3), we see that the effects described here on R are most important when only a small number of partial waves contribute.