Analysis of a Method of Solution of the Problem of Two-Body Interaction in a Common Harmonic-Oscillator Potential with Hard Core*

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In an earlier paper, a method was given for solving the problem of two particles in a common harmonic-oscillator potential modified by an effective two-body potential which includes hard-core, central, spin-orbit, and tensor parts. Using the appropriate Green's function, the wave function was expressible in terms of the harmonic-oscillator wave functions, and energy eigenvalues were obtained. In this paper, the accuracy and convergence of the method is tested by applying it to an exactly solvable (without hard core) case and a partially solvable (with hard core) case, with the intention of applying the method to realistic two-body potentials and calculation of nuclear wave functions and form factors.

1. INTRODUCTION

THE nuclear shell model is based on the concept that the individual nucleons in the nucleus move independently in a common potential. The nuclear many-body theory developed by Brueckner[1] and by Bethe[2] is aimed to provide a justification of the assumption of the shell theory by attempting to derive a self-consistent potential from the two-nucleon potential. Because of the difficulty involved in calculating the self-consistent potential and the many approximations, several authors[3] have assumed the common potential to be the harmonic-oscillator potential. The choice of the harmonic-oscillator potential has been largely made because, as shown by Talmi,[4] in this case the two-nucleon wave function is separable in the relative and center-of-mass (c.m.) coordinates. Thus if the two-body interaction depends only on the relative coordinates, the c.m. motion is factored out.

Since most "realistic" two-nucleon phenomenological potentials[5] include a repulsive hard core (of infinite strength), the usual perturbation theory is not applicable. Bauer and Moshinsky[6] have treated the problem of the two nucleons moving in a common harmonic-oscillator potential plus a hard core by transforming the radial equation for the problem by a simple translation such that the repulsive core disappears, and an interaction potential appears, which can be treated by standard perturbation methods. Nigam[7] has generalized this procedure so that the two-nucleon interaction includes hard-core, central, spin-orbit, and tensor parts. Using the appropriate Green's function, the relevant integral equation is solved to obtain the ground- and excited-state wave functions in terms of the harmonic-oscillator wave functions, and also the corresponding energy eigenvalues. The method thus, provides analytic expressions for the wave function besides giving the energy eigenvalues.

The importance of knowing the wave function with accuracy has increased in recent years because of the precise electron scattering experiments[8] which determine the form factors very accurately. Since the Fourier transform of the form factor gives the nuclear charge distribution, which is also derivable from the wave function, a more sensitive comparison of the wave function with the experimental results can be made than has been available through the energy eigenvalues (binding energies).

The method of Ref. 7 obtains the wave function in terms of harmonic-oscillator wave functions of the zero and higher orders. Recent experiments[9] on the determination of the form factors indicate that the α-particle wave function contains contributions from higher-order harmonic oscillators also. It is therefore expected that the method of solution presented in this paper will be appropriate to apply to such problems. In this paper, however, a practical application will not be considered. We attempt to test and establish the accuracy and convergence of the method by applying it to an exactly solvable case (without hard core) and a partially solvable case (with hard core) for varying strengths of the two-body (harmonic-oscillator) interaction. Application to realistic cases will be reported subsequently.

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2. RESULTS OF NIGAM

The Hamiltonian[7] for the two-particle system in a common harmonic-oscillator potential of frequency ω is given by

$$H_0^{12}=(1/2m)\,p_1^2+(1/2m)\,p_2^2+\tfrac{1}{2}m\omega^2r_1^2+\tfrac{1}{2}m\omega^2r_2^2. \quad (1)$$

Separating the c.m. and relative coordinates, the Schrödinger equation for relative motion is

$$H_0\phi(\mathbf{r})=[-(1/2\mu)\nabla^2+kr^2]\phi(\mathbf{r})=E_0\phi(\mathbf{r}), \quad (2a)$$

where

$$\mu=\tfrac{1}{2}m, \qquad k=\tfrac{1}{2}\mu\omega^2. \quad (2b)$$

The solutions of Eq. (2a) are given by[4,7]

$$\phi_{nl}{}^m(\mathbf{r})=(1/r)R_{nl}(r)\,Y_l{}^m(\hat{r}), \qquad \hat{r}\equiv(\theta,\phi) \quad (3a)$$

where

$$R_{nl}(r)=N_{nl}\,r^{l+1}v_{nl}(r)\,\exp(-\tfrac{1}{2}\nu r^2), \quad (3b)$$

$$\int_0^\infty dr\,R_{nl}(r)R_{n'l}(r)=\delta_{nn'}, \quad (3c)$$

$$N_{nl}{}^2=\frac{2^{l-n+2}(2l+2n+1)\,!!}{\pi^{1/2}n![(2l+1)\,!!]^2}\,\nu^{l+3/2}, \qquad \nu=\mu\omega/\hbar. \quad (3d)$$

The associated Laguerre polynomials $v_{nl}(r)$ are defined by

$$v_{nl}(r)=L_{n+l+1/2}{}^{l+1/2}(\nu r^2)=F(-n\mid l+\tfrac{3}{2}\mid \nu r^2)$$

$$=\sum_{k=0}^{n}(-1)^k2^k\binom{n}{k}\frac{(2l+1)\,!!}{(2l+2k+1)\,!!}(\nu r^2)^k. \quad (3e)$$

The energy eigenvalue corresponding to the eigenfunction $\phi_{nl}{}^m(\mathbf{r})$ is

$$E_0{}^{nl}=(2n+l+\tfrac{3}{2})\hbar\omega, \qquad n=0,1,2,\cdots. \quad (4)$$

In addition to the common harmonic-oscillator potential, the two nucleons interact with an effective two-nucleon potential $v(\mathbf{r})$ which, in general, includes central, spin-orbit, and tensor parts. Including $v(\mathbf{r})$ in Eq. (2), carrying out a partial-wave analysis of the resulting Schrödinger equation, the radial part of the perturbed wave function $u_{nll'}{}^{JS}(r)$ [analog of R_{nl} in Eq. (3)] satisfies the following differential equation[7]:

$$\{E_{nll'}{}^{JS}-[-m^{-1}(d^2/dr^2)+V_{l'}(r)]\}u_{nll'}{}^{JS}(r)$$

$$=\sum_{l''}v_{l'l''}{}^{JS}(r)u_{nll''}{}^{JS}(r), \quad (5a)$$

where

$$V_l(r)=l(l+1)/mr^2+kr^2, \quad (5b)$$

$$v_{l'l''}{}^{JS}(r)=\int d\hat{r}\,\mathcal{Y}_{l'SJ}{}^{M*}(\hat{r})v(\mathbf{r})\mathcal{Y}_{l''SJ}{}^{M}(\hat{r}). \quad (5c)$$

In arriving at Eq. (5) we have used the fact that the total angular momentum J ($\mathbf{J}=\mathbf{l}+\mathbf{S}$) and the total spin S ($S=0$ and 1 for the two-nucleon system) are constants of motion. The second l subscript to u and v is called for because of the complication arising due to the tensor force which, in general, mixes the l values. Thus, for $J=l$, $l'=l$, and for $J=l\pm1$, l' takes the two values $l'=l$ and $l\pm2$. The $\mathcal{Y}_{lSJ}{}^M(\hat{r})=|\,lSJM\rangle$ are the eigenstates[10] corresponding to orbital angular momentum l and total angular momentum J.

The solution of Eq. (5a) can be written using the Green's function $g_{nll'}{}^{JS}(r,r')$ for the differential operator on the left-hand side. We have

$$u_{nll'}{}^{JS}(r)$$

$$=\sum_{l''}\int dr'\,g_{nll'}{}^{JS}(r,r')v_{l'l''}{}^{JS}(r')u_{nll''}{}^{JS}(r'), \quad (6)$$

where

$$g_{nll'}{}^{JS}(r,r')=\sum_{n_1=0}^{\infty}\frac{R_{n_1l'}(r)R_{n_1l'}(r')}{\mathcal{E}_{nll',n_1l'}{}^{JS}}, \quad (7a)$$

$$\mathcal{E}_{nll',n_1l_1}{}^{JS}=E_{nll'}{}^{JS}-E_0{}^{n_1l_1}. \quad (7b)$$

Most realistic two-nucleon potentials[5] include a hard repulsive core, that is, $v(r)=+\infty$ for $0\le r<r_c$, and $v(r)$ for $r_c<r<\infty$. The radial function $u_{nll'}{}^{JS}(r)$ then satisfies the boundary condition $u_{nll'}{}^{JS}(r_c)=0$. The results for this case can be derived from Eq. (5a) by making the transformation $r\to r+r_c$. The integral equation satisfied by the radial function, in this case, is given by

$$u_{nll'}{}^{JS}(r+r_c)=\sum_{l''}\int_0^\infty dr'\,g_{nll'}{}^{JS}(r,r')$$

$$\times[V_{l'}(r',r_c)\delta_{l'l''}+v_{l'l''}{}^{JS}(r'+r_c)]u_{nll''}{}^{JS}(r'+r_c), \quad (8)$$

where

$$V_l(r,r_c)=V_l(r+r_c)-V_l(r). \quad (9)$$

The solution of the integral equation (8) is

$$u_{nll'}{}^{JS}(r+r_c)=\sum_{n_1=0}^{\infty}K_{nll',n_1l'}{}^{JS}R_{n_1l'}(r), \quad (10)$$

where the $K_{nll',n_1l'}{}^{JS}$ satisfy the following infinite set of simultaneous homogeneous equations:

$$\sum_{n_2=0}^{\infty}\sum_{l''}(a_{n_1n_2l'l''}+b_{n_1n_2l'l''}{}^{JS})K_{nll',n_2l''}{}^{JS}$$

$$=\mathcal{E}_{nll',n_1l'}{}^{JS}K_{nll',n_1l'}{}^{JS}, \quad (11)$$

[10] J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley & Sons, Inc., New York, 1952); R. R. Roy and B. P. Nigam, *Nuclear Physics* (John Wiley & Sons, Inc., New York, 1968), p. 553.

TABLE I. Comparison of eigenvalues for the potential $v(r)=\lambda r^2$ (no hard core), using $N\times N$ matrix, with their exact values ($\hbar=\mu=\omega=1$, $k=\frac{1}{2}$, $\nu=1$).

N	E^0	E^1	E^2	E^3	E^4	E^9	E^{19}
			$\lambda=0.1$				
3	1.6431678	3.834301	6.072531				
5	1.6431677	3.834058	6.024956	8.218	10.5		
Exact	1.6431677	3.834058	6.024948	8.216	10.4		
			$\lambda=0.5$				
3	2.121763	5.01497	8.626741				
5	2.121321	4.950072	7.799874	10.94740	15.43129		
10	2.121320	4.949747	7.778175	10.606608	13.43545	33.3	
Exact	2.121320	4.949747	7.778175	10.606602	13.43503	27.6	
			$\lambda=3.0$				
5	3.984725	9.752432	17.706	29.793	48.763		
10	3.968643	9.262696	14.629	20.535	27.866	109.96	
20	3.968627	9.260130	14.55163	19.84319	25.13646	56.92	238.8
Exact	3.968627	9.260130	14.55163	19.84314	25.13464	51.59	104.5
			$\lambda=4.0$				
10	4.500129	10.513766	16.76907	24.16678	33.73	140.7	
20	4.500000	10.500000	16.50002	22.50110	28.522	69.1	306.2
Exact	4.500000	10.500000	16.500000	22.50000	28.500	58.5	118.5

with

$$a_{n_1n_2l'l''}=(V_{l'}(r,r_c))_{n_1l',n_2l''}\delta_{l'l''}$$
$$=\delta_{l'l''}\int_0^\infty dr\, R_{n_1l'}(r)V_{l'}(r,r_c)R_{n_2l''}(r), \quad (12)$$

$$b_{n_1n_2l'l''}{}^{JS}=(v_{l'l''}{}^{JS}(r+r_c))_{n_1l',n_2l''}$$
$$=\int_0^\infty dr\, R_{n_1l'}(r)v_{l'l''}{}^{JS}(r+r_c)R_{n_2l''}(r). \quad (13)$$

Equation (11) can be solved as an eigenvalue equation to determine the eigenvalues $\mathcal{E}_{nll',n_1l'}{}^{JS}$ and the eigenvectors $K_{nll',n_1l'}{}^{JS}$ and hence the perturbed radial solution $u_{nll'}{}^{JS}(r+r_c)$, using Eq. (10).

3. APPLICATION OF THE METHOD TO TWO CASES

In order to test the accuracy and convergence of the method outlined in Sec. 2, we consider first an example which is also exactly solvable. We restrict consideration to a purely central potential and confine our attention to the two particles in a relative angular momentum state $l=0$ ($J=l$). Rewriting Eqs. (10)–(13), after dropping the indices l, J, and S, we have

$$u_n(r+r_c)=\sum_{n_1=0}^{\infty}K_{n_1}{}^nR_{n_1}(r), \quad (14)$$

with

$$\sum_{n_2=0}^{\infty}(a_{n_1n_2}+b_{n_1n_2})K_{n_2}{}^n=\mathcal{E}_{n_1}{}^nK_{n_1}{}^n=(E^n-E_0{}^{n_1})K_{n_1}{}^n, \quad (15)$$

$$a_{n_1n_2}=kr_c{}^2\delta_{n_1n_2}+2kr_c(r)_{n_1n_2}, \quad (16a)$$

$$b_{n_1n_2}=(v(r+r_c))_{n_1n_2}. \quad (16b)$$

The harmonic-oscillator matrix elements that we will need are[7,11]

$$(r)_{n_1n_2}=-(1/2\sqrt{\nu})N_{n_1}N_{n_2}/[4(n_1-n_2)^2-1],$$
$$N_n=2^{-n+2}(2n+1)!!/(\sqrt{\pi})n!, \quad (17a)$$

$$(r^2)_{n_1n_2}=(1/\nu)\{(2n_1+\tfrac{3}{2})\delta_{n_1n_2}-[(n_1+\tfrac{3}{2})(n_1+1)]^{1/2}$$
$$\times\delta_{n_1,n_2-1}-[n_1(n_1+\tfrac{1}{2})]^{1/2}\delta_{n_1,n_2+1}\}. \quad (17b)$$

The examples of potentials for which we carry out calculations are the following:

A. No hard core: (i) $v(r)=\lambda r^2$, (ii) $v(r)=0$, but replace k by $k'=k(1+\lambda/k)$, so that $\omega\to\omega'=\omega(1+\lambda/k)^{1/2}$ and $\nu\to\nu'=\nu(1+\lambda/k)^{1/2}$.

B. Hard core of radius r_c: (i) $v(r)=\lambda r^2$ for $r>r_c$, (ii) $v(r)=0$ for $r>r_c$, but replace k by $k'=k(1+\lambda/k)$, $\omega\to\omega'=\omega(1+\lambda/k)^{1/2}$ and $\nu\to\nu'=\nu(1+\lambda/k)^{1/2}$.

[11] W. H. Shaffer, Rev. Mod. Phys. **16**, 245 (1944).

TABLE II. Comparison of eigenvalues of the potential (i) $v(r)=\infty$ for $0<r<r_c$, $v(r)=\lambda r^2$ for $r>r_c$, elastic constant k, with (ii) $v(r)=\infty$ for $0<r<r_c$, $v(r)=0$ for $r>r_c$, elastic constant $k'=k(1+\lambda/k)$, using $N\times N$ matrix. [$\hbar=\mu=\omega=1$, $k=\frac{1}{2}$, $\nu=1$; the two values of $r_c/(\hbar/\mu\omega)^{1/2}$ chosen correspond to $r_c\approx 0.4\times 10^{-13}$ cm for ω in the region of light nuclei.] The values against $N=(10)$ and (20) are for (i) and those against $N=10$ and 20 are for (ii).

N	E^0	E^1	E^2	E^3	E^4	E^9	E^{19}
			$\lambda=0.1, r_c=0.15$				
(10)	2.073753	4.77616	7.4880	10.2021	12.918	29.0	
(20)	2.0737460	4.77613	7.487982	10.2018	12.916	26.5	60.4
10	2.075456	4.78427	7.50127	10.22020	12.9401	27.7	
20	2.075452	4.78426	7.50122	10.22004	12.9396	26.5	57.0
			$\lambda=0.1, r_c=0.20$				
(10)	2.21706	5.08016	7.9565	10.8360	13.719	31.4	
(20)	2.21705	5.08009	7.9563	10.8351	13.715	28.1	65.7
10	2.218661	5.08945	7.97169	10.8566	13.743	29.9	
20	2.218652	5.08941	7.97156	10.8562	13.742	28.2	62.0
			$\lambda=0.5, r_c=0.15$				
(10)	2.756004	6.3150	9.8956	13.485	17.12	45.8	
(20)	2.755963	6.3148	9.8941	13.477	17.06	35.0	97.5
10	2.754937	6.33816	9.93363	13.532	17.1320	36.9	
20	2.754930	6.33813	9.93353	13.5317	17.1309	35.1	76.2
			$\lambda=0.5, r_c=0.20$				
(10)	2.96994	6.7598	10.580	14.419	18.332	50.0	
(20)	2.96986	6.7592	10.577	14.399	18.223	37.5	106.4
10	2.96515	6.78293	10.6179	14.457	18.299	40.3	
20	2.96513	6.78286	10.6177	14.456	18.296	37.5	83.6
			$\lambda=1.0, r_c=0.15$				
(10)	3.47654	7.92589	12.4227	16.9955	21.8682	67.3	
(20)	3.47635	7.92408	12.4085	16.8980	21.3902	44.6	144.3
10	3.45643	7.93785	12.4362	16.9385	21.4432	46.5	
20	3.45642	7.93780	12.4360	16.9379	21.4413	44.0	96.2
			$\lambda=1.0, r_c=0.20$				
(10)	3.77510	8.53742	13.3714	18.3296	23.694	73.6	
(20)	3.77469	8.53350	13.3428	18.1592	22.980	48.3	157.7
10	3.74166	8.53754	13.3572	18.1828	23.013	51.1	
20	3.74164	8.53742	13.3568	18.1814	23.008	47.2	106.2
			$\lambda=2.0, r_c=0.15$				
(10)	4.70400	10.65264	16.8031	23.4916	21.3495	110.59	
(20)	4.70245	10.6322	16.6359	22.6523	28.6941	64.0	238.4
10	4.61255	10.5655	16.5438	22.5282	28.5168	62.5	
20	4.61253	10.5654	16.5435	22.5272	28.5129	58.5	129.5
			$\lambda=2.0, r_c=0.20$				
(10)	5.16671	11.5922	18.3154	25.728	34.466	120.9	
(20)	5.16358	11.5541	18.0467	24.562	31.128	70.4	260.7
10	5.03187	11.4394	17.8828	24.335	30.797	69.3	
20	5.03182	11.4391	17.882	24.332	30.785	63.1	144.2
			$\lambda=3.0, r_c=0.15$				
(10)	5.78137	13.0594	20.8758	29.9227	40.971	153.7	
(20)	5.77541	12.9757	20.2949	27.6586	35.161	83.9	332.4
10	5.58799	12.7750	19.9953	27.2235	34.458	76.0	
20	5.58796	12.7749	19.9948	27.2218	34.452	70.6	157.8
			$\lambda=3.0, r_c=0.20$				
(10)	6.40614	14.3366	22.9952	33.094	45.37	168.3	
(20)	6.39494	14.1978	22.1689	30.222	38.50	92.9	363.7
10	6.12878	13.8948	21.7081	29.534	37.37	84.8	
20	6.12871	13.8945	21.7069	29.529	37.35	76.5	176.7

Examples (i) and (ii), in each case, correspond to the same situations. Case A(i) can be solved by using the method of Eqs. (14)–(17) with $r_c=0$. However, it also corresponds to a harmonic oscillator with the elastic constant $k'=k(1+\lambda/k)$, as indicated in case A(ii). A comparison of the solution of A(i), using Eqs. (14)–(17), with its exact solution A(ii) will provide a measure of the accuracy and convergence of our method. Similarly, the solutions of B(i) can be compared with the solutions of B(ii).

The exact solution of the Hamiltonian $H_0'=p^2/2\mu+k'r^2$, case A(ii), is given by (with $l=0$)

$$(R_{nl}(r))'=N_{nl}(1+\lambda/k)^{l/4+3/8}r^{l+1}v_{nl}[\nu(1+\lambda/k)^{1/2}r^2] \times\exp[-\tfrac{1}{2}\nu(1+\lambda/k)^{1/2}r^2], \quad (18)$$

$$(E_0^{nl})'=(2n+l+\tfrac{3}{2})(1+\lambda/k)^{1/2}\hbar\omega, \quad (19)$$

$$\int dr\,(R_{nl}(r))'(R_{n'l}(r))'=\delta_{nn'}. \quad (20)$$

Also, we note that

$$(r)_{n_1n_2}'=\int_0^\infty dr\,(R_{n_1})'r(R_{n_2})'=(1+\lambda/k)^{-1/4}(r)_{n_1n_2}. \quad (21)$$

The matrix elements involved in cases B are given as follows (for cases A, they are obtained by putting $r_c=0$):

Case B(i):

$$a_{n_1n_2}=\frac{r_c}{(\hbar/\mu\omega)^{1/2}}\left(1+\frac{\lambda}{k}\right)\times\left[\frac{1}{2}\left(\frac{r_c}{(\hbar/\mu\omega)^{1/2}}\right)\delta_{n_1n_2}+(\nu^{1/2}r)_{n_1n_2}\right]\hbar\omega, \quad (20')$$

$$b_{n_1n_2}=\tfrac{1}{2}(\lambda/k)(\nu r^2)_{n_1n_2}\hbar\omega, \quad (20'')$$

$$(E_0)_{n_1n_2}=E_0^{n_1}\delta_{n_1n_2}=(2n_1+\tfrac{3}{2})\hbar\omega\delta_{n_1n_2}. \quad (20''')$$

Case B(ii):

$$a_{n_1n_2}=\frac{1}{2}\left(\frac{r_c}{(\hbar/\mu\omega)^{1/2}}\right)^2\left(1+\frac{\lambda}{k}\right)\delta_{n_1n_2}\hbar\omega, \quad (21')$$

$$b_{n_1n_2}=\frac{r_c}{(\hbar/\mu\omega)^{1/2}}\left(1+\frac{\lambda}{k}\right)^{3/4}((\sqrt{\nu})r)_{n_1n_2}\hbar\omega, \quad (21'')$$

$$(E_0)_{n_1n_2}=(E_0^{n_1})'\delta_{n_1n_2}=(2n_1+\tfrac{3}{2})[1+(\lambda/k)]^{1/2}\delta_{n_1n_2}\hbar\omega, \quad (21''')$$

with the $R_{n_1}(r)$ in Eq. (14) now replaced by $(R_{n_1}(r))'$ in Eq. (18).

The solution of the problem now involves the diagonalization of the matrix $(a+b+E_0)_{n_1n_2}$, n_1, $n_2=0, 1, 2, \cdots$, whose eigenvalues give the perturbed energies E^n and whose eigenvectors give K_{n1}^n. In doing the numerical calculations we have chosen the units $\hbar=\mu=\omega=1$, so that $k=\frac{1}{2}$ and $\nu=1$. The calculations were done for several values of λ by diagonalizing[12] $N\times N$ matrices, where in order to test the convergence of method, for each value of λ, results for at least two values of N were obtained. The results of the calculations are given in Tables I and II.

In Table I, we have listed the energy eigenvalues E^n for $\lambda=0.1$, 0.5, 1.0, 2.0, 3.0, and 4.0, for case A(i) and also the exact values as obtained from case A(ii). It is clear from the table that the low levels are given to an extremely good accuracy even if the order N of the matrix diagonalized is small; for higher levels, the accuracy can be improved by going to larger values of N. The eigenfunctions $u_n(r)$ were also calculated and compared with their exact values over a range of $r/(\hbar/\mu\omega)^{1/2}$ from 0 to 3. For the higher value of N shown, the agreement of the wave function ranged from 1 to 10%, the best agreement appearing where the wave function is largest and worst in the asymptotic region where the wave function is very small. This is to be expected, since when the wave function is very small, each additional term in Eq. (14) affects the value of $u_n(r)$ by a higher percentage. However, the asymptotic tail of the wave function contributes negligibly compared to the rest. Thus, this in no way constitutes a defect of the method.

In Table II, we have given the eigenvalues for cases B(i) (indicated in the table by putting N, in the first column, within parentheses) and B(ii) (N without parentheses). Again the agreement is very good, becoming less good as λ increases. The wave functions were found to agree within 1 to 10%, as in case A.

4. CONCLUSION

The method, Eqs. (10)–(13), of solving the Schrödinger equation for two particles in a common harmonic-oscillator potential together with a two-particle interaction, which here has been also taken to be harmonic, is found to work with extremely good accuracy as seen by comparison with the exact solution. The order of the matrix required to accomplish the same accuracy increases with the strength of the two-particle interaction, but the method seems to converge fast and it was not necessary to diagonalize matrices of very large order. The method is equally well applicable when a repulsive hard core in the two-particle interaction is included. The application of the method to realistic two-nucleon interactions will be of interest and will be reported subsequently. The method is also applicable to solutions of two-body (and many-body) problems without a common harmonic-oscillator potential, if we are interested in choosing the harmonic-oscillator wave functions as a basis representative.

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[12] S. S. Kuo, *Numerical Methods and Computers* (Addison-Wesley Publishing Co., Reading, Mass., 1965), p. 192.