Inelastic Processes in Particle Transfer Reactions

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It must certainly be true for some levels in all nuclei, and all levels in some nuclei, that the usual treatment of particle-transfer reactions, which neglects inelastic effects, is invalid. Here a practical method for taking these effects into account is described. The method is discussed in terms of the (d, p) reaction, but it has a much broader application.

1. INTRODUCTION

The usual distorted-wave Born approximation (DWBA) for transfer reactions makes three basic assumptions. First, it assumes that the transfer takes place directly from the elastic entrance channel to the residual channel. Only the transferred particles are treated explicitly, while all the others, which we shall refer to collectively as the core, are regarded as passive. The reaction is assumed to proceed only to the extent that the core state is unchanged. Second, it assumes that the elastic optical potential provides wave functions for the relative motion which are good *inside* the nucleus, or at least in the surface region, since this is where the transfer process is concentrated. Third, it assumes that the transfer process itself is weak so that it can be treated in first order. With these assumptions the transition amplitude can be computed for (d, p) reactions, from

$$T_{p,d}=\int \psi_p^{(-)*}\langle\Phi_p((A+1))\mid V_{np}\mid \Phi_d(A)\rangle\psi_d^{(+)}d\mathbf{r}_n d\mathbf{r}_p, \tag{1.1}$$

where ψ_d and ψ_p are distorted waves describing only the elastic scattering in the channels d and p, and Φ_p and Φ_d describe the nuclear states between which the reaction takes place.[1]

There are certainly situations where one or both of the first two assumptions are false, although the third assumption is probably always valid for particle transfer reactions. As concerns the first assumption, the theory will fail to the extent that the reaction of interest does take place between states, one of which is not the parent of the other. This situation is illustrated, in idealized form, for the (d, p) reaction in Fig. 1. In this example the lower states of the residual nucleus $(A+1)$ do have the target A as parent, so that the direct transition can occur. However, the third state has as its parent an excited state of the target, so that it can be reached only through excitation of the core to the parent state either before or after the transfer takes place. The usual DWBA may provide good answers for the former transitions but it cannot treat the latter. As long as one knows which states have the ground state as their parent and restricts attention to them, the usual treatment *may* provide valid results.

However, if there are strongly enhanced inelastic transitions, the usual DWBA will fail in any case because of the second assumption. This can be understood as follows. The (one-channel) optical potential is chosen so as to reproduce the elastic cross section. This assures that the wave function for the relative motion is correct in the *external* region. However, for the purpose of calculating the reaction, it is needed in the *interior* region, or at least in the neighborhood of the nuclear surface. It is here that the one-channel optical model will break down due to deexcitation of other channels back to the elastic channel, if their coupling is sufficiently strong.

One solution to the problem of including excitation of the core in transfer reactions was suggested by Penny and Satchler.[2] They propose evaluating the amplitude for (d, p) reactions from

$$T_{p,d}=\sum_{p',d'}\int \psi_{p,p'}^{(-)*}\langle\Phi_{p'}((A+1))\mid V_{np}\mid \Phi_{d'}(A)\rangle \times\psi_{d,d'}^{(+)}d\mathbf{r}_n d\mathbf{r}_p, \tag{1.2}$$

where $\psi_{d,d'}^{(+)}$ is a generalized distorted wave in the channel d' found by solving the coupled equations for deuteron inelastic scattering by A with an incident wave in the channel d, and $\psi_{p,p'}^{(-)}$ is similarly defined for the proton-$(A+1)$ system. However, a numerical solution to the full problem has never been obtained. Iano and Austern[3] considered the problem from a similar point of view, but solved it only to first order as concerns the inelastic transitions. (That is to say, their generalized distorted waves correspond to the DWBA approximation for the inelastic processes.) Therefore, they do not remedy the second cause of failure of the usual method. Kozlowsky and de-Shalit[4] and Levin[5] also have worked on this problem. They too treat inelastic effects only in first order, and only in the exit channel. The evaluation of (1.2) evidently is an extremely troublesome numerical problem.

In this paper we present a different way of solving the same problem which apparently is more amenable to numerical calculation. We are able to treat inelastic effects to all orders in both entrance and exit channels, and to treat spin-dependent interactions in the optical

[1] See, e.g., N. K. Glendenning, Ann. Rev. Nucl. Sci. **13**, 191 (1963), and references therein.

[2] S. K. Penny and G. R. Satchler, Nucl. Phys. **53**, 145 (1964).
[3] P. J. Iano and N. Austern, Phys. Rev. **151**, 853 (1966).
[4] B. Kozlowsky and A. de-Shalit, Nucl. Phys. **77**, 215 (1966).
[5] F. S. Levin, Phys. Rev. **147**, 715 (1966).

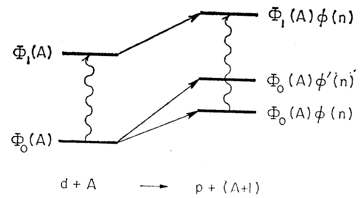


FIG. 1. The (d, p) reaction between idealized nuclei is illustrated. The lowest two levels of $(A+1)$ have the same core configuration $\Phi_0(A)$ as the target and so can be produced directly by the stripping process as indicated by the arrows. The third level, as indicated by its wave function, has an excited core, and so can be reached only through an inelastic collision with the deuteron before or with the proton after the transfer reaction. Only the lowest-order routes are illustrated.

potentials and in the direct interactions.[6] The S-matrix is obtained directly from the application of the physical boundary conditions to the solution of an inhomogeneous system of equations describing the scattering in the residual system. The inhomogeneity is a source term which describes the production of the residual particle by the transfer process. The method is very general for scattering problems because the specification of the type of reaction enters only in the structure of the source term. The equations otherwise describe inelastic scattering.

In this paper, we describe our method in terms of the (d, p) reaction, though it is applicable to all particle transfer reactions. In Sec. 2 the rationale for the equations that we use to describe the reaction is given, followed by a detailed formulation of their structure in Sec. 3. There also it is shown that the solution reduces to the usual DWBA in case the core-excitation effects are neglected. The source term due to the transfer reaction is explicitly calculated in the zero-range approximation. In Sec. 4 the numerical problem posed by the systems of equations used to describe the reaction is discussed.

2. SCHEMATIC PRESENTATION OF SOLUTION

In this section the reasoning is given by which we arrive at the equations we use to describe the transfer reaction where inelastic effects are included. For definiteness we consider the (d, p) reaction

$$d+A\rightarrow p+(A+1). \tag{2.1}$$

For the moment, assume that the system can exist in only a finite, indeed only a few, different states or channels containing the fragments $d+A$ or $p+(A+1)$, such as depicted in Fig. 2. This restriction will be relaxed only in the usual way[7,8] by assuming that the effect of the omitted channels on those of interest can be carried in an optical potential. We shall assume also that the particle transfer process is weak so that channels of the configuration $d+A$ are coupled only weakly to those of $p+(A+1)$, and we shall treat this coupling only in first order.

First, focus attention on a typical deuteron channel d' in Fig. 2. We use d' to label all the quantum numbers needed to define the channel, such as the state of the nucleus A, and the angular momentum of the deuteron. The equation describing the motion in this channel is

$$(T_{d'}+V_{d'd'}-E_{d'})u_{d'}=-\sum_{d''\neq d'}V_{d'd''}u_{d''} \tag{2.2}$$

in which the various terms on the right represent the feeding of the channel d' by inelastic processes from other channels d'', illustrated in the figure by the wiggly arrows. Equation (2.2) together with those describing the other included channels constitute the usual system of coupled equations for inelastic scattering.

Now focus attention on a typical proton channel p'. Again this channel is fed by inelastic processes leading from other proton channels (wiggly arrows) but, as well, it is fed by the transfer reaction from the various deuteron channels. Therefore, there will be two types of source terms in the equations for the proton motion. Accordingly, we write

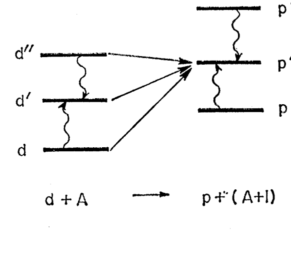
$$(T_{p'}+V_{p'p'}-E_{p'})w_{p'}=-\sum_{p''\neq p'}V_{p'p''}w_{p''}-\sum_{d'}\rho_{p'}{}^{d'}. \tag{2.3}$$

Here $\rho_{p'}{}^{d'}$ represents the source of protons in channel p' as a result of the stripping reaction in channel d', illustrated by the straight arrow in the figure. Of course ρ will depend upon the solutions of the deuteron equations (2.2). We write down its detailed structure in Sec. 3.

Since the reaction is initiated by a beam of deuterons incident on the ground state of nucleus A, the deuteron system (2.2) is to be solved subject to the boundary conditions that only a ground channel has an incoming wave while all channels may have outgoing waves. The proton system is subject to the condition that there are *only* outgoing waves.

The amplitudes of the outgoing proton waves are of course the S-matrix elements for the (d, p) reactions, in which the inelastic scattering is calculated to *all orders* among the retained channels in both the deuteron

FIG. 2. For a typical deuteron channel d' and proton channel p', this figure illustrates by the arrows leading to these levels what processes have to be described by the equations of motion in these channels. See Sec. 2 for a full discussion.



[6] Forthcoming application to the (p, t) reactions. The rationale for our treatment was also described in N. K. Glendenning, Lawrence Radiation Laboratory Report No. UCRL-18225, 1968 [invited paper A.P.S. meeting, Washington, 1968 (unpublished)].

[7] N. K. Glendenning, in *Proceedings of the International School of Physics "Enrico Fermi,"* edited by M. Jean (Academic Press Inc., New York, to be published), Course 40; Lawrence Radiation Laboratory Report No. UCRL-17503, 1967 (unpublished).

[8] N. K. Glendenning, Nucl. Phys. **A117**, 49 (1968).

and proton systems, while the transfer reaction itself is treated as the weak process it is, only in *first order*. That is, we have not included the reaction back on the deuteron channels of pickup reactions. Therefore, the solution to the problem can be obtained in two steps. First, the coupled system (2.2) for the deuterons is solved so that the source terms ρ can be constructed. Second, the inhomogeneous system (2.3) describing the protons is solved.

Of course, one could think of extending the system of equations to include additional particle channels like triton and α channels in which each of these systems is described by equations like (2.3).

3. DETAILED FORMULATION FOR (*d*, *p*) REACTION

In order to illustrate our method without unnecessary detail we consider the (d, p) reaction in the absence of spin-dependent interactions. This means that we can treat the deuteron and proton as spinless particles. The general case is treated in the Appendix.

A. Inelastic Deuteron Scattering

Let the target nucleus (A) be governed by the Hamiltonian H_A, whose eigenfunctions are denoted by $\Phi_{\alpha J}(\mathbf{A})$, where $\mathbf{A}$ denotes all the nuclear coordinates, and α denotes all nuclear quantum numbers additional to J, M such as the parity π_α.

$$(H_A - E_{\alpha_d J_d})\Phi_{\alpha_d J_d}(\mathbf{A}) = 0. \tag{3.1}$$

[We use the subscript d on all quantum numbers referring either to the deuteron itself or the nucleus A, and p for quantum numbers describing the proton and the nucleus $(A+1)$.] For the total system we have

$$H = H_A + T + V(\mathbf{d}, \mathbf{A}), \tag{3.2}$$

where T is the energy of relative motion and V is the deuteron-nucleus interaction. It should be understood that the problem will be solved in a highly truncated space containing only the few interesting channels, and, correspondingly, that V is an effective interaction in the sense discussed elsewhere, which carries the affect of all the channels we neglect on those we do treat explicitly.[7,8]

With the eigenfunction of total angular momentum $\mathbf{I} = l_d + \mathbf{J}_d$ and parity $\pi = (-)^{l_d}\pi_{\alpha_d}$,

$$\phi_{d\pi I}{}^M(\hat{\mathbf{R}}, \mathbf{A}) = \left[Y_{l_d}(\hat{\mathbf{R}})\Phi_{\alpha_d J_d}(\mathbf{A}) \right]_I^M. \tag{3.3}$$

We expand a solution Ψ of

$$(H - E)\Psi = 0 \tag{3.4}$$

corresponding to incident particles in the channel d as

$$\Psi_{d\pi I}{}^M = R^{-1} \sum_{d'} u_{d'}{}^{d\pi I}(R)\phi_{d'\pi I}{}^M(\mathbf{R}, \mathbf{A}), \tag{3.5}$$

where we use d to denote the whole collection of quantum numbers in a deuteron target channel

$$d \equiv \alpha_d J_d l_d \tag{3.6}$$

and by d' some other state of intrinsic motion $\alpha_d' J_d'$ and/or relative motion l_d'. (In this paper $\hat{\mathbf{R}} \equiv \mathbf{R}/R$.) In the usual way from (3.4) and (3.5) we get for each total angular momentum and parity I, π a system of coupled equations for the radial functions $u(r)$. They have the form, for each channel d',

$$[T_{d'} + V_{d'd'}{}^{\pi I}(R) - E_{d'}]u_{d'}{}^{d\pi I}(R) = -\sum_{d'' \neq d'} V_{d'd''}{}^{\pi I}(R) u_{d''}{}^{d\pi I}(R). \tag{3.7}$$

[A sum over a primed subscript will always include the ground (unprimed) channel unless otherwise indicated.] In the above equation,

$$V_{dd'}{}^{\pi I}(R) = \langle \phi_{d\pi I}{}^M(\mathbf{R}, \mathbf{A}) \mid V(\mathbf{d}, \mathbf{A}) \mid \phi_{d'\pi I}{}^M(\mathbf{R}, \mathbf{A}) \rangle \tag{3.8}$$

and

$$T_d = \frac{\hbar^2}{2m_d}\left(-\frac{d^2}{dR^2} + \frac{l_d(l_d+1)}{R^2} \right), \tag{3.9}$$

$$E_d = E - E_{\alpha_d J_d}. \tag{3.10}$$

These are to be solved subject to the boundary conditions that there are incoming waves (I) only in the channels containing the nucleus in its ground state, while all channels may have outgoing waves (O)

$$u_{d'}{}^{d\pi I}(R) \to \delta_{d'd} I_{l_d}(k_d R) - (v_d/v_{d'})^{1/2} S_{d',d}{}^{\pi I} O_{l_{d'}}(k_{d'} R). \tag{3.11}$$

Here

$$I_l^*(kr) = O_l(kr) = G_l(kr) + iF_l(kr) \to \exp\{i[kr - \eta \ln(2kr) - l\pi/2 + \sigma_l]\}, \tag{3.12}$$

where G and F are the irregular and regular Coulomb functions, $\eta = ZZ'e^2/\hbar v$, σ_l is the Coulomb phase shift, and v is the particle velocity.

The total wave function has all angular momentum and both parities, corresponding to the fact that a *beam* of particles is incident on the target.

$$\Psi = \sum_{l_d I M} A_{d\pi I}{}^M \Psi_{d\pi I}{}^M. \tag{3.13}$$

If we choose

$$A_{d\pi I}{}^M = -(2ik_d)^{-1} C_{0M_d M}{}^{l_d J_d I} [4\pi(2l_d+1)]^{1/2} i^{l_d} \exp(i\sigma_d), \tag{3.14}$$

then, in the absence of the Coulomb force,

$$\Psi \to \Phi_{J_d}{}^{M_d}(A) \exp(ik_d Z) + \text{scattered wave},$$

which is the desired solution.

The detailed structure of the coupled equations (3.7) based on microscopic nuclear descriptions has been discussed elsewhere.[7,8]

B. Proton Production and Inelastic Scattering

The nucleus $(A+1)$ we suppose is described by

$$(H_{A+1}-E_{\alpha_p J_p})\Phi_{\alpha_p J_p}((\mathbf{A+1}))=0 \qquad (3.15)$$

and with notation and definitions analogous to the above, the proton motion in any channel p would be described, for each total angular momentum and parity $I\pi$, by

$$[T_p+V_{pp}{}^{\pi I}(r_p)-E_p]w_p{}^{\pi I}(r_p) = -\sum_{p'\neq p} V_{pp'}{}^{\pi I}(r_p)w_{p'}{}^{\pi I}(r_p) \qquad (3.16)$$

were it not for the fact that there is a source of protons in the vicinity of the nucleus due to the stripping of the neutron from the deuteron. We must instead solve the inhomogeneous system

$$[T_p+V_{pp}{}^{\pi I}(r_p)-E_p]w_p{}^{d\pi I}(r_p) = -\sum_{p'\neq p} V_{pp'}{}^{\pi I}(r_p)w_{p'}{}^{d\pi I}(r_p)-\rho_p{}^{d\pi I}(r_p), \qquad (3.17)$$

where ρ denotes the proton source. We calculate this from

$$\rho_p{}^{d\pi I}(r_p)=r_p\sum_{d'}\langle\phi_{p\pi I}{}^{M}(\hat{\mathbf{r}}_p, (\mathbf{A+1}))\mid V_{np}(r)\mid\phi_0(r)\phi_{d'\pi I}{}^{M}(\mathbf{R},\mathbf{A})u_{d'}{}^{d\pi I}(R)/R\rangle \qquad (3.18)$$

and will justify this form later in Sec. 3 D. Here $\mathbf{r}=\mathbf{r}_n-\mathbf{r}_p$ and $2\mathbf{R}=\mathbf{r}_n+\mathbf{r}_p$, and ϕ_0 is the deuteron wave function. In this matrix element, all coordinates (including $\hat{\mathbf{r}}_p$) are integrated except r_p. The various terms in this sum represent the transitions indicated by arrows in Fig. 2 leading from the various channels d' in the deuteron system to the channel p in the proton system.

We have placed d as a superscript on w to correspond to the fact that the reaction is initiated by a *beam* of deuterons in the ground channel d, which information is carried in the source terms through their dependence on solutions of (3.7) with boundary conditions (3.11).

The equations (3.17) are to be solved with the *physical* boundary conditions that proton channels of this reaction have only outgoing waves

$$w_p{}^{d\pi I}\rightarrow-(v_d/v_p)^{1/2}S_{p,d}{}^{\pi I}O_{l_p}(k_pr_p). \qquad (3.19)$$

The amplitude of

$$(v_d/v_p)^{1/2}\Phi_{J_p}{}^{M_p}\exp\{i[k_pr_p-\eta\ln(2k_pr)]\}$$

in $w_p{}^{d\pi I}\phi_{p\pi I}{}^{M}$ is

$$-S_{p,d}{}^{\pi I}i^{-l_p}\exp(i\sigma_p)\sum_{m_p}C_{m_pM_pM}{}^{l_pJ_pI}Y_{l_p}{}^{m_p}(\hat{\mathbf{r}}_p).$$

On the other hand, the sources (3.18) were constructed with unit amplitude for $u_{d'}{}^{d\pi I}$ whereas its actual amplitude in the incident wave is given by (3.14). Hence the total amplitude for the process is

$$f(J_dM_d\rightarrow J_pM_p)=(2ik_d)^{-1}\sum_{l_dl_pm_pIM}[4\pi(2l_d+1)]^{1/2}i^{l_d-l_p}\exp[i(\sigma_d+\sigma_p)] \times C_{0M_dM}{}^{l_dJ_dI}C_{m_pM_pM}{}^{l_pJ_pI}S_{p,d}{}^{\pi I}Y_{l_p}{}^{m_p}(\hat{\mathbf{k}}_p). \qquad (3.20)$$

(We have replaced $\hat{\mathbf{r}}_p$ with $\hat{\mathbf{k}}_p$, since they become coincident asymptotically.) The flux through the surface $r_p{}^2d\Omega$ in the direction $\hat{\mathbf{k}}_p$ is therefore $v_d|f|^2d\Omega$, while the incident deuteron flux is v_d. Hence

$$d\sigma/d\Omega=(2J_d+1)^{-1}\sum_{M_pM_d}|f(J_dM_d\rightarrow J_pM_p)|^2. \qquad (3.21)$$

We should perhaps note that $S_{p,d}{}^{\pi I}$ is a short-hand notation for

$$S_{p,d}{}^{\pi I}\equiv\langle(l_pJ_p)IM\mid S\mid(l_dJ_d)IM\rangle \qquad (3.22)$$

(which is independent of M). It is then clear that

$$\sum_{IM}C_{0M_dM}{}^{l_dJ_dI}C_{m_pM_pM}{}^{l_pJ_pI}S_{p,d}{}^{\pi I} = \langle l_pm_p, J_pM_p\mid S\mid l_d0, J_dM_d\rangle. \qquad (3.23)$$

C. Check with DWBA

We can check the construction of the source term in all its details by considering the limit of our equations when the inelastic effects in both proton and deuteron channels are neglected. Since, in any case, the transfer process is calculated in first order, this limit should lead exactly to the DWBA cross section. Our equations become

$$(T_d+V_{dd}{}^{\pi I}-E_d)u_d{}^{0\pi I}(R)=0, \qquad (3.24)$$

$$(T_p+V_{pp}{}^{\pi I}-E_p)w_p{}^{d\pi I}(r_p)=-\rho_p{}^{d\pi I}(r_p), \qquad (3.25)$$

where d refers to *elastic* deuteron channels and p to *any* proton channel. Introduce the solution of the homogeneous equation corresponding to the second of these,

$$(T_p+V_{pp}{}^{\pi I}-E_p)w_p{}^{0\pi I}(r_p)=0, \qquad (3.26)$$

which is regular at the origin and has the asymptotic behavior

$$w_p{}^{0\pi I}\rightarrow-2i\exp(i\delta_p)\sin(k_pr_p-l_p\pi/2+\delta_p), \qquad (3.27)$$

where δ_p is the phase shift introduced by V_{pp}. Then the asymptotic solution of (3.25) can be written down immediately in terms of $u_d{}^{0\pi I}$ and $w_p{}^{0\pi I}$ (see, for example, Ref. 7). It is

$$u_p{}^{d\pi I}\rightarrow-(v_d/v_p)^{1/2}S_{p,d}{}^{\pi I}O_{l_p}(k_pr_p), \qquad (3.28)$$

where O_{l_p} is an outgoing wave, and S is given by the explicit formula

$$S_{p,d}{}^{\pi I}=\frac{i}{\hbar^2}\left(\frac{m_pm_d}{k_pk_d}\right)^{1/2}\int w_p{}^{0\pi I}\rho_p{}^{d\pi I}dr_p, \tag{3.29}$$

and $k_p=(2m_pE_p/\hbar^2)^{1/2}$ is the wave number. The superscript 0 on the solutions to (3.24) and (3.26) indicate that they describe elastic scattering only. Introducing (3.18) for ρ into this expression for S gives

$$S_{p,d}{}^{\pi I}=\frac{i}{\hbar^2}\left(\frac{m_pm_d}{k_pk_d}\right)^{1/2}\int\frac{w_p{}^{0\pi I}(r_p)}{r_p}\,\phi_{p\pi I}{}^{M*}(\hat{\mathbf{r}}_p,\,(\mathbf{A}+\mathbf{1}))\,V_{np}(r)\phi_0(r)\phi_{d\pi I}{}^{M}(\mathbf{R},\,\mathbf{A})\,\frac{u_d{}^{0\pi I}(R)}{R}\,d\mathbf{r}_nd\mathbf{r}_pd\mathbf{A}. \tag{3.30}$$

Inserting this into (3.20) we have

$$f=(2\hbar^2k_d)^{-1}(m_pm_d/k_pk_d)^{1/2}\sum_{l_dl_pm_p}[4\pi(2l_d+1)]^{1/2}i^{l_d-l_p}\exp[i(\sigma_p+\sigma_d)]Y_{l_p}{}^{m_p}(\hat{\mathbf{k}}_p)$$

$$\times\int\frac{w_p{}^{0\pi I}(r_p)}{r_p}\,Y_{l_p}{}^{m_p*}(\hat{\mathbf{r}}_p)\langle\Phi_{\alpha_pJ_p}{}^{M_p}(\mathbf{A}+\mathbf{1})\mid V_{np}(r)\mid\phi_0(r)\Phi_{\alpha_dJ_d}{}^{M_d}(\mathbf{A})\rangle\frac{u_d{}^{0\pi I}(R)}{R}\,Y_{l_d}{}^{0}(\mathbf{R})\,d\mathbf{r}_nd\mathbf{r}_p. \tag{3.31}$$

Now let us write

$$\psi_d{}^{(+)}\equiv-\sum_{l_d}[4\pi(2l_d+1)]^{1/2}i^{l_d}\exp(i\sigma_d)\frac{u_d{}^{0\pi I}(R)}{2ik_dR}\,Y_{l_d}{}^{0}(\mathbf{R}), \tag{3.32}$$

which is the partial-wave expansion of the deuteron wave, distorted by V_{dd}. In the absence of distortion it goes to

$$\psi_d{}^{(+)}\to\exp(ikZ). \tag{3.33}$$

Also,

$$\psi_p{}^{(-)*}\equiv-4\pi\sum_{l_pm_p}i^{-l_p}\exp(i\sigma_p)[w_p{}^{0\pi I}(r_p)/2ik_pr_p]Y_{l_p}{}^{m_p*}(\hat{\mathbf{r}}_p)\,Y_{l_p}{}^{m_p}(\hat{\mathbf{k}}_p), \tag{3.34}$$

which, in the absence of distortion, goes to

$$\psi_p{}^{(-)*}\to\exp(-i\mathbf{k}_p\cdot\mathbf{r}_p). \tag{3.35}$$

Equation (3.31) can be written now as

$$f=(2\pi\hbar^2)^{-1}[(k_p/k_d)\,m_pm_d]^{1/2}T_{p,d}, \tag{3.36}$$

where

$$T_{p,d}\equiv\int\psi_p{}^{(-)*}(\mathbf{k}_p,\,\mathbf{r}_p)\langle\Phi_{\alpha_pJ_p}{}^{M_p}(\mathbf{A}+1)\mid V_{np}\mid\phi_0(r)\Phi_{\alpha_dJ_d}{}^{M_d}(\mathbf{A})\rangle\psi_d{}^{(+)}(\mathbf{k}_d,\,\mathbf{R})\,d\mathbf{r}_nd\mathbf{r}_p \tag{3.37}$$

is *precisely* the usual DWBA expression for the T-matrix. The cross section according to (3.21) is

$$\frac{d\sigma}{d\Omega}=\frac{m_pm_d}{(2\pi\hbar^2)^2}\frac{k_p}{k_d}(2J_d+1)^{-1}\sum_{M_dM_p}|\,T_{p,d}\,|^2, \tag{3.38}$$

which is the usual DWBA result for the $(d,\,p)$ reaction with spinless nucleons.

Thus we have confirmed, to the last factor, that our construction of the source term is the one which, in the *absence* of inelastic effects leads to the DWBA cross section for the transfer reaction. Consequently, although our equations are approximate, they embody no approximations additional to those of the usual DWBA, whereas they do take account of inelastic effects, to all orders among the retained channels.

D. Source Term

The source term can be explicitly evaluated under the usual zero-range approximation

$$V_{np}(r)\phi_0(r)=D_0\delta(r), \tag{3.39}$$

where D_0 can be related to the binding energy of the deuteron.

To carry out the integration of the coordinates $\mathbf{A}$ we make a parentage expansion

$$\Phi_{J_p}{}^{M_p}(\mathbf{A}+\mathbf{I})=\sum_{l_n\alpha_dJ_d}\beta_{l_n}(J_d,J_p)\times\left[\Phi_{\alpha_dJ_d}(\mathbf{A})\,Y_{l_n}(\hat{\mathbf{r}}_n)\right]_{J_p}^{M_p}\chi_{l_n}(r_n), \tag{3.40}$$

where χ_{l_n} is a single-particle radial function for the neutron in the nucleus $(A+1)$. Inserting these into the source (3.18) gives

$$\rho_p{}^{d\pi I}(r_p)=D_0\sum_{d'}\langle\phi_{p\pi I}{}^{M}(\hat{\mathbf{r}}_p,\,\mathbf{A},\,\mathbf{r}_p)\mid\phi_{d'\pi I}{}^{M}(\hat{\mathbf{r}}_p,\,\mathbf{A})\rangle\times u_{d'}{}^{d\pi I}(r_p), \tag{3.41}$$

where

$$\langle\phi_{p\pi I}{}^{M}(\hat{\mathbf{r}}_p,\,\mathbf{A},\,\mathbf{r}_p)\mid\phi_{d\pi I}{}^{M}(\hat{\mathbf{r}}_p,\,\mathbf{A})\rangle=\sum\beta_{l_n}(J_d,J_p)\chi_{l_n}(r_p)\times\left\langle Y_{l_p}(\hat{\mathbf{r}}_p)\left[\Phi_{J_d}(\mathbf{A})\,Y_{l_n}(\hat{\mathbf{r}}_p)\right]_{J_p}\right]_I^M\left|\left[Y_{l_d}(\hat{\mathbf{r}}_p)\Phi_{J_d}(\mathbf{A})\right]_I^M\right\rangle. \tag{3.42}$$

The integrations are over $\mathbf{A}$ and $\hat{\mathbf{r}}_p$. We write the bracket in this expression symbolically as

$$\langle l_p, (J_d, l_n)J_p; IM \mid l_d J_d; IM\rangle, \tag{3.43}$$

where the arguments of $l_p l_n$ and l_d are $\hat{\mathbf{r}}_p$ while those of J_d are $\mathbf{A}$. To carry out the A integrations we recouple on the left

$$\mid l_p, (J_d, l_n)J_p; IM\rangle = \sum_K (\hat{J}_p\hat{K})^{1/2}(-)^{l_p+J_p+I}\begin{Bmatrix} l_p & l_n & K \\ J_d & I & J_p \end{Bmatrix} \mid (l_p l_n)K, J_d; IM\rangle, \tag{3.44}$$

where $\hat{K}=2K+1$. For the following bracket we have

$$\langle (l_p l_n)K; J_d; IM \mid l_d J_d; IM\rangle = \sum_{m_d M_d} C_{m_d M_d M}{}^{K J_d I} C_{m_d M_d M}{}^{l_d J_d I} \times \langle l_p l_n; K m_d \mid l_d m_d\rangle. \tag{3.45}$$

The $\hat{r}_p$ integration of the last bracket yields

$$\langle l_p l_n; K m_d \mid l_d m_d\rangle = (-)^{l_d}\left(\frac{\hat{l}_p \hat{l}_n}{4\pi}\right)^{1/2}\begin{pmatrix} l_p & l_n & l_d \\ 0 & 0 & 0 \end{pmatrix}\delta_{K l_d}. \tag{3.46}$$

Thus we find for the source term in the channel p

$$\rho_p{}^{d\pi I}(r) = D_0 \sum_{d' l_n} \beta_{l_n}(J_{d'} J_p)(-)^{l_n+J_p+I}\left(\frac{\hat{J}_p \hat{l}_p \hat{l}_n \hat{l}_{d'}}{4\pi}\right)^{1/2} \times \begin{Bmatrix} l_p & l_n & l_{d'} \\ J_{d'} & I & J_p \end{Bmatrix}\begin{pmatrix} l_p & l_n & l_{d'} \\ 0 & 0 & 0 \end{pmatrix}\chi_{l_n}(r)\, u_{d'}{}^{d\pi I}(r). \tag{3.47}$$

Here we see explicitly the radial form of the source which contains the product of the neutron bound state and the deuteron scattering state, as expected. That they have the same coordinate r follows, of course, from the zero-range approximation. For simplicity we have omitted the usual c.m. correction.

Of course, the strength with which the various levels d' contribute to the source in the proton channel p depends on the extent to which each of them is a parent of the level p. In the idealized example of Fig. 1, the first two levels have only the target ground state as parent, while the third state also has a pure parentage, but based on the excited target state. In general the parentage will be spread over several or even many states, though it may still reside dominantly in several states. At any rate, this information is inserted in the source term through the parentage coefficients $\beta_{l_n}(J_{d'}, J_p)$.

E. Inverse Reaction

The reaction inverse to the one of primary interest can be calculated at little extra trouble, as we shall see in Sec. 4. The source for deuterons corresponding to the (p, d) reaction in the channel labeled d is

$$\rho_d{}^{p\pi I}(R) = R\sum_{p'} \langle \phi_{d\pi I}{}^M(\hat{\mathbf{R}}, \mathbf{A})\phi_0(r) \mid V_{np}(r) \mid \phi_{p'\pi I}{}^M(\hat{\mathbf{r}}_p, (\mathbf{A+1}))\, w_{p'}{}^{p\pi I}(r_p)/r_p\rangle \tag{3.48}$$

in which $w_{p'}{}^{p\pi I}$ are solutions of the proton inelastic scattering problem defined by (3.16) with boundary conditions

$$w_{p'}{}^{p\pi I}(r_p) \to \delta_{p'p} I_{l_p}(k_p r_p) - (v_p/v_{p'})^{1/2} S_{p',p}{}^{\pi I} O_{l_{p'}}(k_{p'} r_p). \tag{3.49}$$

The equations describing the deuteron motion following the pickup reaction are, for each channel d,

$$[T_d + V_{dd}{}^{\pi I}(R) - E_d]u_d{}^{p\pi I}(R) = -\sum_{d'\neq d} V_{dd'}{}^{\pi I}(R)\, u_{d'}{}^{p\pi I}(R) - \rho_d{}^{p\pi I}(R), \tag{3.50}$$

which are to be solved with the boundary condition

$$u_d{}^{p\pi I}(R) \to -(v_p/v_d)^{1/2} S_{d,p}{}^{\pi I} O_{l_d}(k_d R). \tag{3.51}$$

The source term can be evaluated explicitly as before, under the zero-range approximation, and is

$$\rho_d{}^{p'\pi I}(R) = D_0 \sum_{p' l_n} \beta_{l_n}(J_d J_{p'})(-)^{l_n+J_{p'}+I}\left(\frac{\hat{J}_{p'} \hat{l}_{p'} \hat{l}_n \hat{l}_d}{4\pi}\right)^{1/2} \times \begin{Bmatrix} l_{p'} & l_n & l_d \\ J_d & I & J_{p'} \end{Bmatrix}\begin{pmatrix} l_{p'} & l_n & l_d \\ 0 & 0 & 0 \end{pmatrix}\chi_{l_n}(R)\, w_{p'}{}^{p\pi I}(R). \tag{3.52}$$

The cross section can be computed from (3.20) and (3.21) if everywhere the subscript d is replaced by p and vice versa.

4. NUMERICAL PROBLEM AND ITS SOLUTION

In brief, the problem of calculating the (d, p) reaction, including inelastic effects, amounts to solving the coupled equations (3.7) describing deuteron elastic and inelastic scattering, constructing the proton source terms (3.47) from this solution, and then solving the inhomogeneous coupled equations (3.17) for the proton scattering by the residual nucleus. Each of these two systems of coupled equations is defined by two boundary conditions: one at the origin, where all solutions must vanish, and one at some suitably large radius (say, R) outside the range of the nuclear interactions. Let us denote the solution to the deuteron equations (3.7) by the *column* vector

$$U(r) = [u_1(r)\cdots u_N(r)], \tag{4.1}$$

where u_1 denotes the entrance channel function, and there are altogether N channels. Clearly, it is not known *a priori* what slope to assign to the components of this vector at the origin, so that in the asymptotic region it will have the required form

$$\mathbf{U}(r)=\mathbf{U}_\infty(r)\equiv\begin{pmatrix} I_1 & -S_{11} & O_1(r) \\ & -S_{21} & O_2(r) \\ & -S_{N1} & O_N(r)\end{pmatrix}, \qquad r\geq R. \tag{4.2}$$

Consequently, a linearly independent set of vector solutions $\mathbf{U}_q(r)$ must be generated from which the desired one can be constructed. This is conveniently done by using the initial values

$$\mathbf{U}_q(r)=\frac{(k_qr)^{l_q+1}}{(2l_q+1)!!}(00\cdots1\cdots0), \qquad r\simeq0, \tag{4.3}$$

where the 1 occurs in the qth position of the *column* vector. Solving the system N times corresponding to $q=1, \cdots, N$ generates the desired set of independent solutions. The particular form and normalization is chosen in accordance with the nature of the solution near the origin, and it ensures that the solution does not grow out of bounds with increasing r. In practice, the integration is started at some value of r (say, $R_{\min}$) chosen to be as small as possible but such that $U_q(R_{\min})$ is not smaller than some assigned small number, say, of the order 10^{-30}.

With this set of solutions $\mathbf{U}_q(r)$, $q=1, \cdots, N$, we seek the linear combination such that

$$\sum_{q=1}^{N} a_q\mathbf{U}_q(R)=\mathbf{U}_\infty(R). \tag{4.4}$$

This, together with the derivative, constitutes $2N$ linear *algebraic* equations in the a_q and S_q, of which there are N each. From these S-matrix elements the elastic and inelastic deuteron cross sections may be calculated, while with the a_q the desired solution can be constructed for use in the proton source term.

The solution of the inhomogeneous problem is obtained in a similar fashion, since a general solution can be obtained from a particular solution say $\mathbf{W}(r)$ of the inhomogeneous system (3.17) plus a general solution of the corresponding homogeneous system (3.16). One may start the particular solution with zero value and slope at the origin. The solutions $\mathbf{W}_q$ to the homogeneous problem are obtained by the same means as above, and the solution with the desired asymptotic boundary condition is found by solving the linear algebraic equations

$$\sum_{q=1}^{M} b_q\mathbf{W}_q(R)+\mathbf{W}(R)=-\begin{pmatrix} S_1 & O_1(R) \\ & \vdots \\ S_M & O_M(R)\end{pmatrix} \tag{4.5}$$

for the unknowns b and S. From these last S-matrix elements the cross sections of the various proton channels in the (d, p) reaction can be calculated.

It is clear that one can calculate the proton elastic and inelastic scattering by the residual nucleus $(A+1)$ by simply solving a set of linear *algebraic* equations.

$$\sum_{q=1}^{M} c_q\mathbf{W}_q(R)=\mathbf{W}_\infty(R), \tag{4.6}$$

where $W_\infty(R)$ is defined similarly to (4.2).

Finally, the inverse (p, d) reaction can be calculated at only the cost of finding a particular solution of (3.50) with the source term (3.48) constructed from the physical solution of (3.16) already found in connection with the scattering boundary conditions (4.6) and then solving a set of algebraic equations analogous to (4.5).

5. SUMMARY

We have shown how to go beyond the usual DWBA treatment of transfer reactions to include the effects of inelastic scattering in both entrance and exit channels. As discussed in more detail in the Introduction, this may be necessary in some cases for one or both of two reasons. (a) The coupling of inelastic channels to the ground is in some nuclei very strong, especially in deformed regions. (b) Most certainly in every nucleus some states produced in particle transfer reactions will have a parentage based more on an excited rather than the ground state of the target. This will be especially true of higher-lying states.

Our basic equations cannot be rigorously derived from the Schrödinger equation for the system. However, we can say that they embody no approximations beyond those made in the usual distorted-wave method, whereas they do carry the effects of inelastic processes on the transfer reaction. These effects are carried to all orders among the retained channels so that both causes of failure of the usual DWBA are covered. However, like the DWBA, the transfer reaction is treated as a weak process, only in first order.[9]

We have illustrated our method by a detailed example, the (d, p) stripping and inverse pickup reaction, in which, for didactic purposes, we have omitted reference to the nucleon and deuteron spins. The S-matrix elements for the deuteron *and* proton elastic and inelastic scattering as well as the stripping and pickup reactions are obtained by applying the appropriate boundary conditions to the solutions of a homogeneous and inhomogeneous system of coupled differential equations for *both* the deuteron and proton channels. One particular solution only is required for each of the two inhomogeneous systems while general solutions of the two homogeneous problems are required.

The method can be applied to other problems besides

[9] There may be some transfer reactions that are stronger than can be treated in first order. See A. P. Stamp, Nucl. Phys. **83**, 232 (1966); also, G. H. Rawitscher, Phys. Rev. **163**, 1223 (1968) who considers the effect of stripping channels on deuteron elastic scattering. As a rule, however, we believe these reactions are weak, although we will investigate this point further in an effort to make a more precise statement.

the one considered here. Our first application is to the (p, t) reaction using a microscopic nuclear description and a spin-dependent direct interaction as well as spin-orbit coupling in the optical potentials.[6]

APPENDIX: GENERAL CASE WITH SPINS

Having considered the (d, p) reaction in detail in the absence of spin-dependent interactions, we now give the changes necessary to include such interactions as spin-orbit terms in the optical potentials and spin-dependent direct interactions.

The channel functions ϕ_d and ϕ_p used in Sec. 3 are now constructed with the use of spin-orbit functions as

$$\phi_{p\pi I}{}^{M}=\left[\mathcal{Y}_{(l_p 1/2)j_p}\Phi_{\alpha_p J_p}\right]_I^M, \tag{A1}$$

$$\phi_{d\pi I}{}^{M}=\left[\mathcal{Y}_{(l_d 1)j_d}\Phi_{\alpha_d J_d}\right]_I^M, \tag{A2}$$

where

$$\mathcal{Y}_{(l_p\frac{1}{2})j_p}{}^{m_p}=\left[Y_{l_p}(\hat{\mathbf{r}}_p)\eta(\boldsymbol{\sigma}_p)\right]_{j_p}^{m_p}, \tag{A3}$$

$$\mathcal{Y}_{(l_d\frac{1}{2})j_d}{}^{m_d}=\left[Y_{l_d}(\hat{\mathbf{R}})[\eta(\boldsymbol{\sigma}_p)\eta(\boldsymbol{\sigma}_n)]_1\right]_{jd}^{m_d}. \tag{A4}$$

Here η is the spin function for a nucleon. The channel quantum numbers d and p now include the additional specification of j_d and j_p, respectively. The source term for the (d, p) reaction becomes

$$\rho_p{}^{d\pi I}(r)=D_0\sum_{d'l_nj_n}\beta_{l_nj_n}(J_{d'},J_p)(-)^{l_{d'}+J_p+j_p+I}\left(\frac{\hat{J}_p\hat{j}_{d'}\hat{l}_p\hat{l}_n}{4\pi}\right)^{1/2}\begin{pmatrix} l_p & l_n & l_{d'}\\ 0 & 0 & 0\end{pmatrix}\begin{Bmatrix} j_p & j_n & j_{d'}\\ J_{d'} & I & J_p\end{Bmatrix}\begin{bmatrix} l_p & \frac{1}{2} & j_p\\ l_n & \frac{1}{2} & j_n\\ l_{d'} & 1 & j_{d'}\end{bmatrix}\chi_{l_nj_n}(r)u_{d'}{}^{d\pi I}(r), \tag{A5}$$

where

$$\begin{bmatrix} a & b & c\\ d & e & f\\ g & h & i\end{bmatrix}=(\hat{c}\hat{f}\hat{g}\hat{h})^{1/2}\begin{Bmatrix} a & b & c\\ d & e & f\\ g & h & i\end{Bmatrix} \tag{A6}$$

is a recoupling coefficient, with { } a 9-j symbol.

Corresponding to the incident wave,

$$\Psi_0=\Phi_{J_d}{}^{M_d}(A)\left[\eta(\boldsymbol{\sigma}_p)\eta(\boldsymbol{\sigma}_n)\right]_1^{\mu_d}\exp(ik_dZ), \tag{A7}$$

the amplitude of $u_{d'}{}^{d\pi I}$ in the incident wave is

$$A_{d\mu_dM_d}{}^{\pi 1M}=-(2ik_d)^{-1}C_{0\mu_d\mu_d}{}^{l_d1j_d}C_{\mu_dM_dM}{}^{j_dJ_dI}\times[4\pi(2l_d+1)]^{1/2}i^{l_d}\exp(i\sigma_d) \tag{A8}$$

Hence the amplitude for the transition is

$$f(J_dM_d\mu_d\to J_pM_p\mu_p)=(2ik_d)^{-1}\sum_{l_dj_dl_pj_pI}[4\pi(2l_d+1)]^{1/2}i^{l_d-l_p}\exp[i(\sigma_d+\sigma_p)]\times C_{0\mu_d\mu_d}{}^{l_d1j_d}C_{\mu_dM_d,M_d+\mu_d}{}^{j_dJ_dI}\sum_{m_pm}C_{m_p\mu_pm}{}^{l_p\frac{1}{2}j_p}C_{mM_p,M_d+\mu_d}{}^{j_pJ_pI}S_{p,d}{}^{\pi I}Y_{l_p}{}^{m_p}(\hat{\mathbf{k}}_p), \tag{A9}$$

so that the cross section is

$$d\sigma/d\Omega=[3(2J_d+1)]^{-1}\sum_{M_d\mu_dM_p\mu_p}|f|^2. \tag{A10}$$

The source term for the inverse reaction can be written down immediately from (A5) by analogy to the relationship between (3.47) and (3.52). The corresponding amplitude can be obtained from (A9) by everywhere interchanging the subscripts d and p, and interchanging 1 and $\frac{1}{2}$ in the two Clebsch-Gordan coefficients.