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Spin Waves in a Degenerate Electron Liquid*

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A formal procedure for calculating the dynamic spin susceptibility of an electron liquid is presented. The spin-dependent oscillations of the system are determined by requiring that the solution of the kinetic equation be consistent with Maxwell's equations. When the small coupling between spin waves and plasma waves is neglected, the condition for spin wave propagation is essentially equivalent to the requirements that the ac magnetic induction vanishes. By making use of this condition in the kinetic equation, we are led to a simple secular equation, valid for arbitrary values of the wavelength of the spin-dependent oscillations. Analytic results are presented for the long wavelength limit of all the spin wave modes.

I, INTRODUCTION

In his classic paper on the oscillations of a degenerate electron liquid Silin[1] first predicted the existence of collective oscillations of the spin magnetization of electrons in nonferromagnetic metals. Recently, Schultz and Dunifer[2] observed a series of side bands in the study of conduction-electron spin resonance of sodium and potassium, and attributed this phenomena to the spin-wave excitation. Platzman and Wolff[3] have analyzed the experimental results with the aid of Landau's Fermi-liquid theory. The agreement between their theory and the experiment data is surprisingly good. However, the method of Platzman and Wolff applied only to simple metals with spherical Fermi surfaces such that the Fermi-liquid interaction function[4] $\psi(\vec{p}, \vec{p}')$ depends only on the angle between $\vec{p}$ and $\vec{p}'$. In addition, their analysis is restricted to long wavelengths such that $qr_c \ll 1$, where q is the wave number and r_c the cyclotron radius. In view of the fact that there is a possibility of observing spin waves in metals with more complicated Fermi surfaces and also at shorter wavelengths, it is desirable to have a more general theory of these spin-wave excitations. The present authors have recently discussed[5] a method of analysis which is valid for arbitrary value of the wavelength. This calculation was restricted to the very simple case in which the Fermi liquid interaction function $\psi(\vec{p}, \vec{p}')$ can be approximated by a constant. Though this is indeed an over simplified model, it is of value in two respects. First, the analysis of Platzman and Wolff[3] indicated that the dominant Fermi liquid effects on spin properties of an electron gas are retained even when only the S-wave interaction is included. This is quite different from the spin-independent case, where[6] we have seen that the procedure of keeping A_0 and A_1 only leads to essentially no change in the propagation of plasma waves. The reason for this is that A_0 is associated with the charge-density distribution and A_1 is associated with current density. There exists a huge restoring force whenever these are out of balance, while there is no corresponding analog in spin properties. Second, the calculation can be carried out for an arbitrary-shaped Fermi surface and thus could be used as a first approximation for the analysis of spin waves in these materials.

In this paper we apply the general method discussed in Ref. 5 to a spherical Fermi surface, for which the function $\psi(\vec{p}, \vec{p}')$ depends only on the angle between $\vec{p}$ and $\vec{p}'$. Instead of retaining only the S-wave interactions, the full interaction function is taken into account. We present in Sec. I the formal method of calculating the dynamic spin susceptibility. In Sec. II, we discuss the propagation of spin waves. If we neglect the small coupling between spin waves and plasma waves, then the condition for spin-wave propagation is essentially $|\overleftrightarrow{\alpha}^{-1}| = 0$, where the tensor $\overleftrightarrow{\alpha}$ relates the dynamic spin magnetization $\vec{m}(\vec{q}, \omega)$ to the ac magnetic induction $\vec{b}(\vec{q}, \omega)$:

$$\vec{m}(\vec{q}, \omega) = \overleftrightarrow{\alpha}(\vec{q}, \omega) \circ \vec{b}(\vec{q}, \omega) \quad . \tag{1.1}$$

Thus the condition for spin-wave propagation in metals is equivalent to setting $\vec{b}(\vec{q}, \omega)$ equal to zero. The kinetic equation under this condition can be transformed into an infinite-size determinantal equation. The various solutions to this equation corresponding to spin wave modes can each be labelled by a pair of subscripts (n, m) correspond to a particular spherical harmonic. General expressions for the elements of the matrix are presented. In Sec. III, we discuss the solutions of the determinantal equation in certain limiting cases. In particular, we present an expression, valid in the long-wavelength limit, for the frequencies $\omega_{n,m}$ of all the different (n, m) modes. The dispersion relation for these modes are extremely similar to the plasma wave modes discussed in the preceding paper. Our method indicates clearly how the various modes are mixed at long wavelengths and reproduces easily all the results that are obtained by perturbation theory.[3] The dispersion relation presented in Sec. III is readily accessible to numerical calculation if only a few terms in the expansion of the interaction function need be kept.

II. DYNAMIC SPIN SUSCEPTIBILITY

We consider our system as a collection of interacting electrons immersed in a neutralizing uniform background of positive charge. A strong dc magnetic field $\vec{B}$ is applied in the Z direction. When a perturbing field of the form

$$\vec{b}(\vec{r}, t) = \vec{b}e^{i(\vec{q}\cdot\vec{r} - \omega t)}$$

is applied to the system, the deviation of the distribution function is represented as $\delta f + \delta\vec{g}\cdot\vec{\sigma}$ and the change in quasi-particle energy is

$$\delta\epsilon_1 + \delta\vec{\epsilon}_2\cdot\vec{\sigma} - \gamma_0\vec{\sigma}\cdot\vec{b}(\vec{r}, t),$$

where $\delta\epsilon_1$ is related to δf and $\delta\vec{\epsilon}_2$ related to $\delta\vec{g}$. According to Landau's theory, the behavior of the system under this condition is described by the famous Landau-Silin transport equations. We have seen in Ref. 6 that the Landau-Silin equations can be separated into two equations, one for the spin-antisymmetric distribution function $\delta\vec{g}$ and another for the spin-symmetric one δf. Now we introduce the following definitions,

$$\delta g_x + i\delta g_y \equiv -g\frac{\partial\rho_0}{\partial\epsilon_{qp}}, \tag{2.1}$$

$$2\gamma B \equiv \Omega_0, \tag{2.2}$$

$$\delta\epsilon_2 \equiv \delta\epsilon_{2x} + i\delta\epsilon_{2y}, \tag{2.3}$$

and

$$b_+ \equiv b_x + ib_y. \tag{2.4}$$

The equation for g can then be written as[6]

$$-i\omega g + \left(i\vec{q}\cdot\vec{v} + \frac{e}{c}(\vec{v}\times\vec{B})\cdot\frac{\partial}{\partial\vec{p}} + i\Omega_0\right)(g + \delta\epsilon_2)$$
$$= \frac{1}{2}\left(-\frac{\partial\rho_0}{\partial\epsilon_{qp}}\right)^{-1}\mathrm{Tr}\sigma_+ I(n) + (\vec{v}\cdot\vec{\nabla} + i\Omega_0)\gamma_0 b_+. \tag{2.5}$$

Here, q is taken in X-Z plane. In writing down these equations, we have followed the notation used in Refs. 3, 5, and 6. Because of the factor $(-\partial\rho_0/\partial\epsilon_{qp})$, all quantities in Eq. (2.5) are understood to be evaluated on the Fermi surface at sufficiently low temperatures. The collision term in Eq. (2.5), following the treatment in Ref. 6, can easily be written in terms of the spherical harmonic expansion coefficients, i.e.,

$$\frac{1}{2}\left(-\frac{\partial\rho_0}{\partial\epsilon_{qp}}\right)^{-1}\mathrm{Tr}\sigma_+ I(n)$$
$$= -\frac{1}{\tau}(g + \delta\epsilon_2) + \sum_{l,\,|m|<l} g_l^{\,m} Y_l^{\,m}/\tau_l. \tag{2.6}$$

We note here that τ_0 does not have any particular relation to τ. The integral of the collision term over momentum space need not vanish because of the possibility of spin-flip collisions. For the case of a spherical Fermi surface, the interaction function $\Psi(\vec{p}, \vec{p}')$ between two quasi-particles can be expanded in a series of Legendre polynomials as

$$\Psi(\vec{p}, \vec{p}') = \sum_l B_l \frac{(2l+1)\pi^2}{m^* p_f} P_l(\cos\xi). \tag{2.7}$$

Here ξ is the angle between $\vec{p}$ and $\vec{p}'$. The B_l's are defined in such a way as to make them dimensionless. We now introduce the position vector $\vec{R}(p_z, \phi)$ of an electron on the Fermi-surface. We can divide R into a periodic and a secular part, $\vec{R} = \vec{R}_p + \vec{R}_s$, where

$$\vec{R}_p(p_z, \phi + 2\pi) = \vec{R}_p(p_z), \tag{2.8}$$

and

$$\vec{R}_s = \vec{v}_z(p_z)\phi/\omega_c. \tag{2.9}$$

The function $g e^{i\vec{q}\cdot\vec{R}_p}$ is also a periodic function of ϕ with a period 2π. Hence it can be expanded as

$$g e^{i\vec{q}\cdot\vec{R}_p} = \sum_l g_l(p_z) e^{il\phi}. \tag{2.10}$$

The function $e^{i\vec{q}\cdot\vec{R}_p}$ and $\vec{v}e^{i\vec{q}\cdot\vec{R}_p}$ can be similarly expanded and their Fermi components have been shown to be[7]

$$e^{i\vec{q}\cdot\vec{R}_p} = \sum_m J_m e^{im\phi}, \tag{2.11}$$

and

$$\vec{v}e^{i\vec{q}\cdot\vec{R}_p} = \sum_l v_f \begin{pmatrix} \frac{1}{2}\sin\theta(J_{l-1} + J_{l+1}) \\ -\frac{1}{2}i\sin\theta(J_{l-1} - J_{l+1}) \\ J_l\cos\theta \end{pmatrix} e^{il\phi}. \tag{2.12}$$

The argument of all the Bessel functions appearing in Eq. (2.12) is $X \equiv q_x v_f/\omega_c$. The kinetic equation can then be written in terms of the Fourier components as

$$g_m = \frac{-\Omega_0\gamma_0 J_m - \gamma_0 \vec{q}\cdot\vec{v}_m - Y_m}{\tilde{\omega} - q_z v_z - \Omega_0 - \omega_c} \quad , \qquad (2.13)$$

where $\tilde{\omega} = \omega + i/\tau$ and Y_m is defined as the integral

$$Y_m = (2\pi)^{-1}\int e^{-im\phi} d\phi\, e^{i\vec{q}\cdot\vec{R}_p} \times\left[\left(\frac{i}{\tau} - \vec{q}\cdot\vec{v} + i\omega_c \frac{\partial}{\partial\phi} - \Omega_0\right)\delta\epsilon_2 + \sum_{l,\,|s|<l} i g_l^{\,s} Y_l^{\,s}/\tau_l\right] . \qquad (2.14)$$

With the aid of Eq. (2.7) and the expression

$$\delta\epsilon_2 = 2\sum_{\vec{p}\,'}\psi(\vec{p},\vec{p}\,')\delta g(\vec{p}\,')$$

given in Ref. 6, the quantity Y_m can easily be expressed in terms of the Fourier components of the distribution function g_m as

$$Y_m = \sum_{l,\,|s|<l} B_l\left(\frac{i}{B_l\tau_l} + \frac{i}{\tau} - q_z v_z - \Omega_0 - m\omega_c\right) \times \theta_l^{\,s} J_{m-s}(2\pi)^{-\frac{1}{2}} g_l^{\,s} , \qquad (2.15)$$

$\theta_l^{\,m}$ being the Legendre polynomial normalized such that

$$\int_{-1}^{+1}(\theta_l^{\,m})^2 d(\cos\theta) = 1 .$$

Substitution of Eq. (2.15) into Eq. (2.13) then gives an equation for each of the Fourier component g_m, i.e.,

$$g_m = \frac{-(\Omega_0\gamma_0 J_m + \gamma_0\vec{q}\cdot\vec{v}_m)b_+}{\tilde{\omega} - q_z v_z - \Omega_0 - \omega_c} + \sum_{l',\,|s'|<l}\left(-1 + \frac{\tilde{\omega}_{l'}}{\tilde{\omega} - q_z v_z - \Omega_0 - m\omega_c}\right) \times B_{l'}\theta_{l'}^{\,s'} J_{m-s'}(2\pi)^{-\frac{1}{2}} g_{l'}^{\,s'} . \qquad (2.16)$$

In Eq. (2.16), $\tilde{\omega} \equiv \omega + i/\tau$ and $\tilde{\omega}_{l'} = \omega + i/B_{l'}\tau_{l'}$. Equation (2.16) looks as complicated as the original transport equation. However, we note that all the effects of the interaction have been grouped into the second term on the right hand side and the form of Eq. (2.16) is most convenient for an approximate calculation keeping only the first few terms in the expansion of $\psi(\vec{p},\vec{p}\,')$. Equation (2.16) can be converted into a system of algebraic equations by the following procedure. First, it follows from Eq. (2.10) and Eq. (2.11) that the relation between g_m and $g_l^{\,s}$ is given by

$$g_l^{\,s} = (2\pi)^{\frac{1}{2}}\sum_m \int_{-1}^{+1}\theta_l^{\,s} J_{m-s} g_m\, d(\cos\theta). \qquad (2.17)$$

Then we substitute Eq. (2.17) into (2.16) multiply both sides by $\theta_l^{\,s} J_{m-s}$ and sum over m. The result can be easily expressed in a matrix form as

$$(\overleftrightarrow{R} - \overleftrightarrow{I})\overleftrightarrow{G} = \overleftrightarrow{H} . \qquad (2.18)$$

The elements of the matrices $\underline{R}$ and $\underline{I}$ are

$$r_{ll'}^{\,ss'} = \sum_m \int_{-1}^{+1} d(\cos\theta) J_{m-s} J_{m-s'}\theta_l^{\,s}\theta_{l'}^{\,s'} \times B_{l'}\left(-1 + \frac{\tilde{\omega}_{l'}}{\tilde{\omega} - q_z v_z - \Omega_0 - m\omega_c}\right) ,$$

and $I_{ll'}^{\,ss'} = \delta_{ll'}\delta_{ss'}$.

$\vec{G}$ and $\vec{H}$ are column vectors with components $G_l^{\,s}$ and $H_l^{\,s}$ respectively, where $H_l^{\,s}$ is given by the expression

$$H_l^{\,s} = b_+ \int\sum_m \theta_l^{\,s} J_{m-s} \times \frac{(\Omega_0\gamma_0 J_m + \gamma_0\vec{q}\cdot\vec{v}_m)(2\pi)^{1/2}}{\tilde{\omega} - q_z v_z - \Omega_0 - m\omega_c} d(\cos\theta) . \qquad (2.19)$$

To obtain the spin susceptibility, the next task is to relate the spin magnetization to the distribution function. The magnetization is obtained by taking the trace of the product of the Pauli spin matrix and the distribution function and integrating over all momentum space, i.e.,

$$\vec{m} = \gamma_0 \mathrm{Tr}\int \vec{\sigma}(\delta f + \delta\vec{g}\cdot\vec{\sigma})d^3p/(2\pi)^3 . \qquad (2.20)$$

The combination $m_+ = m_x + im_y$ is therefore related only to the spherical average of g by the relation

$$m_+ = (2\pi)^{-3} 2m^* p_f g_0^{\,0} . \qquad (2.21)$$

We can easily see from Eq. (2.18) that $g_0^{\,0}$ is proportional to b_+. Hence it follows from Eq. (2.21) that m_+ is simply proportional to b_+ with a scalar proportionality constant. A formal solution for $g_0{}^0$ can easily be obtained from Eq. (2.18). Introducing the definition $\overleftrightarrow{U} \equiv b_+{}^{-1}\overleftrightarrow{H}$, $g_0{}^0$ can be expressed as

$$g_0{}^0/b_+ = |\overleftrightarrow{R}^{00}|/|\overleftrightarrow{R}| \quad . \tag{2.22}$$

Here $|\overleftrightarrow{R}^{00}|$ is the determinant of the matrix obtained by replacing the (0, 0)th column of $\overleftrightarrow{R}$ by the column vector $\overleftrightarrow{U}$, and $|\overleftrightarrow{R}|$ is the determinant of the matrix $\overleftrightarrow{R}$. Combining Eq. (2.21) and Eq. (2.22), we finally obtain an expression for α_+ as

$$\alpha_+ = (\gamma_0/4\pi^3)|\overleftrightarrow{R}^{00}|/|\overleftrightarrow{R}| \quad . \tag{2.23}$$

Since $b_+ = h_+ + 4\pi m_+$, m_+ is also proportional to the magnetic field h_+ with a simple scalar proportionality constant χ_+, given by

$$\chi_+(q,\omega) = \alpha_+(q,\omega)/[1 - 4\pi\alpha_+(q,\omega)] \; .$$

This is just the combination $\chi_{xx} - i\chi_{xy}$ of the elements of the susceptibility tensor. From the equation for g_- and g_z, it can easily be seen that m_- is related to h_- and m_z to h_z in a similar way. Hence we have the important conclusion that to first order in ω_c/ϵ_f, the spin susceptibility has the following properties

$$\begin{aligned} &\chi_{xx} = \chi_{yy}, \quad \chi_{xy} = -\chi_{yx} \;, \\ &\chi_{xz} = \chi_{zx} = \chi_{yz} = \chi_{zy} = 0 \,. \end{aligned} \tag{2.24}$$

This property of the susceptibility tensor is independent of the manner in which we truncate the interaction function $\psi(\vec{p}, \vec{p}\,')$.

III. SPIN WAVES

A. General Dispersion Relation

It is well known that, in general, a propagating electric field or magnetic field with a space-time dependence of the form

$$\exp[i(\vec{q} \cdot \vec{r} - \omega t)]$$

cannot exist in the main bulk of a metal. However, at some particular frequencies and wavelengths, a kind of collective oscillation may be excited and then propagation is possible. Example of these oscillations are the helicons[8] and plasma waves discussed in Ref. 6. The condition for the existence of a finite magnetization of the form

$$\vec{m}(q,\omega) \exp[i(\vec{q} \cdot \vec{r} - \omega t)]$$

corresponds to spin-wave excitations. The dispersion relation of spin wave in metals can be easily obtained by requiring the magnetization to satisfy both the constitutive equation $\vec{b} = \overleftrightarrow{\alpha}^{-1}\vec{m}$ and a relation obtained from Maxwell's equations,[6] i.e.,

$$\vec{b} = \overleftrightarrow{\Omega} \cdot \vec{m} \;, \tag{3.1}$$

with $$\overleftrightarrow{\Omega} = \frac{4\pi c}{\omega} \overleftrightarrow{Q} \left(\frac{\overleftrightarrow{Q}^2 c}{\omega} + \frac{\omega}{c} \overleftrightarrow{I} + \frac{4\pi i \overleftrightarrow{\sigma}}{c} \right)^{-1} \overleftrightarrow{Q}, \tag{3.2}$$

and

$$\overleftrightarrow{Q} = \begin{pmatrix} 0 & -q_z & 0 \\ q_z & 0 & -q_x \\ 0 & q_x & 0 \end{pmatrix} \; . \tag{3.3}$$

The dispersion relation is therefore simply the determinantal equation

$$|\overleftrightarrow{\Omega} - \overleftrightarrow{\alpha}^{-1}| = 0 \; . \tag{3.4}$$

Since elements of σ are of order $\omega_p{}^2/\omega c$, we see readily from Eq. (3.2) and Eq. (2.23) that the elements of Ω are negligible compared with elements of $\underline{\alpha}^{-1}$ provided we are in a region of frequency and wavelengths such that

$$(\omega_p{}^2/c^2q^2)(\omega/\omega_c) \gg 1 \, .$$

The exception to this is near the solution of the equation $|\sigma| = 0$. Under this condition, elements like $\sigma_{zz}{}^{-1}$, for example are no longer of order $\omega_c/\omega_p{}^2$ but can be exceedingly large. Physically, this corresponds to excitation of plasma waves. If we are far away from the crossing point of the dispersion branches of the spin waves and plasma waves the coupling between them may be neglected and the dispersion relation for spin waves reduces then to the simple equation

$$|\overleftrightarrow{\alpha}^{-1}| = 0 \, . \tag{3.5}$$

This can be further factorized into three equations $\alpha_{+1}{}^{-1} = 0$, $\alpha_{-1}{}^{-1} = 0$, and $\alpha_z{}^{-1} = 0$. In this approximation then, the spin-wave excitations occur at the poles of the response function α_+, α_-, etc. Obviously, any component of Eq. (3.5), for example $\alpha_+{}^{-1} = 0$ implies that the corresponding component of magnetic induction b_+ vanishes for the existence of a finite magnetization m_+. Hence a much simpler way of obtaining the dispersion relation is to set b_+ or b_-, etc., equal to zero directly in the kinetic equation and look for the condition for the existence of nonvanishing values for the distribution function $\delta\vec{g}$. Let us investigate the branch corresponding to $\alpha_+{}^{-1} = 0$, for example. Setting $b_+ = 0$ in Eq. (2.18) gives the equation

$$(\overleftrightarrow{R} - \overleftrightarrow{I})\overleftrightarrow{G} = 0. \tag{3.6}$$

Nonvanishing values of g are therefore only possible if q and ω satisfy the relation

$$|\overleftrightarrow{R} - \overleftrightarrow{I}| = 0 \ . \tag{3.7}$$

This is equivalent to the equation $\alpha_+^{-1} = 0$. Other equations corresponding to $\alpha_z^{-1} = 0$ can be similarly derived.

B. Propagation Perpendicular to dc Magnetic Field $\vec{B}$

We now look at the solution of Eq. (3.7) in some limiting cases. In the geometry with $\vec{q}$ perpendicular to $\vec{B}$, i.e., $q_z = 0$, the matrix element of $\underline{R}$ has the following property. $r_{ll'}{}^{ss'}$ vanishes unless $l+s$ and $l'+s'$ are both even or both odd. Hence the secular equation separates into two independent equations: one for the odd modes and another for the even ones. Since the magnetization is only related to the spherical average of $g \equiv g_0{}^0$, we need only consider the even modes in the study of spin waves. For arbitrary value of wavelength, the exact solution of Eq. (3.7) is impossible. To extract information from Eq. (3.7), we need to make some approximation on the interaction function $\psi(\vec{p},\vec{p}\,')$. The easiest way is to truncate the series Eq. (2.7) after a finite number of terms, i.e., to set $B_n = 0$ for $n > n_0$, where n_0 is a fixed number. This will reduce Eq. (3.7) to a finite determinantal equation and the solution can be easily obtained by numerical methods. Another approximation we can make is to keep only terms linear in the parameter B_n; then Eq. (3.7) reduces to

$$1 - \sum_{n,|m|\leq n} r_{nn}{}^{mm} = 0 . \tag{3.8}$$

The sum in Eq. (3.8) is over all pairs of indices (n,m) such that $n+m$ is even. This equation can again be solved numerically with suitable choices for the values of the parameters B_n. It is only a good approximation when the inequality

$$B_n \omega_c \ll |\omega - \Omega_0 - m\omega_c|$$

is satisfied for all n and m. The situation here is very similar to that for the plasma waves discussed in Ref. 6, and the interested reader can refer to that article for the consequences of truncation of the series and also the method of solution at $X \equiv q_x v_f/\omega_c \gg 1$. At the long-wavelength limit where $X \ll 1$, a closed form of the dispersion relation can be easily obtained. Since the argument of the Bessel functions are now small, they can be expanded in a power-series form and the diagonal element is

$$r_{nn}{}^{mm} - 1 = \frac{-\omega + \beta_n(\Omega_0 + m\omega_c)}{\omega - \Omega_0 - m\omega_c} + O(X^2) . \tag{3.9}$$

In Eq. (3.9), $\beta_n \equiv 1 + B_n$, and we have dropped all collision terms for simplicity in writing. The off-diagonal elements are

$$r_{ll'}{}^{ss'} = \frac{\omega\omega_c}{(\omega - \Omega_0 - s\omega_c)(\omega - \Omega_0 - s'\omega_c)} \times X\langle {}^{s'}_{l'} | \sin\theta | {}^{s}_{l} \rangle + O(X^3) \tag{3.10}$$

for $|s - s'| = 1$: all other elements are of order of higher powers in X. In Eq. (3.10)

$$\left\langle {}^{s}_{l} | \cdots | {}^{s'}_{l} \right\rangle \equiv \int_{-1}^{+1} \theta_l{}^s \theta_{l'}{}^s \cdots d(\cos\theta) .$$

It is not difficult to see from Eq. (3.6), Eq. (3.9), and Eq. (3.10) that the solution at the long-wavelength limit of Eq. (3.7) consists of an infinite number of modes each labeled by a pair of indices (n,m). The dispersion relation of each mode is of the form

$$\omega_{nm} = \beta_n(\Omega_0 + m\omega_c) + \gamma_{nm} X^2 . \tag{3.11}$$

as $X \to 0$, the various modes are completely separated from each other. For a particular (n,m) mode, only the spherical harmonic component $g_n{}^m$ remain finite as $X \to 0$, all the other components go to zero as different powers of X. In particular, the magnetization m_+, which is proportional to $g_0{}^0$, varies with X in the (n,m) mode as

$$m_+ \propto X^n . \tag{3.12}$$

This follows from the fact that each component $g_n{}^m$ is coupled by a term of order X to $g_{n-1}{}^{m\pm1}$ and $g_{n+1}{}^{m\pm1}$. The coefficient γ_{nm} in Eq. (3.11) can be easily evaluated from Eq. (3.7). For the (n,m) mode $r_{nn}{}^{mm} - 1$ is of order X^2. Keeping terms independent of X in all other diagonal elements and terms of order X in the (n,m)th row and column, we obtain an expression for γ_{nm} as

$$\gamma_{nm} = \tfrac{1}{2} B_n \left\langle {}^{m}_{n} | \sin^2\theta | {}^{m}_{n} \right\rangle \frac{\beta_n(\Omega_0 + m\omega_c)}{\beta_n{}^2(b+m)^2 - 1} - \tfrac{1}{4}\beta_n{}^2(\Omega_0 + m\omega_c) \sum_{n'm'} \times \frac{\beta_{n'} \left|\left\langle {}^{m'}_{n'} | \sin\theta | {}^{m}_{n} \right\rangle\right|^2}{[\beta_n(b+m) - b - m'][\beta_{n'}(b+m') - \beta_n(b+m)]} . \tag{3.13}$$

Here we have introduced the symbol $b\equiv\Omega_0/\omega_c$. The sum is over $n'=n\pm1$ and $m'=m\pm1$ except for $n=m=0$, where the sum reduces to a single term $n'=m'=1$ only. It follows from (3.12) that the magnetization at the long wavelength limit is strongest in the (0, 0) mode. Hence, this is the mode easiest to excite, and it corresponds to the spin-wave excitation observed by Schultz and Dunnifer.[2] The dispersion relation for this mode is

$$\omega_{00}=\omega_s+\frac{1}{3}X^2\omega_c\beta_0\beta_1\frac{\omega_s-\beta_1\Omega_0}{\beta_1^2-(\omega_s/\omega_c)-\beta_1\Omega_0/\omega_c)^2}. \tag{3.14}$$

The frequency ω_s in Eq. (3.14) is equal to $2\gamma B\times(1+B_0)$ and is just the ESR frequency in the dc field B. Platzman and Wolff's result[3] reduces to Eq. (3.14) in the collisionless limit. These authors found a good fit with experimental results by choosing the values of the parameters as $B_0=-0.1$ and $B_1=0.2$. Equations (3.13) and (3.14) are only valid in the region

$$|\gamma_{nm}X^2|\ll|B_n(\Omega_0-m\omega_c)|.$$

This is so because we have treated the quantity

$$X^2/(\omega-\Omega_0-m\omega_c)$$

as being of order X^2. As X increases, more and more modes become mixed and the solution is extremely complicated. If the series expansion of $\psi(\vec{p},\vec{p}\,')$ converges rapidly, then the behavior of the (0, 0) mode at shorter wavelength can be approximately studied by including B_0 and B_1 only in Eq. (3.7). Equation (3.7) reduces to a 3×3 determinantal equation in this case.

C. Propagation Parallel to the dc Field B

Next we study the geometry with $\vec{q}$ parallel to $\vec{B}$. All the arguments of the Bessel functions appearing in Eq. (2.18) vanish and only the zeroth-order Bessel function survives. Thus

$$r_{nn'}{}^{mm'}\propto\delta_{mm'}$$

and modes with different values of azimuthal index are separated. Since we are only interested in modes of oscillation for which the magnetization (which is proportional to $g_0{}^0$) has a nonvanishing value, we need only consider modes like (0, 0), (1, 0), (2, 0), ..., (3, 0), etc. In a typical matrix element like $r_{nn'}{}^{mm'}$, we can now suppress the indices m and m' and fix them at the value $m=m'=0$. For an arbitrary value of wavelength, we still have to resort to any of the approximation schemes described in Sec. II. We will not go into detailed discussion again. A new feature which is absent in the geometry of the $q_z=0$ case is the possibility of Landau damping. Whenever the wave number increases to such a value that the inequality $q_zv_f>|\omega-\Omega_0|$ is satisfied, then the spin-wave excitation will be severely damped by electrons traveling in phase with the wave. In the long-wavelength limit, a diagonal element has the form

$$r_{nn}=(-\omega+\beta_n\Omega_0)/(\omega-\Omega_0)+B_n\frac{\omega\omega_c^{\,2}}{(\omega-\Omega_0)^3}\times Z^2\left\langle\begin{matrix}0\\n\end{matrix}\middle|\cos^2\theta\middle|\begin{matrix}0\\n\end{matrix}\right\rangle+O(Z^4). \tag{3.15}$$

Here $Z\equiv q_zv_f/\omega_c$. An off-diagonal element has the form

$$r_{nn'}=B_{n'}\frac{\omega\omega_c}{(\omega-\Omega_0)^2}Z\left\langle\begin{matrix}0\\n\end{matrix}\middle|\cos\theta\middle|\begin{matrix}0\\n'\end{matrix}\right\rangle+O(Z^2). \tag{3.16}$$

The dispersion relation for the $(n, 0)$ mode can then be obtained by the same method used in arriving at Eq. (13). It is

$$\omega_{n0}=\beta_n\Omega_0+\gamma_{n0}Z^2$$

with

$$\gamma_{n0}=(\beta_n\omega_c^{\,2}/B_n\Omega_0)\left\langle\begin{matrix}0\\n\end{matrix}\middle|\cos^2\theta\middle|\begin{matrix}0\\n\end{matrix}\right\rangle+\sum_{n'}\frac{\beta_n^{\,2}\omega_c^{\,2}B_{n'}\left|\left\langle\begin{matrix}0\\n\end{matrix}\middle|\cos\theta\middle|\begin{matrix}0\\n'\end{matrix}\right\rangle\right|^2}{B_n\Omega_0(B_n-B_{n'})}. \tag{3.17}$$

The sum in n' goes through $n'=n\pm1$. Owing to the way in which the expansion in powers of q_z is done, Eq. (3.17) is valid only in the rather restricted region

$$|\gamma_{n0}Z^2|\ll|B_{n'}-B_n|\Omega_0\,,$$

$$\text{and}\quad Z\omega_c\ll|B_n|\Omega_0\,. \tag{3.18}$$

When the wave-vector makes an arbitrary angle with the magnetic field, all the modes are mixed and the resulting dispersion relation became more complicated. The dependence of the spin-wave frequency on the angle between $\vec{q}$ and $\vec{B}$ can furnish useful information on the interaction function and provide further confirmation of the theory. This situation will be discussed elsewhere.[9]

IV. SUMMARY

We have presented a general method of evaluating the spin-susceptibility tensor in the Fermi-liquid theory. We have also shown how to obtain the dispersion relation of spin waves from the response functions $\overleftrightarrow{\sigma}$ and $\overleftrightarrow{\alpha}$. When the coupling between spin waves and plasma waves can be neglected, the dispersion relation can be simply obtained from the kinetic equation by first setting b_+ (or b_-, b_z) = 0. The resulting dispersion relations are very similar to those of plasma waves. Unlike the situation in plasma waves, the first two expansion coefficients of the interaction function B_0 and B_1 are not canceled out, and they are responsible for the dominant terms showing Fermi-liquid effects. It is important to note that a spin wave is primarily a many-body effect which does not exist in the absence of correlations. This was first pointed out by Platzman and Wolff[3] for the particular case of the (0, 0) mode at the long-wavelength limit. The general proof of this result can be readily obtained from Eq. (3.7). If we set all the B_n equal to 0 in Eq. (3.7), the dispersion relation reduces to $|\underline{I}| = 0$ which has no solution of any kind. In principle, all the expansion coefficients of the spin-antisymmetric part of the interaction function $\psi(\vec{p}, \vec{p}\,')$ can be determined by a study of these spin-wave excitations.

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