Velocity-Dependent Potentials in the Heisenberg Picture*

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Velocity-dependent potentials are investigated in both the Langrangian and Hamiltonian formalisms. A canonical transformation is introduced so that a consistent formulation is achieved. It is found that the proper Hamiltonian operator should be $H=K+\frac{1}{16}f^{-1}(r)[f'(r)]^2$ rather than the customarily used form of $K=\frac{1}{2}\mathbf{p}\cdot f(r)\mathbf{p}+V(r)$.

I. INTRODUCTION

In the last few years there have been many studies and applications of velocity-dependent potentials in nuclear physics.[1] The motivation for introducing such potentials originally stems from the desirability of replacing the hard core in nucleon-nucleon interactions. Velocity-dependent potentials can arise from the Taylor expansion of a nonlocal potential or from nonstatic effects. The two most commonly used forms are $\mathbf{p}\cdot f(r)\mathbf{p}$ and $\mathbf{p}^2g(r)+g(r)\mathbf{p}^2$. The functions $f(r)$ and $g(r)$ are usually taken for convenience to be square-well, exponential, or Gaussian. Since these two types of velocity-dependent potential are essentially the same, because of $\mathbf{p}^2g+g\mathbf{p}^2=2\mathbf{p}\cdot g\mathbf{p}-(2/r)g'-g''$, we will only discuss the type $\mathbf{p}\cdot f(r)\mathbf{p}$.

Recently, Razavy[2] discussed a fundamental problem concerning velocity-dependent potentials. Specifically, by means of Hamilton's canonical equations he studied the relation between the Hamiltonian and the energy of the system. However, his argument involves some misleading statements. In the present paper, we pursue this subject further and point out that the Hamiltonian obtained by means of the canonical method from a Lagrangian for a velocity-dependent potential does not satisfy the canonical equation of motion. In Sec. II, we propose a consistent method to get the canonical equation of motion.

In quantum mechanics, the velocity-dependent potentials mentioned above present a new problem which arises from noncommutativity of physical quantities. Let us consider the following Lagrangian in the Heisenberg picture:

$$\mathcal{L}(r,\dot{r})=\tfrac{1}{2}\dot{\mathbf{r}}f^{-1}(r)\dot{\mathbf{r}}-V(r), \tag{1}$$

which satisfies various invariance requirements, and where $f^{-1}(r)$ and $V(r)$ are functions of $|\mathbf{r}|$ only.[3] The canonical momentum for $\mathbf{r}$ would be defined as

$$\mathbf{p}=\partial\mathcal{L}/\partial\dot{\mathbf{r}}=\tfrac{1}{2}\{\dot{\mathbf{r}}f^{-1}(r)+f^{-1}(r)\dot{\mathbf{r}}\}, \tag{2}$$

where $\partial\dot{r}$ can be regarded as a c number, so the differentiation in (2) is done in the usual manner. The fundamental commutation relation is

$$[p_i, r_j]=-i\delta_{ij}. \tag{3}$$

From this equation we have

$$[\delta p_i, r_j]+[p_i,\delta r_j]=0.$$

Thus we can regard δp_i and δr_i as c numbers, provided that $\mathbf{p}$ and $\mathbf{r}$ are independent variables. From (2) and (3) the commutator

$$[\dot{r}_i, r_j]=-if(r)\delta_{ij} \tag{4}$$

is obtained.

With the help of (4), $\dot{\mathbf{r}}$ can be expressed in terms of $\mathbf{p}$ and $\mathbf{r}$:

$$\dot{\mathbf{r}}=\tfrac{1}{2}\{\mathbf{p}f(r)+f(r)\mathbf{p}\} \tag{5}$$

and the Hamiltonian corresponding to the Lagrangian (1) is given by

$$\begin{aligned}K(r,p)&=\tfrac{1}{2}(\dot{\mathbf{r}}\mathbf{p}+\mathbf{p}\dot{\mathbf{r}})-\mathcal{L}\\&=\tfrac{1}{2}\mathbf{p}\cdot f(r)\mathbf{p}-\tfrac{1}{8}f^{-1}(r)[f'(r)]^2+V(r),\end{aligned} \tag{6}$$

where f' means the derivative of f with respect to r.

On the other hand, when we derive the Euler-Lagrange equation from the Lagrangian (1) or the Hamilton canonical equation from the Hamiltonian (6), by means of the variational principle, $\delta\dot{\mathbf{r}}$ is no longer a c number. In the Lagrangian formalism, although $\delta\dot{\mathbf{r}}$ is induced by a virtual displacement $\delta\mathbf{r}$ for a fixed time, $\delta\dot{\mathbf{r}}$ is not necessarily a c number for the velocity-dependent potential.[4] Also, in the Hamiltonian formalism,

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[1] See, for example, R. E. Peierls, in *Proceedings of the International Conference on Nuclear Structure, Kingston, Canada, 1960*, edited by D. A. Bromley and E. W. Vogt (The University of Toronto Press, Toronto, Canada, 1960), p. 12; J. S. Levinger and O. Rojo, Phys. Rev. **123**, 2177 (1961); M. Razavy, G. Field, and J. S. Levinger, *ibid.* **125**, 269 (1962); A. M. Green, Nucl. Phys. **33**, 218 (1962); R. C. Herndon, E. W. Schmid, and Y. C. Tang, *ibid.* **42**, 113 (1963); R. K. Bhaduri and M. A. Preston, Can. J. Phys. **42**, 696 (1964); A. E. S. Green and R. D. Sharma, Phys. Rev. Letters **14**, 380 (1965); G. Darewych and A. E. S. Green, Phys. Rev. **164**, 1324 (1967); J. E. Brolley, *ibid.* **171**, 1439 (1968); E. M. Ferreina, N. Guillén, and J. Sesma, J. Math. Phys. **9**, 1210 (1968).

[2] M. Razavy, Phys. Rev. **171**, 1201 (1968).

[3] The form of $f(r)$ is not specified, except that we require it to approach a constant as r becomes greater than the force range and to have derivatives up to third order.

[4] In the definition (2) for the canonical momentum, $\partial\dot{r}$ is not the variation induced by δr. Since the equation of motion is of second order, we have two freedoms for the initial condition r_0 and $\dot{r}_0$. The proper interpretation is that $\partial\dot{r}$ in (2) corresponds to the freedom for $\dot{r}_0$ and is different from $\partial\dot{r}$ in the variational principle in the derivation of equation of motion.

the variation is taken with $\mathbf{p}$ and $\mathbf{r}$ as the independent variables:

$$\delta I=\int_{t_1}^{t_2}[\tfrac{1}{2}(\delta\dot{\mathbf{r}}\mathbf{p}+\dot{\mathbf{r}}\delta\mathbf{p}+\delta\mathbf{p}\dot{\mathbf{r}}+\mathbf{p}\delta\dot{\mathbf{r}}) -(\partial K/\partial\mathbf{r})\delta\mathbf{r}-(\partial K/\partial\mathbf{p})\delta\mathbf{p}]dt=0. \quad (7)$$

From (5), the variation $\delta\dot{\mathbf{r}}$ induced by $\delta\mathbf{r}$ and $\delta\mathbf{p}$ is obtained, and we can easily see by using (3) that $\delta\dot{\mathbf{r}}$ is not commutable with $\mathbf{r}$ and $\mathbf{p}$ except when f is a constant. Therefore, in (7) we cannot put $\delta\dot{\mathbf{r}}=(d/dt)\delta\mathbf{r}$ because $\delta\mathbf{r}$ is a c number. Thus the usual canonical equations of motion

$$\dot{r}_i=\partial K/\partial p_i \quad\text{and}\quad \dot{p}_i=-\partial K/\partial r_i \quad (8)$$

are inconsistent with the canonical commutation relation.[5] Indeed, if we treat $\delta\dot{\mathbf{r}}$ as a c number and put $\delta\dot{\mathbf{r}}=(d/dt)\delta\mathbf{r}$, the resulting equation of motion in the Hamiltonian formalism is in general different from that obtained in the Lagrangian formalism. As will be seen, both equations of motion are also different from the equation derived by the consistent method outlined in Sec. II.

II. CONSISTENT FORMULATION

The main reason of the inconsistency mentioned in Sec. I lies in the fact that the canonical momentum $\mathbf{p}$ is not proportional to $\dot{\mathbf{r}}$ and hence the commutator $[\dot{\mathbf{r}}, r]$ is not a c number. Therefore, to avoid this let us introduce the canonical transformation below. By a canonical transformation we mean one which preserves the fundamental commutation relation (3). For the sake of simplicity, we will discuss the s-wave scattering, so the vectors $\mathbf{r}$ and $\mathbf{p}$ are replaced by the scalars r and p in all the previous expressions and $V(r)$ by $v(r)\equiv V(r)+f'/r$.

The canonical transformation $(r, p)\rightarrow(R, \pi)$ is given by the generating function

$$W(R, p)=-\tfrac{1}{2}\{g(R)p+pg(R)\}, \quad (9)$$

and

$$r=-\partial W/\partial p=g(R),$$
$$\pi=-\partial W/\partial R=\tfrac{1}{2}\{g'(R)p+pg'(R)\}, \quad (10)$$

where

$$g'(R)=dr/dR=f^{1/2}(r) \quad (11)$$

and $g'(R)$ is the derivative of $g(R)$ with respect to R. Equivalently, we can write

$$R=\int^r f^{-1/2}(s)\,ds,$$
$$\pi=\tfrac{1}{2}\{pf^{1/2}(r)+f^{1/2}(r)p\}. \quad (12)$$

[5] Razavy assumed that these canonical equations are still valid. However, with these equations, the Hamiltonian $K(r, p)$ becomes the generator of time development for the system: $\dot{r}=-i[r, K]$, $\dot{p}=-i[p, K]$, and hence $dF(r, p, t)/dt=\partial F/\partial t-i[F, K]$.

The new Lagrangian is

$$\mathfrak{L}(R, \dot{R})=\tfrac{1}{2}\dot{R}^2-\tfrac{1}{4}i[\dot{R}, g''(R)(g'(R))^{-1}] +\tfrac{1}{32}f'^2f^{-1}-v(r), \quad (13)$$

where

$$\dot{R}=\tfrac{1}{2}\{\dot{r}f^{-1/2}(r)+f^{-1/2}(r)\dot{r}\}. \quad (14)$$

The canonical momentum π for R has the desired form

$$\pi=\partial\mathfrak{L}/\partial\dot{R}=\dot{R} \quad (15)$$

and is consistent with (12). The commutation relation retains the form invariance

$$[\pi, R]=[\dot{R}, R]=-i. \quad (16)$$

Thus we have the new Hamiltonian

$$H(R, \pi)=\tfrac{1}{2}(\pi\dot{R}+\dot{R}\pi)-\mathfrak{L}(R, \dot{R})$$
$$=\tfrac{1}{2}\pi^2+\tfrac{1}{4}(\ln g'(R))''-\tfrac{1}{8}[(\ln g'(R))']^2+v(R). \quad (17)$$

Although the Hamiltonian $H(R, \pi)$ satisfies the condition for canonical transformation

$$-\tfrac{1}{2}(\dot{p}r+r\dot{p})-K(r, p)=\tfrac{1}{2}(\pi\dot{R}+\dot{R}\pi)-H(R, \pi) +dw(R, p)/dt, \quad (18)$$

$H(R, \pi)$ is not numerically equal to $K(r, p)$, despite the fact that $W(R, p)$ does not involve time explicitly. This fact is due to the noncommutativity of $\dot{R}$ with $g'(R)=f^{1/2}(r)$ and p in the expression

$$dW(R, p)/dt=-\tfrac{1}{2}\{g(R)\dot{p}+\dot{p}g(R)\} -\tfrac{1}{4}\{\dot{R}g'(R)+g'(R)\dot{R}\}p-\tfrac{1}{4}p\{\dot{R}g'(R)+g'(R)\dot{R}\}.$$

Using (10)–(12) and (16), this becomes

$$dW(R, p)/dt=-\tfrac{1}{2}(r\dot{p}+\dot{p}r)-\tfrac{1}{2}(\dot{R}\pi+\pi\dot{R})+\tfrac{1}{16}f^{-1}(f')^2.$$

Therefore we have

$$H(R, \pi)=K(r, p)+\tfrac{1}{16}f^{-1}(r)[f'(r)]^2. \quad (19)$$

Equation (19) is confirmed by the direct calculation of transforming R and π in (17) into r and p. From (19) we know that there is no unitary transformation corresonding to this canonical transformation.

According to the argument in Sec. I, we can regard $\delta\dot{R}$ as a c number owing to the commutation relation (16) and we can put $\delta\dot{R}=(d/dt)\delta R$. Hence the usual equations of motion are derived for R and π:

$$\frac{\partial\mathfrak{L}(R, \pi)}{\partial R}-\frac{d}{dt}\frac{\partial\mathfrak{L}(R, \pi)}{\partial\dot{R}}=0 \quad (20)$$

or

$$\dot{R}=\partial H(R, \pi)/\partial\pi$$

and

$$\dot{\pi}=-\partial H(R, \pi)/\partial R. \quad (21)$$

Equations (20) and (21) give the same equation of motion, which is different from the one derived from

(8). Transforming R, π back into r, p, Eq. (21) becomes

$$\dot{r}=\partial H(r,p)/\partial p=\partial K(r,p)/\partial p, \tag{22a}$$

$$\dot{p}=-\partial H(r,p)/\partial r\neq-\partial K(r,p)/\partial r. \tag{22b}$$

Equation (22a) is in accord with (2) and (5).

From (22), $H(r, p)$ is the proper Hamiltonian; that is, it is the generator of time for the system:

$$dF(r,p,t)/dt=(\partial F/\partial t)-i[F,H], \tag{23}$$

where $F(r, p, t)$ is an arbitrary function of r, p, and t. But $K(r, p)$ is not the proper Hamiltonian. Obviously, $H(r, p)$ is also constant in time:

$$dH/dt=0, \tag{24}$$

but $K(r, p)$ is not. Really, H is the first integral of the equation of motion, with the integrating factor $\frac{1}{2}p$:

$$dH/dt=\tfrac{1}{2}p[\ddot{r}-\tfrac{1}{2}f'\dot{r}f^{-1}\dot{r}+\tfrac{1}{2}if''r+fv' +\tfrac{1}{8}(2ff'''-3f'f''+\tfrac{3}{2}f^{-1}f'^3)]+\text{H.c.}, \tag{25}$$

where H.c. denotes the Hermitian conjugate. In view of (23)–(25), it is clear that the operator H is the proper Hamiltonian, the eigenvalue of which is the energy of the system, and the system is conservative.

III. DISCUSSION

Contrary to the case in classical mechanics, the system with the velocity-dependent potential $\frac{1}{2}pf(r)p$ is dissipative in the quantum-mechanical case. The operator $K(r, p)$ does not represent the energy of the system and is not a constant of motion. When we employ such a velocity-dependent potential, the associated term $\frac{1}{16}f^{-1}(f')^2$ should be added to the Hamiltonian. The resulting Hamiltonian $H(r, p)$ thus obtained should be used in the Schrödinger equation for a stationary state. Only in special cases, for example, for

$$f(r)=\text{const} \quad \text{or} \quad f(r)=cr^2, \tag{26}$$

does the above extra term have no contribution.

It should be emphasized that the above discussion does not invalidate the work cited in Ref. 1 or open it to question. This is because in most of these applications the potential parameters are determined phenomenologically, and therefore one may consider, at least for the s wave, the omission of the associated terms [like $\frac{1}{16}f^{-1}(f')^2$ in our example] as amounting to a redesignation of the meaning of $V(r)$.

When we interpret $H(r, p)$ as (kinetic energy)+(potential energy), the Lagrangian $\mathcal{L}(r, \dot{r})$ is not of the form (KE)−(PE), and $\frac{1}{2}\dot{r}f^{-1}\dot{r}$ shows up as the effective kinetic energy. However, this fact is not peculiar, because we have already a similar example in relativistic mechanics.

Although the canonical transformation (9)–(12) keeps the form of the canonical commutation relation and the generating function does not contain time explicitly, the Hamiltonian is changed and hence there is no corresponding unitary transformation. This is the new aspect for this velocity-dependent potential in quantm meuchanics.

After completing this work, we became aware of the paper by Fujiwara[6] in which he got an extra term similar to ours in the quantum-mechanical Hamiltonian, using a modified Feynman path integral method. The numerical coefficient of the extra term is different from that of our extra term, but this fact is due to the ambiguity in making his classical Lagrangian Hermitian.

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[6] I. Fujiwara, Progr. Theoret. Phys. (Kyoto) **21**, 902 (1959).