When $H=0$, we have in the first place

$$\Gamma_{1,N}(\mathbf{r})\equiv 0\,, \quad \text{all } \mathbf{r},\, N\,, \tag{A11}$$

and the argument then yields, in place of (A10), the result

$$\liminf_{N\to\infty} \hat{\chi}_N(T,0) \geq \sum_{\mathbf{r}} \Gamma_2(T,0;\,\mathbf{r})\,. \tag{A12}$$

To obtain a corresponding *upper* bound for this case we use the inequality (A7) in (A2) together with (A11) which yields

$$\hat{\chi}_N(T,0) \leq N^{-1} \sum_{\mathbf{r}_1,\,\mathbf{r}_2\subset\Omega_N}\sum \Gamma_2(\mathbf{r}_2-\mathbf{r}_1,\, T,\, 0)\,. \tag{A13}$$

Since Γ_2 is non-negative, we can extend the sum on $\mathbf{r}_2$ to all those points outside Ω_N (but in Ω_∞) satisfying $|\mathbf{r}_2-\mathbf{r}_1| \leq R_N$ where R_N is the diameter of Ω_N. Then the sum on $\mathbf{r}_1$ may be performed yielding

$$\hat{\chi}_N(T,0) \leq \sum_{|\mathbf{r}|\leq R_N} \Gamma_2(T,0,\mathbf{r}) \leq \sum_{\mathbf{r}} \Gamma_2(T,0,\mathbf{r})\,, \tag{A14}$$

where the last sum might diverge to $+\infty$. On allowing N to approach ∞ and combining with (A12), we obtain the desired result

$$\hat{\chi}_\infty(T,0) = \lim_{N\to\infty} \hat{\chi}_N(T,0) = \sum_{\mathbf{r}} \Gamma_2(T,0,\mathbf{r})\,, \tag{A15}$$

where the limit may take the value $+\infty$.

It is clear that a completely analogous proof will establish the lower bound corresponding to (A10) for the energy fluctuations and the specific heat when $H\neq 0$. The analysis to establish an upper bound for $H=0$ along the same lines fails because $U(T,0)\neq 0$.

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Ising Model: Field-Theoretic and Functional-Integral Aspects

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A functional-integral formulation of the Ising model is used as a link between the usual approach in terms of summation over spin variables and field-theory-like formulas. The latter take the form of Feynman-diagram expansions, Dyson integral equations, or Schwinger functional-derivative field equations. Basic to the theory is a spinless-nucleon-like Green's function, related to the Ising spin-spin correlation function. By means of an infinite renormalization technique, a mesonlike propagator, related to local energy-energy correlations, is bootstrapped. Finally, a vertex function, associated with energy-spin-spin correlations, is introduced. To zeroth order this theory bears similarity to the spherical model, but vital differences are also noted. A brief discussion is presented of the relation of methods employed in field theory for the treatment of the infrared divergence, and approximations which might be of value for critical correlations.

I. INTRODUCTION

In recent years there has been a close interplay between developments in field theory and the many-body problem.[1] Generally, methods devised in one discipline have found applications in the other. There are a number of notable exceptions. For example, no clear relation has been established between the critical transition properties of many-body systems and a fundamental particle effect. On the other hand, no many-body analog to the infrared divergence exists. [There is a striking similarity between the exponent modification of critical correlations and the form of the Green's function of certain particles for $p^2\sim m^2$. For two scalar fields, ψ with mass m and φ which is massless, coupled by a Lagrangian term $g\psi^2(\mathbf{x})\varphi(\mathbf{x})$, the ψ particle has a Green's function which behaves for $p^2\sim m^2$ as

$$(p^2-m^2)^{-1}|1-p^2/m^2|^{g^2/16\pi^2m^2}.$$

In quantum electrodynamics, the electron Green's function goes like[2]

$$(p-m)^{-1}|1-p^2/m^2|^{-e^2(3-d_l)/8\pi^2}$$

for $p^2\sim m^2$.]

In this paper we shall develop several formulations of the Ising model which resemble in many ways the field theory[2] of a spinless nucleon and scalar meson. The results may be expressed as a Feynman-diagram expansion,[3] employing, as bare propagator lines, G_{ij}^0 (related to the spin-spin correlation function of the spherical model) and D_{ij}^0 (which starts off infinite). After renormalization, G_{ij} turns out to be a progenitor of the spin-spin correlation function $\langle\mu_i\mu_j\rangle_\mu$, where μ_i is the spin variable of site i, which may have value ± 1. The infinitely renormalized function D_{ij} is associated

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[2] N. N. Bogoliubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields* (Interscience Publishers, Inc., New York, 1959).

[3] Such a coupling scheme for the Ising model was first considered by F. H. Stillinger, Jr., Phys. Rev. 126, 1239 (1962).

with energy-energy correlations $\langle \Delta e_i \Delta e_j \rangle_\mu$, where

$$\Delta e_i = e_i - \langle e_i \rangle_\mu , \tag{1.1}$$

$$e_i = -\sum_j v_{ij}\mu_i\mu_j . \tag{1.2}$$

Further resummation is achieved by introducing a vertex function $\Gamma(ij;k)$ and developing Dyson equations.[1,2] The vertex function is related to a spin-spin-energy correlation function $\langle \mu_i\mu_j\Delta e_k\rangle_\mu$.

Alternatively, we may make a Schwinger functional-derivative formulation[2] of the Ising model. A potential-energy coupling parameter plays the role of the charge density (scalar component of four-current).[3]

The link between field theory and the Ising model will be a functional-integral formulation of the latter, first proposed by Montroll and Berlin.[4] This formulation was central to a study made by Langer and the present author[5] of critical correlations in the Ising model. Note that functional-integral techniques have achieved a number of notable successes in recent years—such as in the study of the polymer excluded-volume problem,[6] knotting problems,[7] the density of states in a random medium,[8] the analysis of first-order phase transitions,[9] the study of many-body systems with long-ranged forces,[10] and the theory of the infrared divergence.[11]

In Sec. II, general features of the functional-integral approach to the Ising model will be discussed. The analytic properties in an energylike variable z are examined. In previous work[5] similarities to the spherical model[12] were noted. Here it will be shown that there are vital qualitative differences in the analytical structure of the spherical and Ising models. In Sec. III, a series expansion will be made which will lead to the Feynman-Dyson diagram theories. In the process an infinite renormalization will be performed, and a mesonlike propagator bootstrapped (the D function). A Luttinger-Ward[13] formula is developed for the partition function. The Schwinger formulation is next produced. In Sec. VI, remarks are made about approximations of the type introduced in the treatment of the infrared divergence.

In the course of the work a number of averages or integral operations will be employed. These are denoted by subscripted braces or brackets. The definitions are located at:

$\langle\ \rangle_\mu$: Eq. (2.2);
$\langle\ \rangle_t$: Eq. (2.14);
$[\ ]_\varphi$: Eq. (3.7);
$\langle\ \rangle_\varphi$: Eq. (3.8);
$\langle\ \rangle_\varphi^\lambda$: Eq. (3.35).

II. FUNCTIONAL-INTEGRAL FORMULATION

A. Partition Function

Consider the Ising-model partition function (in this paper only the field-free case will be discussed)

$$Z = \sum_{\{\mu=\pm1\}} \exp(\beta \sum_{ij} v_{ij}\mu_i\mu_j) , \tag{2.1}$$

where $\beta = J/2k_BT$. The interaction between a spin on site i and one on site j is $-Jv_{ij}\mu_i\mu_j$. The average of any function of the set $\{\mu\} \equiv \mu_1 \cdots \mu_N$ is given by

$$\langle \mathcal{A}\{\mu\}\rangle_\mu = \frac{1}{Z} \sum_{\{\mu=\pm1\}} \mathcal{A}\{\mu\} \exp(\beta \sum_{ij} v_{ij}\mu_i\mu_j) . \tag{2.2}$$

The μ summation may be converted to integrations by introducing δ functions employed in the form

$$\delta(1-\mu^2) = \frac{\beta}{2\pi i}\int_{-i\infty}^{i\infty} dt\ e^{\beta t(1-\mu^2)} . \tag{2.3}$$

The partition function becomes

$$Z = \left(\frac{\beta}{\pi i}\right)^N \int_{-\infty}^{\infty} d\{\mu\} \int_{-i\infty}^{i\infty} d\{t\} \times \exp(\beta \sum_j t_j) \exp[-\beta \sum_{ij} (\mathbf{T}-\mathbf{V})_{ij}\mu_i\mu_j] , \tag{2.4}$$

where $\mathbf{T}$ is the diagonal matrix with t_i in the ii position and $\mathbf{V}$ is the cyclic matrix with elements v_{ij}. The objective of achieving an integral over a Gaussian form in the μ's has been attained.

Before continuing the systematic development, let us look for familiar aspects in the type of expression to which the partition function has been reduced. Perhaps we can best recognize the Z of Eq. (2.4) as a field-theory-like object by writing

$$\exp[-\beta(-\sum_j t_j - \sum_{ij} v_{ij}\mu_i\mu_j + \sum_i t_j\mu_j^2)] = \exp[-\beta(\mathcal{H}_\mu{}^0 + \mathcal{H}_t{}^0 + \mathcal{H}_{\text{int}})] , \tag{2.5}$$

$$\mathcal{H}_\mu{}^0 = \sum_{ij} (z\delta_{ij} - v_{ij})\mu_i\mu_j , \tag{2.6}$$

$$\mathcal{H}_t{}^0 = -\sum_j t_j , \tag{2.7}$$

$$\mathcal{H}_{\text{int}} = \sum_j (t_j - z)\mu_j{}^2 . \tag{2.8}$$

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Later we will go into the reason for dividing the terms in this fashion, i.e., the introduction and significance of z. For the moment we merely wish to point out that $\mathcal{H}_\mu{}^0$ is a quadratic in the μ's. Although it is not local, for the critical problem we believe that the range of the forces is short compared with the important distances for correlated fluctuations, so that the essense of $\mathcal{H}_\mu{}^0$ would be contained in a local-field-theory-like term,

$$\mathcal{H}_\mu{}^0=\int d\mathbf{r}\,[a\mu^2(\mathbf{r})+b|\nabla\mu|^2]+\cdots, \tag{2.9}$$

where the μ is regarded as being a function of a continuous, rather than discrete, variable.

The term $\mathcal{H}_t{}^0$ is linear, rather than quadratic, in t_j. We shall find it convenient to write

$$\mathcal{H}_t{}^0=-\sum_j t_j-\lim_{\epsilon\to0}\tfrac{1}{2}\sum_{ij}\epsilon_{ij}(t_i-z)(t_j-z), \tag{2.10}$$

and to renormalize before allowing $\epsilon\to0$.

The term $\mathcal{H}_{\text{int}}$ is quadratic in the μ_j's and linear in t_j-z. It is the clue to the type of diagram theory to be suspected, viz., one with vertices out of which emanate two lines related to some sort of weighted, tracelike operation on $\mu_m\mu_n$, and one line related to this "trace" operation on $(t_m-z)(t_n-z)$.

In creating the "field-theoretic" formulation we shall find the functional-integral techniques most useful, so let us return to the partition function (2.4). If the t contours are shifted sufficiently far to the right, the μ and t integrations may be interchanged. The choice of $\{t\}$ contours $\mathcal{C}$ will be $-i\infty+\gamma$ to $i\infty+\gamma$, with

$$\gamma>\sum_j v_{ij}\equiv\tilde{v}_0. \tag{2.11}$$

Equation (2.4) yields[4]

$$Z=i^{-N}\left(\frac{\beta}{\pi}\right)^{N/2}\int_{\mathcal{C}}d\{t\}\,\exp(\beta\sum_j t_j)|\mathbf{T}-\mathbf{V}|^{-1/2}. \tag{2.12}$$

(As $N\to\infty$, an integral of this type is equivalent to a functional integral; cf. Sec. II E, where this is made explicit by going to Fourier-transform variables.)

B. Pair Correlations

Of central interest in the development of the theory are several correlation functions. The first is the spin-pair correlation $\langle\mu_n\mu_m\rangle_\mu$. By methods completely analogous to the above, one obtains[5]

$$\langle\mu_m\mu_n\rangle_\mu=\frac{1}{2\beta Z}i^{-N}\left(\frac{\beta}{\pi}\right)^{N/2}\int_{\mathcal{C}}d\{t\}\times\exp(\beta\sum_j t_j)|\mathbf{T}-\mathbf{V}|^{-1/2}\mathcal{G}_{mn}=(1/2\beta)\langle\mathcal{G}_{mn}\rangle_t, \tag{2.13}$$

where the matrix $\mathcal{G}_{mn}$ is the inverse (Green's function) of $\mathbf{T}-\mathbf{V}$. The integral operation $\langle\;\rangle_t$ is defined by

$$\langle\mathcal{A}[t]\rangle_t=\frac{1}{Z}i^{-N}\left(\frac{\beta}{\pi}\right)^{N/2}\int_{\mathcal{C}}d\{t\}\times\exp(\beta\sum_j t_j)|\mathbf{T}-\mathbf{V}|^{-1/2}\mathcal{A}\{t\}. \tag{2.14}$$

C. Energy-Energy Correlation

To study the energy-energy correlation, it will be convenient to introduce a more general coupling constant (equivalently, a local reduced temperature) into the Ising model so that the probability of a configuration $\{\mu\}$ is

$$\exp[\sum_{ij}(\beta+\xi_i)^{1/2}(\beta+\xi_j)^{1/2}v_{ij}\mu_i\mu_j]/Z^*,$$
$$Z^*=\sum_{\{\mu=\pm1\}}\exp[\sum_{ij}(\beta+\xi_i)^{1/2}(\beta+\xi_j)^{1/2}v_{ij}\mu_i\mu_j]. \tag{2.15}$$

One easily shows that (asterisk indicates general $\{\xi\}$, while no asterisk would indicate $\{\xi\}=0$)

$$\frac{\partial\ln Z^*}{\partial\xi_m}=\sum_j\left(\frac{\beta+\xi_j}{\beta+\xi_m}\right)^{1/2}v_{mj}\langle\mu_m\mu_j\rangle_\mu{}^*. \tag{2.16}$$

Of special interest is the limit of $\{\xi\}\to0$, in which case Eq. (2.16) reduces to

$$\partial\ln Z/\partial\xi_m=-\langle e_m\rangle_\mu\equiv-\langle e\rangle_\mu. \tag{2.17}$$

Taking a second derivative of Eq. (2.16) and again letting $\{\xi\}\to0$ yields

$$\frac{\partial^2\ln Z}{\partial\xi_m\partial\xi_n}=\langle\Delta e_m\Delta e_n\rangle_\mu+(1/2\beta)v_{mn}\langle\mu_m\mu_n\rangle_\mu+(1/2\beta)\langle e\rangle_\mu\delta_{mn}. \tag{2.18}$$

These formulas assume an interesting and simple functional-integral form. Changing β to $\beta+\xi_i$ in Eq. (2.3), one obtains

$$Z^*=i^{-N}\prod_i\left(\frac{\beta+\xi_i}{\pi}\right)^{1/2}\int_{\mathcal{C}}d\{t\}\times\exp[\sum_i(\beta+\xi_i)t_i]|\mathbf{T}-\mathbf{V}|^{-1/2}. \tag{2.19}$$

Now ξ derivatives lead to

$$\frac{\partial\ln Z^*}{\partial\xi_m}=\tfrac{1}{2}(\beta+\xi_m)^{-1}+\langle t_m\rangle_t{}^*, \tag{2.20}$$

which implies that

$$-\langle e\rangle_\mu=1/2\beta+\langle t_m\rangle_t. \tag{2.21}$$

Furthermore,

$$\frac{\partial^2 \ln Z^*}{\partial \xi_m \partial \xi_n} = -\tfrac{1}{2}(\beta+\xi_m)^{-2}\delta_{mn} + \langle (t_m - \langle t_m\rangle_t^*)(t_n - \langle t_n\rangle_t^*)\rangle_t^*, \quad (2.22)$$

which yields the relation

$$\langle \Delta e_m \Delta e_n\rangle_\mu = -\tfrac{1}{2}\beta^{-2}\delta_{mn} + \langle (t_m - \langle t_m\rangle_t)(t_n - \langle t_n\rangle_t)\rangle_t. \quad (2.23)$$

We will find it convenient later to introduce variables

$$z = \frac{1}{N}\sum_j t_j \quad (2.24a)$$

and

$$\varphi_j = t_j - z. \quad (2.24b)$$

In terms of these variables Eq. (2.23) may be written in the relevant form

$$\langle \Delta e_m \Delta e_n\rangle_\mu = \langle \varphi_m \varphi_n\rangle_t - (1/2\beta^2)\delta_{mn} - (1/N)\beta^{-2}(c_H/k - \tfrac{1}{2}), \quad (2.25)$$

where c_H is the specific heat per particle.

An alternative relation between energy-energy correlations and functional integrals was given in Sec. V of Ref. 5, and will not be discussed here.

D. Spin-Spin-Energy Correlation

A central role in the theory will be played by a vertex function which is related to a spin-spin-energy correlation function. To develop the necessary formulas consider $\partial\langle \mu_m\mu_n\rangle_\mu^*/\partial\xi_k$ from the dual points of view of μ summations and functional integrals. One finds that

$$-\langle \mu_m\mu_n\Delta e_k\rangle_\mu = (1/2\beta)\langle \varphi_k \mathcal{G}_{mn}\rangle_t - (1/2\beta)^2\langle \mathcal{G}_{mn}\rangle_t[\delta_{mk}+\delta_{nk}] + (1/N)(1/2\beta)[\partial\langle \mathcal{G}_{mn}\rangle_t/\partial\beta]. \quad (2.26)$$

E. Fourier-Transform Variables and z

It is convenient to make an orthogonal change of variables to the fourier components $\tilde{t}_p$, with $\mathbf{p}$ in the Brillouin zone of the reciprocal lattice:

$$\tilde{t}_p = N^{-1/2}\sum_j t_j e^{i\mathbf{p}\cdot\mathbf{r}_j}, \qquad t_j = N^{-1/2}\sum_p \tilde{t}_p e^{-i\mathbf{p}\cdot\mathbf{r}_j}. \quad (2.27)$$

For $\mathbf{p}=0$ define the special variable

$$z = N^{-1}\sum_i t_i = N^{-1/2}\tilde{t}_0, \quad (2.28)$$

and for $\mathbf{p}\neq 0$ introduce the names [cf. Eq. (2.24)]

$$\tilde{\varphi}_p = \tilde{t}_p, \quad \mathbf{p}\neq 0, \qquad \varphi_j = N^{-1/2}\sum_{p\neq 0}\tilde{\varphi}_p e^{-i\mathbf{p}\cdot\mathbf{r}_j}. \quad (2.29)$$

The partition function may be written

$$Z = \int_{\mathcal{C}} dz\, e^{N\beta z} e^{N\Omega(z)}, \quad (2.30)$$

where

$$e^{N\Omega(z)} = i^{-N} N^{1/2}\left(\frac{\beta}{\pi}\right)^{N/2} \int d\{\tilde{\varphi}\}' |\mathbf{T}-\mathbf{V}|^{-1/2}. \quad (2.31)$$

The prime on $\{\tilde{\varphi}\}'$ indicates omission of the $p=0$ variable. As $N\to\infty$ and $\mathbf{p}$ becomes a continuous variable this is truly a functional integral. The meaning of the $\{\tilde{\varphi}\}'$ integral and the contours are discussed in Appendix A.

In order to arrange the integrals in an order in which the $\{\tilde{\varphi}\}'$ integrals are done first, it may be necessary to introduce a convergence factor. This is clarified in Appendix B for the ideal lattice model and is also exhibited in Eq. (3.5).

Equation (2.30) bears a resemblance to a formula arising from the spherical model,[12] wherein

$$e^{N\Omega_{\rm SM}(z)} = (N^{1/2}\beta/i\pi^{1/2})(2/\beta e)^{N/2}|z\mathbf{1}-\mathbf{V}|^{-1/2}. \quad (2.32)$$

We shall now point out, however, that there must be vital differences in the analtyical structure of Ω and $\Omega_{\rm SM}$.

For the spherical model, one performs the z integration by a saddle-point method, with z_S determined by

$$\beta + \Omega_{\rm SM}'(z_S) = 0 \quad (2.33)$$

for $z_S > \tilde{v}_0$ [$z=\tilde{v}_0$ is a branch point of $\Omega_{\rm SM}(z)$]. At a critical temperature the solution of Eq. (2.33) has $z_S=\tilde{v}_0$. For lower temperatures the proper evaluation of Eq. (2.30) requires that one select[12]

$$z_S = \tilde{v}_0, \quad \beta \geq \beta_{\rm SM}{}^c. \quad (2.34)$$

This results in a third-order phase transition.

Let us imagine that a similar technique applied for the correct Ω, i.e.,

$$\beta + \Omega'(z_S) = 0, \quad z_S \geq \tilde{v}_0, \quad (2.35)$$

has a solution for $\beta<\beta_c$. [Recall the condition (2.11) that the z contours be to the right of $\tilde{v}_0$.] The free energy per particle, $F/N=f$, would be

$$f/k_BT = -\beta z_S - \Omega(z_S), \quad (2.36)$$

and the energy per particle, $u=\partial(\beta f)/\partial\beta$, is

$$u = -\tfrac{1}{2}J(z_S + \tfrac{1}{2}\beta^{-1}). \quad (2.37)$$

If $z_S \geq \tilde{v}_0$, this implies, erroneously, that

$$u < -\tfrac{1}{2}J\tilde{v}_0,$$

i.e., that the energy per particle is more negative than it would be under conditions of perfect order in the system. [The difficulty does not arise in the spherical model, where

$$u_{\rm SM} = -\tfrac{1}{2}J(z_S - \tfrac{1}{2}\beta^{-1}).] \quad (2.38)$$

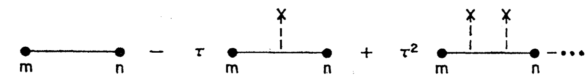

FIG. 1. Diagram series for $\mathcal{G}_{mn}(\tau)$ [Eq. (3.14)].

At a later point we shall return to the question of whether the z integration can be performed by a saddle technique, rather than as the straightforward, but complicated, integral along $\mathcal{C}$. Unfortunately we shall find difficulties associated with the few simple schemes which might be proposed.

III. DIAGRAMMATIC PERTURBATION THEORY

A. Introductory Development

The basic link between the functional integrals and a Feynman-diagram expansion is a theorem for moments of a Gaussian distribution which is analogous to Wick's theorem. In this section we shall show how this rule may be exploited. First, it is necessary to broaden our definition of the partition and correlation functions.

In Ref. 5, it is demonstrated that the determinant $|\mathbf{T}-\mathbf{V}|$ may be reexpressed in terms of a Green's function:

$$|\mathbf{T}-\mathbf{V}|^{-1/2}=|z\mathbf{1}-\mathbf{V}|^{1/2}\times\exp\left(-\frac{1}{2}\int_0^1 d\tau\sum_i \varphi_i\mathcal{G}_{ii}(\tau)\right), \quad (3.1)$$

$$|z\mathbf{1}-\mathbf{V}|^{-1/2}=\exp\{-\tfrac{1}{2}\sum_{p\epsilon BZ}\ln[z-\tilde{v}(\mathbf{p})]\}, \quad (3.2)$$

$$\tilde{v}(\mathbf{p})=\sum_j v_{ij}\exp[i\mathbf{p}\cdot(\mathbf{r}_i-\mathbf{r}_j)], \quad (3.3)$$

and the Green's function $\mathcal{G}_{ij}(\tau)$ is the inverse of the matrix

$$(z+\tau\varphi_i)\delta_{ij}-v_{ij}. \quad (3.4)$$

Equation (3.1) is derived by taking τ logarithmic derivative of the determinant of Eq. (3.4) and reintegrating. A generalization will be useful. Define

$$\mathcal{S}=\mathcal{S}^0\exp\left(-\frac{1}{2}\int_0^1 d\tau\sum_i \varphi_i\mathcal{G}_{ii}(\tau)\right), \quad (3.5)$$

$$\mathcal{S}^0=\exp(\tfrac{1}{2}\sum_p{}'\,\epsilon_p\tilde{\varphi}_p\tilde{\varphi}_{-p}). \quad (3.6)$$

We choose $\epsilon_p=\epsilon_{-p}>0$, and eventually $\epsilon_p\rightarrow 0$. Meanwhile the ϵ's provide convergence. The prime on the $\mathbf{p}$ summation means to omit the $\mathbf{p}=0$ term. We are interested in functional integrals of the type

$$[\mathcal{B}]_\varphi=i^{-N+1}\int d\{\tilde{\varphi}\}'\mathcal{B}(z,\{\tilde{\varphi}\}'), \quad (3.7)$$

$$\langle\mathcal{A}\rangle_\varphi=[\mathcal{A}\mathcal{S}]_\varphi/[\mathcal{S}]_\varphi. \quad (3.8)$$

In particular,

$$e^{N\Omega(z)}=-iN^{1/2}(\beta/\pi)^{N/2}|z\mathbf{1}-\mathbf{V}|^{-1/2}[\mathcal{S}]_\varphi \quad (3.9)$$

is needed to calculate the partition function. The correlation function $\langle\mu_m\mu_n\rangle_\mu$ may be obtained from the "Green's function"

$$G_{mn}(z)\equiv\langle\mathcal{G}_{mn}(z,\{\varphi\}')\rangle_\varphi. \quad (3.10)$$

The energy-energy correlations are related to

$$D_{mn}(z)\equiv-\langle\varphi_m\varphi_n\rangle_\varphi. \quad (3.11)$$

The diagrammatic perturbation theory will be in powers of φ. Thus either the range of φ in the integration must be effectively small or a resummation must be performed. Basic to the development is the integral-equation-like version of the Green's-function equation:

$$\mathcal{G}_{mn}(\tau)=G_{mn}{}^0-\tau\sum_k G_{mk}{}^0\varphi_k\mathcal{G}_{kn}(\tau), \quad (3.12)$$

where the zeroth-order Green's function is

$$G_{mn}{}^0\equiv[(z\mathbf{1}-\mathbf{V})^{-1}]_{mn}. \quad (3.13)$$

Equation (3.12) may be iterated and the result expressed in the form of a series of diagrams:

$$\mathcal{G}_{mn}(\tau)=(\text{expression in Fig. 1}). \quad (3.14)$$

The solid lines stand for G^0 and the prongs (dashed lines with a cross) indicate φ. At each intermediate vertex, a summation is performed.

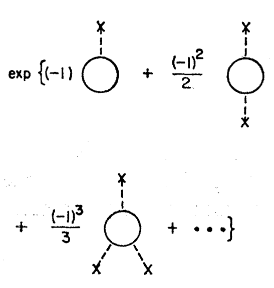

FIG. 2. Diagram series for $\mathcal{S}/\mathcal{S}^0$ [Eq. (3.16)].

B. Ω Function

The integral term in the exponent of $\mathcal{S}$ can be reexpressed as a diagrammatic expansion (Fig. 2) with closed loops of G^0 bonds:

$$\frac{\mathcal{S}}{\mathcal{S}^0}=\exp\left(-\frac{1}{2}\int_0^1 d\tau\sum_i \varphi_i\mathcal{G}_{ii}(\tau)\right) \quad (3.15)$$

$$=(\text{expression in Fig. 2}). \quad (3.16)$$

Note that since $G_{ii}{}^0$ is independent of i, and

$$\sum_i \varphi_i=0,$$

the first diagram is zero. We adopt the convention that with each closed loop of G bonds there is associated a factor of $\frac{1}{2}$.

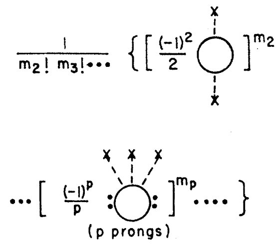

FIG. 3. Typical terms in the expansion of $\mathcal{S}/\mathcal{S}^0$.

In order to perform the $\tilde{\varphi}$ integrations it is necessary to expand the exponential. A typical term, composed of m_2 two-prong loops, m_3 three-prong loops, etc., is given in Fig. 3. The only $\tilde{\varphi}$ integrals are the multidimensional Gaussian moments

$$\langle\varphi_{i_1}\cdots\varphi_{i_{2n}}\rangle_{\varphi}{}^0 \equiv \left[\prod_p{}' \left(\frac{\epsilon_p}{2\pi}\right)^{1/2}\right]\int d\{\varphi\}'\varphi_{i_1}\cdots\varphi_{i_{2n}} \times\exp(\tfrac{1}{2}\sum_p{}' \epsilon_p\tilde{\varphi}_p\tilde{\varphi}_{-p})$$
$$=\sum_{\text{all pairings}}\langle\varphi_{f_1}\varphi_{s_1}\rangle_{\varphi}{}^0\langle\varphi_{f_2}\varphi_{s_2}\rangle_{\varphi}{}^0\cdots\times\langle\varphi_{f_n}\varphi_{s_n}\rangle_{\varphi}{}^0, \quad (3.17)$$

where $f_1s_1f_2s_2\cdots f_ns_n$ indexes the pairings of the subscripts $i_1\cdots i_{2n}$.[14] In graphical terms, we perform the $\langle\ \rangle_{\varphi}{}^0$ "average" of a given graph by drawing all the graphs which can be made by linking pairs of prongs in all ways. The average (linkage) of a pair of prongs is represented by a dashed line between the two vertices, which stands for

$$\langle\varphi_i\varphi_j\rangle_{\varphi}{}^0 \equiv -D_{ij}{}^0 = -N^{-1}\sum_p{}' \epsilon_p{}^{-1}e^{-ip\cdot(r_i-r_j)}. \quad (3.18)$$

When the prongs of Fig. 3 are linked together, there are repetitions of the same diagram due to two causes. The first is the $m_p!$ possible permutations of the m_p p-loops. We shall include such diagrams only once, thus cancelling the prefactor $1/m_2!m_3!\cdots$. Secondly, any p-loop may be placed in p proper rotational positions. By not duplicating these graphs the factors $1/p$ are eliminated. On the other hand, it will be necessary in the ensuing discussion to include duplicate diagrams which arise from inversion of a loop, such as those shown in Fig. 4.

A linked-cluster theorem can be developed[15] for the evaluation of $\log[\mathcal{S}]_{\varphi}$. We shall find it more profitable to proceed in a Luttinger-Ward[13] fashion. First, however, it is necessary to examine several other functional integrals.

[14] Equation (3.17) is easily derived by taking derivatives of the characteristic function

$$\langle\exp(\sum_p{}' \tilde{k}_p\tilde{\varphi}_p)\rangle_{\varphi}{}^0=\exp(-\tfrac{1}{2}\sum_p{}' \tilde{k}_p\tilde{k}_{-p}/\epsilon_p).$$

[15] Reference 1, p. 130 ff.

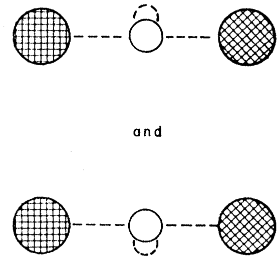

FIG. 4. Example of diagrams which have equal values, but which must both be included.

C. D Function

Consider

$$D_{ij}\equiv -\langle\varphi_i\varphi_j\rangle_{\varphi} \quad (3.19)$$
$$=-[\varphi_i\varphi_j\mathcal{S}]_{\varphi}/[\mathcal{S}]_{\varphi}. \quad (3.20)$$

The $\mathcal{S}$ in the numerator is expanded as in Eq. (3.16) and Fig. 3. The $\tilde{\varphi}$ integral again amounts graphically to connecting the prongs of the loops to each other, and now also to the two prongs which stand for φ_i and φ_j. After performing a particular linkage we obtain a diagram, some of whose loops are connected directly or indirectly to φ_i and/or φ_j, and some of which form a part of the diagram not so connected. Associated with any diagram connected to φ_i and/or φ_j there can be any of a set of diagrams which is identical to the set which generates $[\mathcal{S}]_{\varphi}$. The factor arising from the sum over this set just cancels the $[\mathcal{S}]_{\varphi}$ in the denominator of Eq. (3.20). Thus we get a linked-cluster theorem. D_{ij} is equal to a sum, the terms of which are in correspondence with all connected diagrams formed by linking the prongs of loops with the two prongs φ_i and φ_j. The value to be assigned to each diagram is determined as follows. Associate a summation index $k_1\cdots k_{2n}$ with each intermediate vertex. Include a factor $G_{kk'}{}^0$ for a solid line connecting vertex k to k', and $-D_{kk'}{}^0$ for a dashed line. Sum over $k_1\cdots k_{2n}$. Since the φ's occur in pairs, all the powers of $(-)$ are even. Finally, include a factor of $\frac{1}{2}$ for each closed loop of G^0 bonds. Then we obtain

$$-D_{ij}=\text{(expression in Fig. 5)}; \quad (3.21)$$

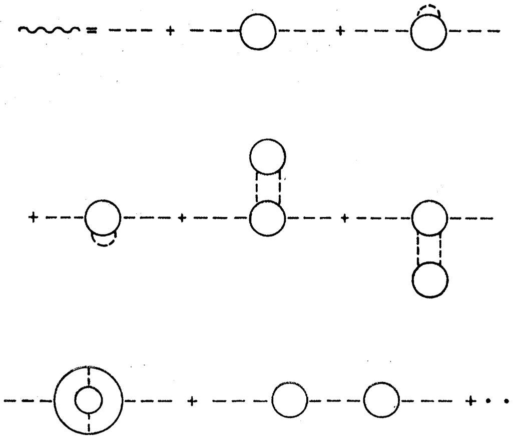

FIG. 5. Diagram series for $-D_{ij}$ [Eq. (3.21)].

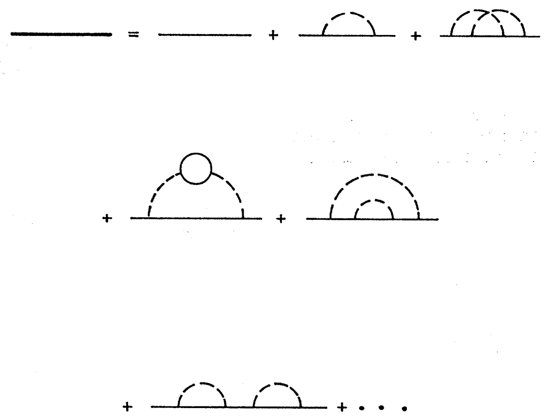

FIG. 6. Diagram series for G_{mn} [Eq. (3.22)].

in diagrams this quantity will be represented by a wavy line as shown. The diagrams in which φ_i and φ_j are not directly linked lead to a term $\langle\varphi_i\rangle\langle\varphi_j\rangle$ which is zero in the absence of a field. (More generally, define

$$D_{ij}=-\langle\varphi_i\varphi_j\rangle+\langle\varphi_i\rangle\langle\varphi_j\rangle.)$$

D. G Function

Next let us examine the series for $G_{mn}\equiv\langle\mathcal{G}_{mn}\rangle_\varphi$. $\mathcal{G}_{mn}$ can be expanded into a series, a typical term of which is represented by $(-1)^l$ times a base line of $l+1$ G^0 factors and l prongs [Eq. (3.14)]. These prongs, and the prongs of the loops arising from the expansion of $\mathcal{S}$, are to be connected in pairs to give the result of φ integration. The sum over all pieces not connected to the G base line cancels $[\mathcal{S}]_\varphi$ again. Diagrammatically, we obtain

$$G_{mn}=(\text{expression in Fig. 6}); \qquad (3.22)$$

in diagrams this quantity will be represented by a heavy line as shown.

E. Self-Energies

The functions D_{ij} and G_{mn} may be expressed in terms of proper self-energies. In the evaluation of $-D_{ij}$, when the prong for φ_i is linked through a group of loops to φ_j, it may be done in such a way that there are no single intermediate D^0 linkages which, if cut, separate the

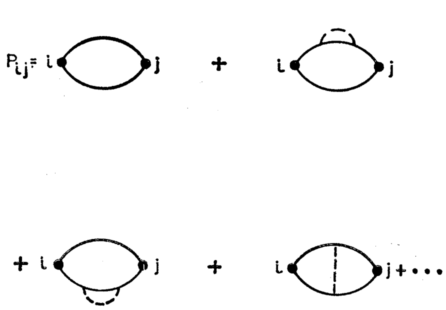

FIG. 7. Diagram series for P_{ij} [Eq. (3.23)].

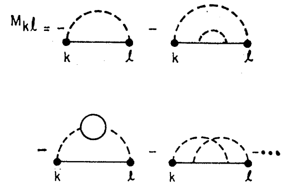

FIG. 8. Diagram series for M_{kl} [Eq. (3.28)].

diagram into two disconnected parts. We represent the sum over such graphs by

$$\sum_{kl} D_{ik}{}^0 P_{kl}(-D_{lj}{}^0)$$

(sign chosen to make $\mathbf{P}$ positive definite). The series for P_{ij} begins

$$\begin{aligned} P_{ij}&=(\text{expression in Fig. 7})\\ &=\tfrac{1}{2}G_{ij}{}^0G_{ij}{}^0+\cdots. \end{aligned} \qquad (3.23)$$

All but the last diagram in Eq. (3.21) is of this type. On the other hand, there may be one intermediate dashed line which, when cut, separates the diagram, or two such lines, or three, etc. This leads to the series

$$D_{ij}=D_{ij}{}^0+\sum_{kl} D_{ij}{}^0P_{kl}(-D_{lj}{}^0)$$
$$+\sum_{k_1l_1k_2l_2} D_{ik_1}{}^0P_{k_1l_1}(-D_{l_1k_2}{}^0)P_{k_2l_2}(-D_{l_2j}{}^0)+\cdots. \qquad (3.24)$$

This equation may be represented compactly in a matrix notation, but since these matrices are cyclic, it is best to use the Fourier-transform representation. This is a completely conventional procedure in diagram theory.[1] A momentum $\mathbf{p}_i$ is associated with each bond in such a way that momentum is conserved at the vertices (arbitrary directions are also assigned to the lines). Summations (integrations) are performed over all intermediate moments. Equation (3.24) takes the form

$$\tilde{D}(\mathbf{p})=1/\{[\tilde{D}^0(\mathbf{p})]^{-1}+\tilde{P}(\mathbf{p})\}, \qquad (3.25)$$

where $\tilde{D}^0(\mathbf{p})=1/\epsilon_p$. Now the limit $\epsilon_p\to 0$ may be taken:

$$\lim_{\epsilon\to 0}\tilde{D}(\mathbf{p})=\lim_{\epsilon\to 0}\tilde{P}(\mathbf{p}). \qquad (3.26)$$

The fact that $\tilde{D}^0(\mathbf{p})\to\infty$ is of no consequence. It is of interest that in this calculation one can see how the infinite renormalization arises from the initial development.

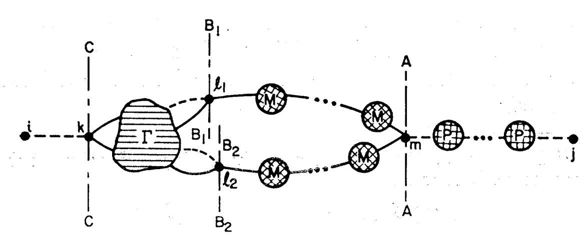

FIG. 9. Structure of a typical diagram in the expansion of D_{ij}.

FIG. 10. Diagrammatic representation of Eq. (3.29) for D.

We omit the details of the analogous and routine[1] proof that G may be expressed in terms of a self-energy $\tilde{M}(\mathbf{p})$:

$$\tilde{G}(\mathbf{p})=1/\{[\tilde{G}^0(\mathbf{p})]^{-1}+\tilde{M}(\mathbf{p})\}. \quad (3.27)$$

The series for M begins (note the convention of minus sign)

$$\begin{aligned} M_{kl} &= (\text{expression in Fig. 8}) \\ &= D_{kl}^0 G_{kl}^0 - \cdots. \end{aligned} \quad (3.28)$$

F. Dyson Theory and the Vertex Function

In the infinite diagram series for P and M it is possible to completely eliminate G^0 and D^0 bonds in favor of G and D. Rather than pursue this course, we will follow the procedure suggested by Dyson, and center attention on a vertex function. First, consider D_{ij}, a typical graph of which has the structure given in Fig. 9. We proceed from right to left along the graph until we come to the last place at which the diagram can be cut along a D^0 (cut A). At the end of this line, vertex m, the diagram branches into two G^0 lines (part of a G^0-bond loop). We proceed along one branch until we come to the last place at which the section from l_1 to m may be removed by a cut of a G^0 line (cut B_1). Cut B_2 on the lower branch is similarly defined. Finally, make cut C to remove the last D^0 bond. In the various regions a variety of diagrams may appear and these may be summed over, independently of the other regions. Thus in the region from j to cut A, the diagrams of D appear and summing yields D_{mj}. In A to B_1 and A to B_2 appear the G graphs yielding G_{l_1m} and G_{l_2m}. The sum over all diagrams which may appear in CB_1B_3 will be termed the vertex function $\Gamma(l_1l_2;m)$, and will be denoted diagramatically by a triangle. This leads to Fig. 10 and Eq. (3.29) for D:

$$D_{ij}=D_{ij}^0-\tfrac{1}{2}\sum_{kl_1l_2m} D_{ik}^0\Gamma(l_1l_2;k)G_{l_1m}G_{l_2m}D_{mj}, \quad (3.29)$$

which can also be written

$$\tilde{P}(\mathbf{q})=\frac{1}{2N}\sum_{p}\tilde{\Gamma}(\mathbf{p},\mathbf{p}-\mathbf{q};\mathbf{q})\tilde{G}(\mathbf{p})\tilde{G}(\mathbf{p}-\mathbf{q}). \quad (3.30)$$

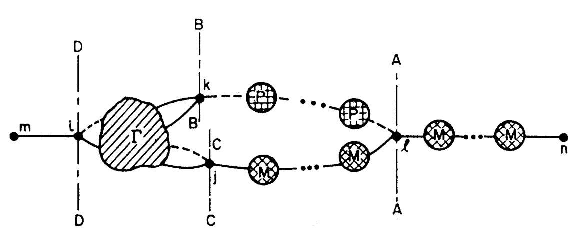

FIG. 11. Structure of a typical diagram in the expansion G_{mn}.

FIG. 12. Diagrammatic representation of Eq. (3.31) for G.

In like manner, the typical diagram for G can be divided as shown in Fig. 11. Summing over the diagrams which may appear between the cuts leads to the results shown in Fig. 12 and Eq. (3.31):

$$G_{mn}=G_{mn}^0-\sum_{ijkl} G_{mi}^0\Gamma(ij;k)D_{kl}G_{jl}G_{ln}, \quad (3.31)$$

or

$$\tilde{M}(\mathbf{p})=\frac{1}{N}\sum_{q}{}' \,\tilde{\Gamma}(\mathbf{p},\mathbf{p}-\mathbf{q};\mathbf{q})\tilde{D}(\mathbf{q})\tilde{G}(\mathbf{p}-\mathbf{q}). \quad (3.32)$$

To close the set of Dyson relations, an equation for Γ must be obtained. Unfortunately, this equation again involves an infinite series.[1] A typical diagram for Γ can be reduced to a skeleton by making cuts which isolate D terms, G terms, and Γ terms. Summing over all possible inserts between these cuts results in an "integral" equation for Γ. There is not a single skeleton, as for D and G, but an infinite number. The first few terms in the series equation for Γ are those given in Fig. 13 and Eq. (3.33):

$$\begin{aligned}\Gamma(ij;k)=\delta_{ij}\delta_{jk}-\sum_{i_1i_2j_1j_2k_1k_2}&\Gamma(ii_1;i_2)\Gamma(jj_1;j_2)\\ &\times D_{i_2j_2}G_{i_1k_1}G_{j_1k_2}\Gamma(k_1k_2;k)+\cdots. \end{aligned}\quad (3.33)$$

G. Luttinger-Ward Theory for Ω

At this point we may profitably return to a consideration of a diagrammatic expression for Ω in terms of the G and D functions. A further extension of definitions to introduce a coupling parameter λ will be of value.

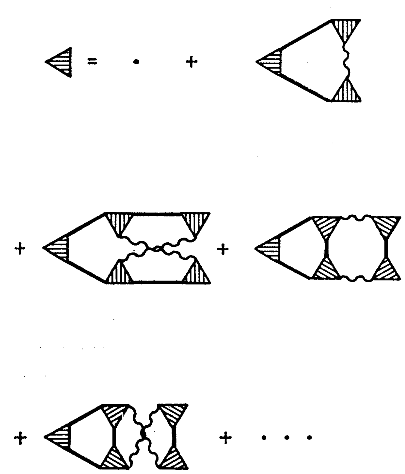

FIG. 13. Diagrammatic equation for Γ; cf. Eq. (3.33).

Define the function

$$\mathcal{S}^\lambda=\mathcal{S}^0 \exp\left(-\frac{1}{2}\int_0^\lambda d\tau \sum_i \varphi_i \mathcal{G}_{ii}(\tau)\right). \quad (3.34)$$

The bracket $\langle \mathfrak{a} \rangle_\varphi{}^\lambda$ has the weighting factor $\mathcal{S}^\lambda$ [16]:

$$\langle \mathfrak{a} \rangle_\varphi{}^\lambda=[\mathfrak{a}\mathcal{S}^\lambda]/[\mathcal{S}^\lambda]. \quad (3.35)$$

Generalizations of G and D are

$$G_{mn}{}^\lambda=\langle \mathcal{G}_{mn}(\lambda)\rangle_\varphi{}^\lambda \quad (3.36)$$

and

$$D_{ij}{}^\lambda=\langle \varphi_i\varphi_j\rangle_\varphi{}^\lambda. \quad (3.37)$$

Digrammatically, the effect of the above λ insertions is to associate a factor of λ with each vertex arising from expansion of $\mathcal{G}$. By differentiating $[\mathcal{S}^\lambda]_\varphi$ with respect to λ it is easy to show that

$$N\Omega=\ln\frac{N^{1/2}}{i}+\tfrac{1}{2}N\ln\frac{\beta}{\pi}+\tfrac{1}{2}\sum_p \ln\tilde{G}^0(\mathbf{p})$$
$$+\tfrac{1}{2}\lim_{\epsilon\to 0}\{\sum_p{}' \ln[2\pi\tilde{D}^0(\mathbf{p})]-I\}, \quad (3.38)$$
$$I=\int_0^1 d\lambda\frac{1}{\lambda}\sum_i \langle \lambda\varphi_i\mathcal{G}_{ii}(\lambda)\rangle_\varphi{}^\lambda.$$

Employing diagrammatic analysis or methods introduced in Sec. III H, one can show that

$$\sum_i \langle \lambda\varphi_i\mathcal{G}_{ii}(\lambda)\rangle_\varphi{}^\lambda=\sum_{ijmn} G_{im}{}^\lambda G_{in}{}^\lambda D_{ij}{}^\lambda \Gamma^\lambda(mn;j) \quad (3.39)$$
$$=\sum_p \tilde{G}^\lambda(\mathbf{p})\tilde{M}^\lambda(\mathbf{p}) \quad (3.40)$$
$$=2\sum_p{}' \tilde{D}^\lambda(\mathbf{p})\tilde{P}^\lambda(\mathbf{p}). \quad (3.41)$$

The dangers of proceeding in a fashion which interchanges the $\epsilon\to 0$ limit and λ integration in Eq. (3.38) are easily seen. Since

$$\tilde{D}^\lambda(\mathbf{p})=[\epsilon_p+\tilde{P}^\lambda(\mathbf{p})]^{-1}, \quad (3.42)$$

in the limit $\epsilon\to 0$ the right side of Eq. (3.41) is $2(N-1)$. The λ integral in Eq. (3.38) diverges. Actually, because $\tilde{P}(\lambda)$ has at least two vertices, for small λ it is of order λ^2. Thus the λ integral effectively cuts off at $\lambda\approx\epsilon^{1/2}$, avoiding the divergence. A convenient method of performing the λ integration first is to employ techniques introduced by Luttinger and Ward,[13] although we parallel Klein's argument.[17]

In considering the graphs contributing to $\tilde{M}^\lambda$ of Eq. (3.40) or $\tilde{P}^\lambda$ of Eq. (3.41) we first go to skeletal graphs such that the full series is recovered by summing over these skeletal graphs but with each bond a fully renormalized G or D. The skeletal graphs are then grouped by the number of vertices $2n$:

$$\sum_i \langle \lambda\varphi_i\mathcal{G}_{ii}(\lambda)\rangle_\varphi{}^\lambda=\sum_{n=1}^\infty \lambda^{2n}B_n(\lambda), \quad (3.43)$$
$$B_n(\lambda)=\sum_p \tilde{G}^\lambda(\mathbf{p})\tilde{M}_n[\tilde{G}^\lambda,\tilde{D}^\lambda] \quad (3.44)$$
$$=2\sum_p \tilde{D}^\lambda(\mathbf{p})\tilde{P}_n[\tilde{G}^\lambda,\tilde{D}^\lambda]. \quad (3.45)$$

$\tilde{M}_n$ is composed of the skeletal M graphs with $2n$ vertices. The λ factor at each vertex has been extracted in Eq. (3.43) but $\tilde{M}_n$ is still a function of λ by virtue of its functional dependence on $\tilde{G}^\lambda$ and $\tilde{D}^\lambda$. Identical statements hold for $\tilde{P}_n$.

Differentiation of $B_n(\lambda)$ involves the successive differentiation of each of the n bonds $\tilde{D}^\lambda$ and $2n$ bonds $\tilde{G}^\lambda$ of each graph. When one of the $\tilde{G}^\lambda$ bonds of $\tilde{M}_n$ is isolated for differentiation, the first factor $\tilde{G}^\lambda$ closes the graph again to make an $\tilde{M}_n$ diagram. Detailed consideration[13,17] shows that

$$\frac{dB_n}{d\lambda}=2\sum_p n\tilde{M}_n\frac{\partial\tilde{G}^\lambda}{\partial\lambda}+2\sum_p{}' n\tilde{P}_n\frac{\partial\tilde{D}^\lambda}{\partial\lambda}. \quad (3.46)$$

The λ integral of I can now be done by parts, the explicit λ's of Eq. (3.43) being integrated and the B_n differentiated:

$$I=\sum_n \frac{1}{2n}B_n(1)-\int_0^1 d\lambda\left(\sum_p \tilde{M}^\lambda(\mathbf{p})\frac{\partial\tilde{G}^\lambda}{\partial\lambda}+\sum_p{}' \tilde{P}^\lambda(\mathbf{p})\frac{\partial\tilde{D}^\lambda}{\partial\lambda}\right)$$
$$=2N\Omega^\dagger-\sum_p\left(\ln\frac{\tilde{G}}{\tilde{G}^0}+\tilde{M}(\mathbf{p})\tilde{G}(\mathbf{p})\right)$$
$$-\sum_p{}'\left(\ln\frac{\tilde{D}}{\tilde{D}^0}+\tilde{P}(\mathbf{p})\tilde{D}(\mathbf{p})\right). \quad (3.47)$$

We have determined the diagrams of $\Omega^\dagger$ from the skeletal diagrams of M or P, with renormalized $\tilde{D}$ and $\tilde{G}$ bonds. One can also show[13] that $\Omega^\dagger$ is composed of the skeletal diagrams of Ω with renormalized $\tilde{D}$ and $\tilde{G}$ bonds, when one carries the arguments of Sec. III B through to the point of constructing a linked-cluster theorem, in the usual way.[15]

Finally, one obtains for Ω an expression totally in terms of the renormalized G and D functions:

$$N\Omega=\ln\frac{N^{1/2}}{i}+\tfrac{1}{2}N\ln\frac{\beta}{\pi}+\tfrac{1}{2}\sum_p \ln\tilde{G}(\mathbf{p})$$
$$+\tfrac{1}{2}\sum_p{}' \ln[2\pi\tilde{D}(\mathbf{p})]+\tfrac{3}{4}(N-1)+N\Omega^\dagger. \quad (3.48)$$

H. Further Discussion of the z Integral

In Sec. II D, it was shown that if one were to perform the z integration in Eq. (2.30) for the partition function

[16] The λ introduced here is similar to the usual scaling of strength of interaction. Such a parallelism is induced by the change of variables $\varphi_i'=\lambda\varphi_i$, $\tau'\to\tau/\lambda$.

[17] A. Klein, Phys. Rev. **121**, 950 (1961).

by a direct saddle-point procedure, the saddle point would have to lie to the left of $\tilde{v}_0$. This conclusion was drawn from examination of

$$u=-\tfrac{1}{2}J(z_S+\tfrac{1}{2}\beta^{-1}). \qquad (2.37)$$

After considering the Luttinger-Ward equation (3.48) one might conclude that the singularity of Ω does not retain any trace of the singularity of each of the terms in its perturbation series. The terms of the perturbation series have a singularity at $z=\tilde{v}_0$ due to this singularity in G^0. One expects that the comparable singularity of Ω is at the singularity of G, i.e., where

$$z_0-\tilde{v}_0+\tilde{M}(\mathbf{p}=0,\, z_0)=0. \qquad (3.49)$$

We expect this to be the same point at which

$$\tilde{P}(\mathbf{p}=0,\, z_0)=0. \qquad (3.50)$$

If the point z_0 defined by Eqs. (3.49) and (3.50) lies to the left of $\tilde{v}_0$, one might speculate that this accounts for z_S being to the left of $\tilde{v}_0$. This is probably not the explanation, however, since from Eq. (2.37) we can also conclude that

$$\frac{dz_S}{d\beta}=\frac{1}{2\beta^2}-\frac{2}{J}\frac{\partial u}{\partial\beta}>0. \qquad (3.51)$$

This means that z_S moves from left to right with decreasing temperature (unlike the spherical model). The critical phenomena cannot be due to the saddle point moving to the left to strike z_0.

There is another type of saddle-point integration which might be valid for the z integral and is important for the ideal case (Appendix B). Imagine that the singularities of $e^{N\Omega(z)}$ lie to the left of $\tilde{v}_0$ on the real axis. Distort the z contour to run from $-\infty-i\epsilon$, counterclockwise around $\tilde{v}_0$, to $-\infty+i\epsilon$. Then the partition function can be written

$$Z=\int_{-\infty}^{\tilde{v}_0} dz\, e^{N\beta z}e^{N\Upsilon(z)}, \qquad (3.52)$$

$$e^{N\Upsilon(Z)}=\lim_{\alpha\to 0}(e^{N\Omega(z-i\alpha)}-e^{N\Omega(z+i\alpha)}). \qquad (3.53)$$

The saddle z would be determined by

$$\beta+\Upsilon'(z_S)=0, \qquad (3.54)$$

and one can show in the previous manner that the z_S must be to the left of $\tilde{v}_0$ and move from left to right with decreasing temperature. It is difficult under these circumstances to see what analytic feature accounts for the critical point, if we continue to assume that the critical feature is associated with the zero of Eqs. (3.49) and (3.50). One possibility is that the singularity in z of $\tilde{G}$ for $\mathbf{p}=0$ is to the left of the singularities for nonzero $\mathbf{p}$. This is the opposite of the behavior of $\tilde{G}^0$.

Until we have a clear idea of how the z integrations are performed, it is impossible to confirm the suspicion that $G_{mn}(z)$ has the same r_{nm} dependence as the spin-spin correlations, or that $D_{ij}(z)$ has the same r_{ij} dependence as the energy-energy correlations.

IV. SCHWINGER FORMULATION

An alternative formulation of field theory is that due to Schwinger,[2] involving functional derivatives. To examine the analog of this approach, we return to the Ising model with generalized coupling, Eq. (2.15). An Ω^* may be defined by

$$e^{N\Omega^*(z)}=-iN^{1/2}(\beta/2\pi)^{N/2}|z\mathbf{1}-\mathbf{V}|^{-1/2} \times\exp(z\sum_i \xi_i)[\exp(\sum_i \xi_i\varphi_i)\mathcal{S}]_\varphi. \qquad (4.1)$$

The $\langle\ \rangle_\varphi$ operation is generalized to

$$\langle\mathcal{A}\rangle_\varphi{}^*=[\mathcal{A}\exp(\sum_i \xi_i\varphi_i)\mathcal{S}]_\varphi/[\exp(\sum_i \xi_i\varphi_i)\mathcal{S}]_\varphi. \qquad (4.2)$$

From the Green's function (3.13) we obtain for $G_{mn}{}^*\equiv\langle\mathcal{G}_{mn}\rangle_\varphi{}^*$

$$\sum_k (G^0)^{-1}{}_{mk}G_{kn}{}^*+\langle\varphi_m\mathcal{G}_{mn}\rangle_\varphi{}^*=\delta_{mn}. \qquad (4.3)$$

The term $\langle\varphi_k\mathcal{G}_{kn}\rangle_\varphi{}^*$ in Eq. (4.3) may be replaced by a ξ derivative operation, since

$$\frac{\partial}{\partial\xi_k}G_{ij}{}^*=\langle\varphi_k\mathcal{G}_{ij}\rangle_\varphi{}^*-\Phi_kG_{ij}{}^*, \qquad (4.4)$$

with

$$\Phi_k\equiv\langle\varphi_k\rangle_\varphi{}^*. \qquad (4.5)$$

One obtains the Schwinger equation[2] for the Green's function:

$$\sum_k (G^0)^{-1}{}_{mk}G_{kn}{}^*+\left(\Phi_m+\frac{\partial}{\partial\xi_m}\right)G_{mn}{}^*=\delta_{mn}. \qquad (4.6)$$

For large distances, if $G_{kn}{}^*$ can be treated as a continuous function of the variable r_k, then the first term may be approximated by[5] (for an s-dimensional isotropic lattice)

$$\sum_k (G^0)^{-1}{}_{mk}G_{kn}{}^*\equiv\sum_k (z-v_{mk})G_{kn}{}^* \qquad (4.7)$$

$$\equiv(z-\tilde{v}_0)G^*(\mathbf{r}_m,\mathbf{r}_n) -\sigma^2\nabla_{r_m}{}^2G^*(\mathbf{r}_m,\mathbf{r}_n)+\cdots, \qquad (4.8)$$

where

$$\sigma^2=\frac{1}{2s}\sum_j |\mathbf{r}_i-\mathbf{r}_j|^2v_{ij}. \qquad (4.9)$$

The Schwinger equation, through second derivative terms, assumes the more familiar form

$$\left((z-\tilde{v}_0)-\sigma^2\nabla^2+\Phi(\mathbf{r})+\frac{\partial}{\partial\xi(\mathbf{r})}\right) \times G^*(\mathbf{r},\mathbf{r}',\{\xi\})=\delta(\mathbf{r}-\mathbf{r}'). \qquad (4.10)$$

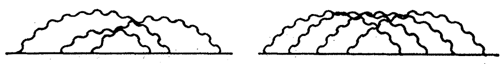

FIG. 14. Diagrams with no inelastic scattering of D with the vacuum.

One sees that $\xi(\mathbf{r})$ (or ξ_m, if we continue to work at the level of discrete variables) plays the role of a charge density (scalar component of a four-current) and $\Phi(\mathbf{r})$ acts like the conjugate potential of the field which is induced.

One frequently eliminates the current in favor of the field. This can be done by noting that

$$D_{ij}^* \equiv -\langle \varphi_i \varphi_j \rangle_\varphi^* = -\partial \Phi_j / \partial \xi_i \tag{4.11}$$

so that

$$\frac{\partial}{\partial \xi_i} = -\sum_j D_{ij}^* \frac{\partial}{\partial \Phi_j}. \tag{4.12}$$

Thus this Schwinger equation also takes the form

$$\sum_k (G^0)^{-1}{}_{mk} G_{kn}^* + \Phi_m G_{mn}^* - \sum_k D_{mk}^* \frac{\partial G_{kn}^*}{\partial \Phi_k} = \delta_{mn}. \tag{4.13}$$

The field equations for Φ_i and D_{ij}^* have an interesting aspect in this formulation, too. Such field equations are derived in field theory by use of Bose commutation relations. Here we must introduce an alternative theoretical technique. Consider the differential operator

$$\left(\frac{\partial}{\partial \varphi_m}\right)' \equiv \sum_p{}' \frac{\partial \tilde{\varphi}_p}{\partial \varphi_m} \frac{\partial}{\partial \tilde{\varphi}_p} = N^{-1/2} \sum_p{}' e^{i\mathbf{p}\cdot\mathbf{r}_m} \frac{\partial}{\partial \tilde{\varphi}_p}. \tag{4.14}$$

If one has a function $F(\varphi_1 \cdots \varphi_N)$, then

$$\left(\frac{\partial}{\partial \varphi_m}\right)' F = \frac{\partial F}{\partial \varphi_m} - \frac{1}{N} \sum_j \frac{\partial F}{\partial \varphi_j}. \tag{4.15}$$

Since $(\partial/\partial\varphi_m)'$ is a sum of derivatives $\partial/\partial\tilde{\varphi}_p$, the quantity

$$\langle (\partial/\partial\varphi_m)' \rangle_\varphi^* \equiv \left[\left(\frac{\partial}{\partial \varphi_m}\right)' [\exp(\sum_i \xi_i \varphi_i) \mathcal{S}] \right]_\varphi \Big/ [\exp(\sum_i \xi_i \varphi_i) \mathcal{S}]_\varphi \tag{4.16}$$

vanishes. Explicit performance of the operation on the right-hand side, using Eq. (4.15), yields

$$0 = \sum_j (D^0)^{-1}{}_{mj} \Phi_j - \tfrac{1}{2} G_{mm}^* + \xi_m + \beta - L(z), \tag{4.17}$$

$$L(z) = \frac{1}{N} \sum_j (\tfrac{1}{2} G_{jj}^* - \xi_j) + \beta. \tag{4.18}$$

$(\mathbf{D}^0)^{-1}$ is defined in Eq. (4.20). Equation (4.18), with the exception of $\beta - L(z)$, is analogous to a Schwinger equation.[2] It should be pointed out that if we perform the z integration of Eq. (1.4), then

$$\langle L \rangle_z = \int_{\mathcal{C}} dz \frac{d}{dz} e^{N\beta z + N\Omega^*(z)} = 0, \tag{4.19}$$

so that the $L(z)$ term does not appear in an equation obtained from Eq. (4.17) by multiplying by $e^{N\beta z + N\Omega^*(z)}$ and integrating over z. If one performs this operation and allows $\epsilon \to 0$, the resulting equation merely states that $\langle \mu_m^2 \rangle_\mu = 1$.

There is a more interesting way in which the $L(z)$ term can be avoided. Up to this point we have carefully circumvented difficulties associated with the fact that $\tilde{D}^0(p)$ is a $N-1$ dimensional matrix but D_{ij}^0 is N-dimensional with rank $N-1$. $\mathbf{D}^0$ and $(\mathbf{D}^0)^{-1}$ as used are defined by

$$D^0{}_{ij} = N^{-1} \sum_p{}' \epsilon_p^{-1} e^{-i\mathbf{p}\cdot(\mathbf{r}_i - \mathbf{r}_j)}, \tag{3.18}$$

$$(D^0)^{-1}{}_{ij} = N^{-1} \sum_p{}' \epsilon_p e^{-\mathbf{p}\cdot(\mathbf{r}_i - \mathbf{r}_j)}. \tag{4.20}$$

Note that this implies

$$\sum_k D^0{}_{ik} (D^0)^{-1}{}_{kj} = \delta_{ij} - 1/N. \tag{4.21}$$

Thus Eq. (4.11) can be written

$$\Phi_i - \tfrac{1}{2} \sum_m D^0{}_{im} G_{mm}^* + \sum_m D^0{}_{im} \xi_m = 0, \tag{4.22}$$

which is an "integral" form of the usual Schwinger equation.

A field equation for D_{ij}^* is obtained by taking the derivative of Eq. (4.22) with respect to ξ_j. One obtains

$$D_{ij}^* = D^0{}_{ij} + \tfrac{1}{2} \sum_{mk} D^0{}_{im} D_{jk}^* \frac{\partial G_{mm}^*}{\partial \Phi_k}. \tag{4.23}$$

The equations for G and D may be rendered identical to the Dyson equations (3.31) and (3.29) by taking the limit of all $\xi \to 0$ and identifying[2]

$$\frac{\partial G_{mn}}{\partial \Phi_l} = -\sum_{ij} G_{mi} G_{nj} \Gamma(ij; l). \tag{4.24}$$

A final word may be said about the significance of the vertex function. Equation (4.24) may be multiplied by D_{kl} and summed on l, to yield, in conjunction with Eq. (4.12),

$$\langle \varphi_k \mathcal{G}_{mn} \rangle_\varphi = \sum_{ijl} G_{mi} G_{nj} D_{kl} \Gamma(ij; l). \tag{4.25}$$

This relation is also easily verified by drawing the diagrams for both sides. One may now return to Eq. (2.26) to show that

$$\sum_{ijl} \langle G_{mi}G_{nj}D_{kl}\Gamma(ij;\,l)\rangle_z = -2\beta\langle\mu_m\mu_n\Delta e_k\rangle_\mu + \langle\mu_m\mu_n\rangle_\mu \times[\delta_{mk}+\delta_{nk}] - (1/N)[\partial(2\beta\langle\mu_m\mu_n\rangle_\mu)\partial/\beta]. \quad (4.26)$$

V. DEVELOPMENT BY ANALOGY WITH THE TREATMENT OF THE INFRARED DIVERGENCE

In field theory the difficulties which arise in the infrared region may be handled by taking account of processes in which large numbers of low-energy photons are emitted and absorbed. Processes in which these photons undergo inelastic scattering with the vacuum to be reabsorbed as several lower-energy photons may be ignored. In the context of the present diagrams this means that we must account for processes involving multiple D lines, such as those shown in Fig. 14, but that processes which involve loops from $\mathcal{S}$, such as shown in Fig. 15, are to be ignored (except insofar as they renormalize D). From the functional-integral point of view, this is equivalent to replacing the $\tilde{\varphi}$ dependence of $\mathcal{S}$ by a Gaussian in $\tilde{\varphi}$:

$$\mathcal{S}_G = -iN^{1/2}(\beta/\pi)^{N/2}\exp[\tfrac{3}{4}(N-1)+\Omega^\dagger(z) + \tfrac{1}{2}\sum_p \ln\tilde{G}(\mathbf{p})]\exp[\tfrac{1}{2}\sum_p{}' \tilde{D}^{-1}(\mathbf{p})\tilde{\varphi}_p\tilde{\varphi}_{-p}]. \quad (5.1)$$

With this approximation and normalization factor, $\langle\varphi_i\varphi_j\rangle_\varphi$ and Ω are correctly given, and G_{mn} is approximated by the sum over the subset of diagrams indicated above.

It is interesting to note that to this approximation G is related to an "excluded-volume" random-walk problem of polymer physics. Assume that we are interested in distances $R=|\mathbf{r}_m-\mathbf{r}_n|$ sufficiently large that $\mathcal{G}$ may be approximated by solution of the differential equation[5]

$$[(z-\bar{v}_0)-\sigma^2\nabla^2+\varphi(\mathbf{r})]\mathcal{G}(\mathbf{r},\,\mathbf{r}',\,z-\bar{v}_0)=\delta(r-r'). \quad (5.2)$$

A Feynman integral may be written for

$$\hat{\mathcal{G}}(\mathbf{r},\mathbf{r}',t)=\frac{1}{2\pi i}\int_{-i\infty+\alpha}^{i\infty+\alpha} e^{st}\mathcal{G}(\mathbf{r},\mathbf{r}',s)ds, \quad (5.3)$$

and is

$$\mathcal{G}(\mathbf{r},\mathbf{r}',t)=\mathfrak{N}\int_{\mathbf{r}'}^{\mathbf{r}}\delta\mathbf{r}(\tau) \times\exp\left(-\frac{1}{4\sigma^2}\int_0^t|\dot{\mathbf{r}}(\tau)|^2d\tau-\int_0^t\varphi[\mathbf{r}(\tau)]d\tau\right). \quad (5.4)$$

The paths $\mathbf{r}(\tau)$ go from $\mathbf{r}'$ to $\mathbf{r}$ in "time" t; $\mathfrak{N}$ is a normalization constant. The function φ now appears exclusively as a linear term in the exponent, so that with the Gaussian $\mathcal{S}_G$ the φ functional integral may be performed.

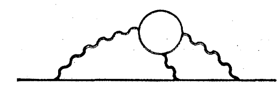

FIG. 15. A diagram with an elastic scattering of D with the vacuum.

The result for the inverse Laplace transform of the averaged Green's function is

$$\hat{G}(R,t)=\mathfrak{N}\int_0^{\mathbf{R}}\delta\mathbf{r}(\tau)\exp\left(-\frac{1}{4\sigma^2}\int_0^t d\tau|\dot{\mathbf{r}}(\tau)|^2 -\frac{1}{2}\int_0^t\int_0^t d\tau\,d\tau'\,D[|\mathbf{r}(\tau)-\mathbf{r}(\tau')|]\right). \quad (5.5)$$

This may be interpreted physically as the canonical ensemble probability density that a walk which starts at a time zero at the origin will be at $\mathbf{R}$ at time t if the walk is random except for a repulsive energy $D(r)$ between segments separated by a distance r.

The infrared divergence of the electron Green's function may also be treated as such an interacting walk problem, where the interaction is the time-dependent Coulomb potential. Because of the long-range nature of Coulomb forces it appears to be appropriate, to leading order, to use as the weighting factor an interaction D, averaged over free random walks.[11] For short-range interactions it is necessary to use a more self-consistent averaging scheme, as in the work of Edwards.[6] Neither of these approaches appears to yield meaningful results in application to this Ising problem if it is assumed that D_{ij} has the same r_{ij} dependence as the energy-energy correlations.

VI. CONCLUSIONS

A great deal of our physical intuition about many-body systems is due to our ability to think of the fundamental processes in quasiparticle or collective-mode terms. To the degree that the results presented here cast the Ising model into this familiar form they may be of value. It appears to be possible to create a parallel formalism, and hence a similar interpretation, of other many-body systems[18,19] for which the appropriate quasiparticles may or may not be clear.

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APPENDIX A: $\{\tilde{\varphi}\}'$ INTEGRATION

For the sake of actually performing the integrals over the set $\{\tilde{\varphi}\}'$, it is useful to change to sine and cosine

[18] A. J. F. Siegert, Physica Suppl. **26**, S30 (1960); R. Hirota, Ph.D. thesis, Northwestern University, Evanston, Ill., 1961 (unpublished).

[19] As a nontrivial example, L. Dworin and the author have constructed the preliminary formalism for the Anderson magnetic-impurity model.

transform variables defined by (assuming the number of sites in each direction to be odd, to avoid modes at the vertices of the Brillouin zone)

$$\varphi_p = i/\sqrt{(2)}[\tilde{X}_p{}^c + i\tilde{X}_p{}^s], \quad \text{(A1)}$$

$$\varphi_{-p} = i/\sqrt{(2)}[\tilde{X}_p{}^c - i\tilde{X}_p{}^s]. \quad \text{(A2)}$$

The Jacobian of the transformation from the $\tilde{\varphi}$ to the $\tilde{X}$ is i^{N-1}. The **p** label for the $\tilde{X}$'s runs over half the Brillouin zone excluding zero. The $\tilde{X}$ contours are from $-\infty$ to ∞.

As an application, consider

$$i^{-N+1}\left[\prod_p{}'\left(\frac{\epsilon_p}{2\pi}\right)^{1/2}\right]\int d\{\tilde{\varphi}\}'\exp(\tfrac{1}{2}\sum_p{}'\epsilon_p\tilde{\varphi}_p\tilde{\varphi}_{-p})$$

$$=\left[\prod_p{}'\left(\frac{\epsilon_p}{2\pi}\right)^{1/2}\right]\int_{-\infty}^{\infty}(\prod_p{}'' d\tilde{X}_p{}^c d\tilde{X}_p{}^s)$$

$$\times\exp\{-\tfrac{1}{2}\sum_p{}''\epsilon_p[(\tilde{X}_p{}^c)^2+(\tilde{X}_p{}^s)^2]\}=1. \quad \text{(A3)}$$

APPENDIX B: IDEAL SYSTEM

The noninteracting lattice model is a special case which is illustrative of some of the analytic features which enter in the type of formulation introduced in this paper. It is trivial to handle the ideal case by Eq. (2.12). If one wishes to make a formulation paralleling the general case, it is necessary to include a convergence parameter ζ in the definition of $\Omega(z)$. Only let $\zeta\to 0$ after the z integration of Eq. (2.30) (actually, ζ may be set to zero after the z contour is distorted). We define Ω by (a more convenient convergence factor than S^0 is employed)

$$e^{N\Omega}=(N^{1/2}/i)(\beta/\pi)^{N/2}J, \quad \text{(B1)}$$

$$J=i^{1-N}\int d\{\tilde{\varphi}\}'[\prod_i(z+\varphi_i)^{-1/2}]\exp(\zeta\sum_{i=2}^N\varphi_i). \quad \text{(B2)}$$

A $\tilde{\varphi}_0$ term may be added to the definition of φ_i and a $\tilde{\varphi}_0$ integration included by means of a δ function in integral representation. Then,

$$J=\frac{N^{1/2}}{2\pi i^N}\int_{-\infty}^{\infty}dk\int d\{\tilde{\varphi}\}\prod_i(z+\varphi_i)^{1/2}$$

$$\times\exp(\zeta\sum_{i=2}^N\varphi_i+N^{1/2}k\tilde{\varphi}_0), \quad \text{(B3)}$$

where φ_i now has a $\tilde{\varphi}_0$ term, unlike Eq. (2.30). A change of variables back to φ_i, $i=1\cdots N$, may be made, in which form the φ integration can be performed:

$$J=\frac{N^{1/2}}{2\pi i^N}\int_{-\infty}^{\infty}dk\left(\int_{-i\infty}^{i\infty}d\varphi_1\,(z+\varphi_1)^{-1/2}e^{k\varphi_1}\right)$$

$$\times\left(\int_{-i\infty}^{i\infty}d\varphi\,(z+\varphi)^{-1/2}e^{(k+\zeta)\varphi}\right)^{N-1} \quad \text{(B4)}$$

$$=\frac{N^{1/2}}{2\pi}(2\pi^{1/2})^N e^{\zeta z}\int_0^{\infty}k^{-1/2}(k+\zeta)^{(1-N)/2}e^{-N(\zeta+k)z}dk. \quad \text{(B5)}$$

The k integration will be performed by the change of variables $q=k+\zeta$ and expansion of $(q=\zeta)^{-1/2}$. Integration by parts yields

$$J=\frac{N^{1/2}}{2\pi}(2\pi^{1/2})^N e^{\zeta z}\sum_{j=0}^{\infty}\frac{\Gamma(\frac{1}{2}+j)(-\zeta)^j}{\Gamma(\frac{1}{2})j!}$$

$$\times\left(e^{-N\zeta z}\sum_{l=1}^{\frac{1}{2}N+j-m}\frac{(-1)^l\Gamma(\frac{1}{2}N+j-l)(Nz)^{l-1}}{\Gamma(\frac{1}{2}N+j)\zeta^{\frac{1}{2}N+j-l}}\right.$$

$$\left.+\frac{(-1)^{\frac{1}{2}N+j-m}\Gamma(m)(Nz)^{\frac{1}{2}N+j-1}}{\Gamma(\frac{1}{2}N+j)}\int_{N\zeta z}^{\infty}\frac{e^{-q}}{q^m}dq\right) \quad \text{(B6)}$$

$=1,$ N even

$=\frac{1}{2},$ N odd.

The important point to notice is that the terms of the l summation represent an entire function. The z integration of $e^{N\beta z}$ times these terms yields zero by closing the contour in the left half-plane. Thus the divergence for $\zeta\to 0$ is avoided. For the remaining terms, only $j=0$ survives.

$J(z)$ has a branch point at $z=0$, logarithmic for N even, and half-power for N odd. Distort the z contour to $-\infty$, counterclockwise around zero, to $-\infty$, as in Sec. III H. We find that

$$e^{N\Upsilon(z)}=2^N(N\beta)^{N/2}(-z)^{1/2N-1}/\Gamma(\tfrac{1}{2}N), \quad \text{(B7)}$$

which yields $Z=2^N$. It is of interest to note that for this ideal case $e^{N\beta z+N\Upsilon(z)}$ has a saddle point at $z_S=-1/2\beta$.