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Spin-Independent Oscillations of a Degenerate Electron Liquid*

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A formal procedure, valid for arbitrary wavelengths, for calculating the conductivity tensor of an electron liquid is presented. The spin-independent magnetoplasma modes are determined by requiring that the constitutive equation (solution of the kinetic equation) be consistent with Maxwell's equations. If the small coupling between plasma waves and spin waves is neglected, Maxwell's equations can be used to eliminate the ac electric and magnetic fields from the kinetic equation. The magnetoplasma modes can then be determined rather simply by solving the kinetic equation. The secular equations determining the dispersion relations for propagation both perpendicular and parallel to the dc magnetic field are given. Analytic results are given for the long wavelength limit, and numerical results are presented for the case in which the interaction function is approximated by the first-three terms in an expansion in Legendre polynomials.

I. INTRODUCTION

It is well known that the free electron theory of metals is able to account successfully for a wide variety of phenomena in metals as, for example, the deHaas–van Alphen effect, the temperature dependence of specific heat, and cyclotron resonance. However, the interaction between the electrons is far from weak. It is then natural to ask what modification of the free-electron results and what new phenomena will arise if we include this interaction properly. There has been considerable literature on a system of interacting electrons starting from microscopic many-body theory,[1] but usually these formal microscopic calculations do not lend themselves easily to comparison with experiment. In this respect, a semiphenomenological approach, first formulated by Landau, seems to be more fruitful.

In his classic papers on Fermi liquids,[2] Landau proposed a theory which applies for a system of fermions with short-range interactions as, for example, He^3. Later, Silin[3] showed that the Landau theory is also applicable, with slight modification, to a system of interacting electrons even though the Coulomb force between electrons is long range in nature. Although the Fermi liquid theory involves some *ad hoc* assumptions, it has been justified by microscopic theory.[4] Provided certain restrictions are satisfied, the theory is rigorous. In this section we present a brief outline of the Landau theory as applied to the electrons in a metal. In particular, we sketch the derivation of the Landau-Silin transport equation for an electron liquid. In Sec. II we use this transport equation to study the electrical conductivity tensor of a metal in the presence of a dc magnetic field of induction $\vec{B}$. We first consider the case in which the Fermi-liquid interaction function $\Phi(\vec{p}, \vec{p}')$ can be approximated by a constant. This simple case displays many of the important features of the method of analysis which we use, while at the same time it is relatively free of mathematical complexity. In addition, this calculation can be carried out for a Fermi surface of arbitrary shape, so that it may actually be a useful first approximation in studying some effects in metals with complicated Fermi surfaces. For the general case in which the Fermi-liquid interaction function cannot be approximated by a constant, we limit our considerations to spherical Fermi surfaces and assume that the interaction function $\Phi(\vec{p}, \vec{p}')$ depends only on the angle between $\vec{p}$ and $\vec{p}'$. The formal method for determining the conductivity tensor is discussed in detail, but calculations are presented only for the case in which $\Phi(\vec{p}, \vec{p}')$ is approximated by the first-three terms in an expansion in spherical harmonics.[5] In Sec. III we investigate the magnetoplasma modes of the electron liquid. This involves making Maxwell's equations consistent with the constitutive equations which relate the response of the system to the ac electric and magnetic fields. The general dispersion relation for the magnetoplasma modes is quite complicated. However, if we neglect the small coupling between spin waves and "plasma" waves, we can make use of Maxwell's equations from the start and considerably simplify the calculation. For the case of plasma waves we use Maxwell's equations to express the electric field $\vec{E}$ in terms of the electron current density $\vec{j}$, and then write $\vec{j}$ in terms of the distribution function. The transport equation then becomes a homogeneous equation whose nontrivial solutions are obtained by setting an infinite determinant equal to zero.[6] A similar procedure has been applied to the case of spin waves and will be reported in a later publication. We give explicit expressions for the elements of the infinite determinant whose zeros

correspond to plasma waves. The long wavelength limit of the dispersion relations for propagation perpendicular to and parallel to the dc magnetic field are determined by expanding the elements of the determinant in powers of qv_F/ω_c, where q is wave number, v_F the Fermi velocity and ω_c the electrons cyclotron frequency. Dispersion relations valid for short wavelengths ($qv_F/\omega_c \gg 1$) can be obtained by similar approximate methods. The final section of this paper contains a summary of our results and some discussion of the interpretation of experimental results.

The basic assumption of the Landau theory is that the low-lying eigenstates for the real interacting system can be obtained from certain eigenstates of the ideal noninteracting system by switching on the interaction "adiabatically". In particular, the ground state of the real system can be obtained in this manner and will also possess a sharp Fermi surface. It is well known that this assumption is not satisfied by some systems of fermions. For example, electrons in metals with attractive interaction can have a ground-state characteristic of a superconductor. This state is completely different from the free-electron case and cannot be obtained by "adiabatic switching". A system which obeys the basic assumption in Landau theory is, by definition, a normal Fermion system.

Using the same adiabatic switching procedure, we can define the concept of a quasiparticle or a quasihole which is obtained by starting with the ground state plus a particle outside or a hole inside the Fermi surface. The quasiparticles and quasiholes thus appear as elementary excitations of the real system, which, when combined, give rise to a large class of excited states. Because of the interaction, a quasiparticle will decay with a finite lifetime. Obviously, the concept is only useful if the lifetime of a quasiparticle is large compared with the time of switching and the characteristic time involved in the particular phenomena being studied. In a pure system at zero temperature, the lifetime of a quasiparticle has been shown[7] to vary as the inverse square of the energy separation from the Fermi surface. Hence, the concept of quasiparticle is useful if only low-lying excited states are involved. In what follows, we shall omit the spin index σ of a quasiparticle and include it in the momentum $\vec{p}$, unless necessary for clarity. Each excited state can then be characterized by a distribution function $n(\vec{p})$ of the quasiparticles. It should be emphasized here that the physically meaningful quantity is the departure from the equilibrium distribution $-\delta n(\vec{p})$, rather than $n(\vec{p})$ itself, because it does not make sense to define a distribution function $n(\vec{p})$ in a range of momentum where the quasiparticles are unstable. Only $\delta n(\vec{p})$ enters the final equations.

The second assumption of Landau is that the state energy is a functional of the distribution function. For $\delta n(\vec{p})$ sufficiently small, we can write the total energy of the system as

$$E[n(\vec{p})] = E_0 + \sum_p \mathcal{E}_{qp}\delta n(\vec{p}) + \tfrac{1}{2}\sum_{\vec{p}\,\vec{p}'}\Phi(\vec{p},\vec{p}')\delta n(\vec{p})\delta n(\vec{p}') + O(\delta n^3) + \cdots \quad . \tag{1.1}$$

In (1.1), $\mathcal{E}_{qp}$ is the first functional derivative of E, and $\Phi(\vec{p},\vec{p}')$ is the second derivative, etc. If $\delta n(\vec{p})$ describes a state with one extra quasiparticle of momentum $\vec{p}$, the energy of the system is given by (1.1) as $E_0 + \mathcal{E}_{qp}$. This means that the energy of a single quasiparticle of momentum $\vec{p}$ is just $\mathcal{E}_{qp}$. The gradient of $\mathcal{E}_{qp}$,

$$v_p \equiv \nabla_p \mathcal{E}_{qp} \quad , \tag{1.2}$$

is the group velocity of the quasiparticle. $\Phi(\vec{p},\vec{p}')$ in (1.1), being the second variational derivative of E with respect to $n(\vec{p})$, is symmetric in $\vec{p}$ and $\vec{p}'$. It is the interaction energy between the pair of quasiparticles with momentum $\vec{p}$ and $\vec{p}'$. We shall see later that it is this quantity which is responsible for a considerable modification of the physical properties of the system from those predicted by the free-electron theory.

So far $\delta n(\vec{p})$ has been considered as independent of position and time. Consider now a distribution of the form

$$n(\vec{p},\vec{r},t) = n^0(\vec{p}) + \delta n(\vec{p},\vec{r},t) \quad . \tag{1.3}$$

In (1.3), $n(\vec{p},\vec{r},t)$ gives the distribution in a unit volume centered at a point $\vec{r}$. Obviously, such a description works only in the regime where the uncertainty principle is of no concern. Within linear response theory, we can write, without loss of generality, the departure from the equilibrium distribution in a plane wave form as

$$\delta n(\vec{p},\vec{r},t) = \delta n(\vec{q},\omega)\exp[i(\vec{q}\cdot\vec{r} - \omega t)] \quad . \tag{1.4}$$

The uncertainty principle gives rise to an uncertainty $\hbar q$ in the momentum $\vec{p}$ and $\hbar\omega$ in the energy $\mathcal{E}_p$. If q and ω satisfy the relations,

$$qv_F \ll \mu, \quad \text{and} \quad \omega \ll \mu \quad , \tag{1.5}$$

then the uncertainty principle can be ignored.[8] In other words, Landau's theory may be applied only to a

macroscopic perturbation. It is important to bear this point in mind as we proceed. As before, the total energy is a functional $E[n(\vec{p}, \vec{r}, t)]$ of the distribution, i.e.,

$$E = E_0 + \sum_p \int d^3r \mathcal{E}_{qp} \delta n(\vec{p}, \vec{r}) + \tfrac{1}{2} \sum_{pp'} \int d^3r \int d^3r' \Phi(\vec{p}, \vec{p}', \vec{r} - \vec{r}') \delta n(\vec{p}, \vec{r}) \delta n(\vec{p}', \vec{r}') + \cdots \quad . \tag{1.6}$$

Here, we have assumed that $\mathcal{E}_{qp}$ is independent of $\vec{r}$ and that Φ is a function of $\vec{r} - \vec{r}'$ only. This is true for a translationally invariant system. For electrons in a real metal, additional complications arise because of the periodic structure. If the wavelength of the disturbance is long enough so that (1.5) is satisfied, $\delta n(\vec{r})$ will vary only slightly throughout a unit cell. One can then take average over a unit cell, and (1.6) still holds. For short range forces, $\delta n(\vec{p})$ may be considered as constant over the range of interaction, so that in (1.6) $\delta n(\vec{p}', \vec{r}')$ may be replaced by $\delta n(\vec{p}', \vec{r})$. The total energy of the system can then be written as

$$E = E_0 + \int d^3r \delta E(\vec{r}) \quad , \tag{1.7}$$

$$\text{with} \quad \delta E(\vec{r}) = \sum_p \mathcal{E}_{qp} \delta n(\vec{p}, \vec{r}) + \tfrac{1}{2} \sum_{\vec{p}\vec{p}'} \Phi(\vec{p}, \vec{p}') \delta n(\vec{p}, \vec{r}) \delta n(\vec{p}', \vec{r}) \quad . \tag{1.8}$$

$\Phi(\vec{p}, \vec{p}')$ in (1.8) is defined by the expression

$$\Phi(\vec{p}, \vec{p}') = \int d^3r' \Phi(\vec{p}, \vec{p}', \vec{r} - \vec{r}') \quad . \tag{1.9}$$

The volume Ω is assumed to be unity throughout, and the momentum $\vec{p}$ is quantized accordingly. According to (1.8), the local excitation energy of a quasiparticle of momentum $\vec{p}$ is equal to

$$\mathcal{E}(\vec{p}, \vec{r}) = \mathcal{E}_{qp} + \sum_{\vec{p}'} \Phi(\vec{p}, \vec{p}') \delta n(\vec{p}', \vec{r}) \quad . \tag{1.10}$$

The theory above is not directly applicable to electrons in metals because of the long-range nature of Coulomb interaction, which leads to divergences in expressions like (1.9). As Silin[3] has first shown, these difficulties are removed if we allow for the dynamic screening of the particle motion self-consistently. First, the departure from equilibrium will give rise to an average density fluctuation and hence to a space charge electrostatic field $E(\vec{r}, t)$ which is given by the Maxwell's equation

$$\vec{\nabla} \cdot \vec{E}(\vec{r}, t) = 4\pi e \sum_{\vec{p}} \delta n(\vec{p}, \vec{r}, t) \quad . \tag{1.11}$$

Hence, a part of the interaction between the quasiparticles can be regarded as a quasiparticle interacting with an additional applied electric field given by (1.11). The interaction between any given pair of quasiparticles is thus screened by the polarization cloud surrounding each particle, and the residual short-range correlation between them can then be treated in exactly the same manner as described in the standard Landau theory. With this new interpretation of the function $\Phi(\vec{p}, \vec{p}')$, (1.8), (1.9), and (1.10) apply to a system of charged fermions such as electrons in a normal metal as well as to uncharged fermions.

Let us now consider the transport properties of a Fermi liquid. For this purpose, it is easier to work with an expression for $\Phi(\vec{p}, \vec{p}')$ showing explicit spin dependence. We write

$$\Phi(\vec{p}, \vec{p}') = \phi(\vec{p}, \vec{p}') + \psi(\vec{p}, \vec{p}') \vec{\sigma} \cdot \vec{\sigma}' \quad . \tag{1.12}$$

As Pines and Nozieres[8] have argued, $\Phi(\vec{p}, \vec{p}')$ described by (1.12) is the most general form possible if the Fermi surface has inversion symmetry and time reversal invariance is assumed. With a dc magnetic field present, time reversal invariance is no longer valid. However, we shall simply assume that the interaction function $\Phi(\vec{p}, \vec{p}')$ is not modified by the presence of the magnetic field. This should be a good approximation if the magnetic field is weak enough so that the inequality

$$\hbar \omega_c \ll \mathcal{E}_F \tag{1.13}$$

is satisfied. In (1.13), ω_c is the usual cyclotron resonance frequency and $\mathcal{E}_F$ is the Fermi energy.

The Landau-Silin transport equation can be derived by regarding the quasiparticles as independent, described by a classical Hamiltonian $\mathcal{E}(\vec{p}, \vec{r})$. However, the spin part of the Hamiltonian must be treated quantum mechanically, since it has no classical analog. Thus we combine the classical Liouville's equation with the quantum-mechanical equation of motion for the spin part and write the basic transport equation as[9]

$$\frac{\partial n}{\partial t} + i[\mathcal{E}, n] + \{n, \mathcal{E}\} = I(n) \quad . \tag{1.14}$$

In (1.14), $[\mathcal{E}, n]_-$ is the commutator of $\mathcal{E}$ and n, and $\{\cdots\}$ is the symmetrized Poission bracket defined as

$$\{A,B\}=\frac{1}{2}\left(\frac{\partial A}{\partial \vec{r}}\cdot\frac{\partial B}{\partial \vec{p}}+\frac{\partial B}{\partial \vec{p}}\cdot\frac{\partial A}{\partial \vec{r}}\right)-\frac{1}{2}\left(\frac{\partial A}{\partial \vec{p}}\cdot\frac{\partial B}{\partial \vec{r}}+\frac{\partial B}{\partial \vec{r}}\cdot\frac{\partial A}{\partial \vec{p}}\right) \quad . \tag{1.15}$$

$I(n)$ is a phenomenological collision term which describes collision between quasiparticles. It cannot be derived rigorously within the scope of the Landau theory. We assume that a large dc magnetic field $\vec{B}$ is applied in the z direction. Let us denote the energy of a quasiparticle of momentum $\vec{p}$ in the absence of the dc field $\vec{B}$ as $\mathcal{E}_{qp}$. For a metal with a spherical Fermi surface, $\mathcal{E}_{qp}$ is equal to $p^2/2m^*$. Here, m^* is the effective mass of a quasiparticle. According to (1.2), it is the ratio of Fermi momentum p_f to the Fermi velocity v_f at the Fermi surface. When the magnetic field $\vec{B}$ is switched on, the equilibrium distribution and the quasiparticle energy can be shown (to first order in $\omega_c/\mathcal{E}_f$) to be

$$n_0=\rho_0-\gamma B\sigma_z(\partial\rho_0/\partial\mathcal{E}_{qp}) \quad , \tag{1.16}$$

and $$\mathcal{E}_0=\mathcal{E}_{qp}-\gamma\sigma_z B \quad . \tag{1.17}$$

Here γ is a renormalized gyromagnetic ratio,[10] and ρ_0 is the usual Fermi distribution function

$$\rho_0=\{\exp[(\mathcal{E}_{qp}-\mu)/kT]+1\}^{-1} \quad . \tag{1.18}$$

It should be emphasized that the quantities n_0, ρ_0 are matrices in spin space. Suppose now the system experiences a small perturbing ac magnetic field of induction $\vec{b}$ and an electric field $\vec{E}$. These are of course the self-consistent magnetic and electric fields inside the metal rather than the applied field. The fields $\vec{b}$ and $\vec{E}$ will create quasiparticles and change the distribution to

$$n=n_0+\delta n \quad . \tag{1.19}$$

It is convenient to write δn as

$$\delta n=\delta f+\delta\vec{g}\cdot\vec{\sigma} \quad . \tag{1.20}$$

The local quasiparticle energy is changed from the value $\mathcal{E}_0$ to $\mathcal{E}_0+\delta\mathcal{E}$, which can be written as

$$\mathcal{E}_p=\mathcal{E}_0+\delta\mathcal{E}_1+\delta\vec{\mathcal{E}}_2\cdot\vec{\sigma}-\tfrac{1}{2}g_s\beta\vec{\sigma}\cdot\vec{b} \quad . \tag{1.21}$$

Here g_s is the spectroscopic splitting factor, and β is the Bohr magneton. In (1.21), the terms $\delta\mathcal{E}_1$ and $\delta\vec{\mathcal{E}}_2\cdot\vec{\sigma}$ represent the change in quasiparticle energy brought about by the change in the distribution of all the other quasiparticles. Making use of (1.10), we can easily express them in terms of the distribution function as

$$\delta\mathcal{E}_1=2\textstyle\sum_{\vec{p}'}\phi(\vec{p},\vec{p}')\delta f(\vec{p}') \quad , \tag{1.22}$$

and $$\delta\vec{\mathcal{E}}_2=2\textstyle\sum_{\vec{p}'}\psi(\vec{p},\vec{p}')\delta\vec{g}(\vec{p}') \quad . \tag{1.23}$$

The last term on the right-hand side of (1.21) is the direct interaction of the magnetic moment of the bare electron with the perturbing field $\vec{b}$. It is convenient to combine this term with the term $\delta\vec{\mathcal{E}}_2\cdot\vec{\sigma}$ and define a factor γ_1 by the relation

$$-\gamma_1\vec{\sigma}\cdot\vec{b}=\delta\vec{\mathcal{E}}_2\cdot\vec{\sigma}-\tfrac{1}{2}g_s\beta\vec{b}\cdot\vec{\sigma} \quad . \tag{1.24}$$

Substituting (1.20) and (1.21) into the Landau-Silin kinetic equation and keeping linear terms only gives

$$\frac{\partial}{\partial t}\delta n+i([\delta\mathcal{E},n_0]_-+[\mathcal{E}_0,\delta n]_-)+\{n_0,\delta\mathcal{E}\}+\{\delta n,\mathcal{E}_0\}=I(n) \quad . \tag{1.25}$$

We now make use of the following properties:

$$\frac{\partial\mathcal{E}}{\partial\vec{p}}=\vec{v}+\frac{\partial}{\partial\vec{p}}\delta\mathcal{E}_1-\frac{\partial}{\partial\vec{p}}\gamma_1\vec{\sigma}\cdot\vec{b} \quad , \tag{1.26}$$

$$-\frac{\partial\mathcal{E}}{\partial\vec{r}}=\frac{e}{c}\left(\frac{\partial\mathcal{E}}{\partial\vec{p}}\times(\vec{B}+\vec{b})\right)-\frac{\partial}{\partial\vec{r}}(\delta\mathcal{E}_1-\gamma_1\vec{\sigma}\cdot\vec{b})+e\vec{E} \quad , \tag{1.27}$$

and $$i[\mathcal{E},n]_-=-i\gamma B[\sigma_z,\vec{\sigma}]\cdot\vec{g}-i\gamma_1\vec{b}\cdot[\vec{\sigma},\sigma_z]_-\left(-\gamma B\frac{\partial\rho_0}{\partial\mathcal{E}_{qp}}\right)$$

$$= -2\gamma\beta(\sigma_x g_y - \sigma_y g_x) + 2\gamma_1 \left(+\gamma\beta \frac{\partial \rho_0}{\partial \mathcal{E}_{qp}}\right)(\sigma_y b_x - \sigma_x b_y) \ . \tag{1.28}$$

The insertion of the two terms $(e/c)[(\partial \mathcal{E}/\partial \vec{p})\times(\vec{B}+\vec{b})]$ and $e\vec{E}$ in (1.27) as the driving force due to the external field is not obvious as it may seem as first sight. Since we are dealing with quasiparticles and not bare electrons, the validity of these Lorentz force terms requires some investigations. Pines and Nozieres[8] have shown that (1.27) is correct. Using (1.26)–(1.28) in (1.25) gives

$$\frac{\partial \delta f}{\partial t} + \frac{\partial}{\partial t}\delta\vec{g}\cdot\vec{\sigma} - 2\gamma B(\sigma_x g_y - \sigma_y g_x) + 2\gamma_1 \frac{\partial \rho_0}{\partial \mathcal{E}_{qp}}(+\gamma\beta)(\sigma_y b_x - \sigma_x b_y) + \vec{v}\cdot\frac{\partial}{\partial \vec{r}}(\delta f + \delta\vec{g}\cdot\vec{\sigma})$$

$$+ \frac{\partial n_0}{\partial \vec{p}}\left(\frac{e}{c}\vec{v}\times\vec{b} + e\vec{E} + \frac{e}{c}\frac{\partial \delta \mathcal{E}_1}{\partial \vec{p}}\times\vec{B} - \frac{\partial}{\partial \vec{r}}(\delta \mathcal{E}_1 - \gamma_1 \vec{\sigma}\cdot\vec{b}) - \frac{e}{c}\frac{\partial}{\partial \vec{p}}(\gamma_1 \vec{\sigma}\cdot\vec{b})\times\vec{B}\right) + \frac{\partial}{\partial \vec{p}}(\delta f + \delta\vec{g}\cdot\vec{\sigma})\frac{e}{c}\vec{v}\times\vec{B} = I(n). \tag{1.29}$$

This can be further separated into two equations, one for the spin-symmetric distribution f and the other for the spin-antisymmetric part $\vec{g}$. On taking half the trace of (1.29), we obtain

$$\frac{\partial \delta f}{\partial t} + \vec{v}\,\frac{\partial}{\partial \vec{r}}\left(\delta f - \delta \mathcal{E}_1 \frac{\partial \rho_0}{\partial \mathcal{E}_{qp}}\right) + \frac{e}{c}\vec{v}\times\vec{B}\cdot\frac{\partial}{\partial \vec{p}}\left(\delta f - \delta \mathcal{E}_1 \frac{\partial \rho_0}{\partial \mathcal{E}_{qp}}\right) + e\vec{E}\cdot\vec{v}\frac{\partial \rho_0}{\partial \mathcal{E}_{qp}} = \tfrac{1}{2}\mathrm{Tr} I(n) \ . \tag{1.30}$$

Now multiply (1.29) by $\frac{1}{2}(\sigma_x + i\sigma_y)$ and take the trace. All terms which do not contain an even power of σ will vanish. Let $\sigma_+ = \sigma_x + i\sigma_y$, $\delta g_+ = \delta g_x + i\delta g_y$, $b_+ = b_x + ib_y$. Then we obtain

$$\frac{\partial}{\partial t}\delta g_+ + v\cdot\frac{\partial}{\partial r}\left(\delta g_+ + \gamma_1 b_+ \frac{\partial \rho_0}{\partial \mathcal{E}_{qp}}\right) + \frac{e}{c}\vec{v}\times\vec{B}\cdot\frac{\partial}{\partial \vec{p}}\left(\delta g_+ + \gamma_1 b_+ \frac{\partial \rho_0}{\partial \mathcal{E}_{qp}}\right) + 2i\gamma B\left(\delta g_+ + \gamma_1 b_+ \frac{\partial \rho_0}{\partial \mathcal{E}_{qp}}\right) = \tfrac{1}{2}\mathrm{Tr}\sigma_+ I(n) \ . \tag{1.31}$$

Similar equations hold for the function $\delta g_- = \delta g_x - i\delta g_y$ and δg_z. The two equations (1.30) and (1.31) are basic in the study of transport properties of Fermi liquid. The quantities defined by

$$\tilde{f} = \delta f - (\partial \rho_0/\partial \mathcal{E}_{qp})\delta \mathcal{E}_1 \ , \tag{1.32}$$

and

$$\tilde{\vec{g}} = \delta\vec{g} - (\partial \rho_0/\partial \mathcal{E}_{qp})\delta \mathcal{E}_2 \ , \tag{1.33}$$

play a role similar to δf and δg in the noninteracting system. If $n_0(\mathcal{E}_{qp})$ represents the equilibrium distribution function, $n_0(\mathcal{E}_p)$ will represent a local equilibrium. The difference between $n(\vec{p})$ and $n_0(\mathcal{E}_p)$, which measures the departure from local equilibrium, can easily be seen to be just $\tilde{f}$ and $\tilde{\vec{g}}\cdot\vec{\sigma}$, i.e.,

$$n(\vec{p}) = n_0(\mathcal{E}_p) + \tilde{f} + \tilde{\vec{g}}\cdot\vec{\sigma} \ . \tag{1.34}$$

These two functions occur frequently in our further discussion.

II. MAGNETOCONDUCTIVITY TENSOR

In this section, the magnetoconductivity tensor of a metal will be calculated by solving the transport equation (1.30). This tensor, being the response function of the system to an external perturbing electromagnetic field, is an essential quantity in nearly all transport properties of metals. Cohen, Harrison, and Harrison[11] have evaluated this quantity for a simple metal with a spherical Fermi surface in the absence of electron correlations. Recently, Platzman and Walsh[12] have extended their calculation to include the S-wave Fermi-liquid interaction, but no explicit expression for the conductivity tensor was given. Hamilton and McWhorter[13] have also evaluated the tensor including both S and P wave interactions. All these results are applicable only for metals with a spherical Fermi surface like Na and K. However, most metals have complicated Fermi surface, and many transport properties such as magnetoresistance, depend critically on the geometry of the Fermi surface. For this reason, we feel that it is worthwhile to evaluate here the conductivity tensor for a metal with an arbitrary Fermi surface, while including at the same time the effect of the Fermi-liquid interaction. In general, the interaction function $\phi(\vec{p},\vec{p}')$ will depend on both $\vec{p}$ and $\vec{p}'$ in a complicated way. The complete solution, although possible in principle, will involve too many unknown parameters and hence of little use. We shall simplify the problem by replacing $\phi(\vec{p},\vec{p}')$ by a constant ϕ_0, i.e., including only the S-wave interactions. This is not as bad an approximation as it may appear at first sight. Theoretical calculation for the interaction function by Rice[14] indicates that the S-wave interaction is the dominant term. Provided that the many-body effect on a particular transport property does not come primarily from the angular dependent terms[15] in the expansion of $\phi(\vec{p},\vec{p}')$, we can still get

a fairly good estimate of the interplay of band structure and many-body correlations effects on transport properties of metals via the conductivity tensor evaluated in this model.

Let us first introduce the function $f(\vec{p})$ defined by

$$\delta f(\vec{p}) = (-\partial\rho_0/\partial\mathcal{E}_{qp}) f(\vec{p}) \quad . \tag{2.1}$$

Now we assume all quantities appearing in (1.30) have a space time dependence of the form $\exp[i(\vec{q}\cdot\vec{r}-\omega t)]$. The collision term is hard to obtain from first principle. We shall discuss it in more detail when we consider simple metals with spherical Fermi surface. For the calculation here, we approximate the collision integral by introducing a single collision time t as

$$\tfrac{1}{2}\mathrm{Tr}I(n) = (\partial\rho_0/\partial\mathcal{E}_{qp})[(f+\delta\mathcal{E}_1) - \langle f+\delta\mathcal{E}_1\rangle]/\tau \quad . \tag{2.2}$$

The bracket $\langle\cdots\rangle$ in (2.2) denotes the average over the Fermi surface. As first shown by Silin,[16] the local departure from the equilibrium distribution $-f+\delta\mathcal{E}_1$ appears in (2.2) instead of f, because in a collision with impurity,[17] it is the local energy of the quasiparticle and not the equilibrium energy which is conserved. The integral of (2.2) over the momentum space vanishes. Hence, it satisfies the criterion that the number of quasiparticles has to be conserved in a collision process. Substituting (2.1) and (2.2) into (1.30) gives

$$\left(-i\omega+\frac{1}{\tau}+i\vec{q}\cdot\vec{v}-\frac{e}{c}\vec{v}\times\vec{B}\cdot\frac{\partial}{\partial\vec{p}}\right)f+\left(i\vec{q}\cdot\vec{v}+\frac{1}{\tau}-\frac{e}{c}\vec{v}\times\vec{B}\cdot\frac{\partial}{\partial\vec{p}}\right)\delta\mathcal{E}_1 - e\vec{E}\cdot\vec{v} = \frac{\langle f+\delta\mathcal{E}_1\rangle}{\tau} \quad . \tag{2.3}$$

In arriving at (2.3), we have cancelled out a common factor $-\partial\rho_0/\partial\mathcal{E}_{qp}$. At sufficiently low temperature, $-\partial\rho_0/\partial\mathcal{E}_{qp}$ behaves like a delta function, i.e.,

$$-\partial\rho_0/\partial\mathcal{E}_{qp} \approx \delta(\mathcal{E}_{qp}-\mathcal{E}_F) \quad . \tag{2.4}$$

Hence, all quantities appearing in (2.3) are understood to be evaluated at the Fermi surface. Following Eckstein,[18] we find it convenient to use the parameters $\mathcal{E}$, p_z, and s in place of p_x, p_y, p_z. Here $\mathcal{E}$ is energy, p_z momentum in direction of $\vec{B}$, and s is a parameter with the dimensions of time which locates the position of an electron on its orbit. The equation of motion of an electron in the presence of the dc magnetic field $\vec{B}$ is

$$\partial\vec{p}/\partial s = (e/c)\vec{v}\times\vec{B} \quad . \tag{2.5}$$

Making use of (2.5), (2.3) can be written in a much simpler form as

$$-\omega f + (\tau^{-1}+i\vec{q}\cdot\vec{v}+\partial/\partial s)(f+\delta\mathcal{E}_1) - e\vec{E}\cdot\vec{v} = \langle f+\delta\mathcal{E}_1\rangle/\tau \quad . \tag{2.6}$$

We now introduce the position vector $\vec{R}(p_z, s)$ of an electron on the Fermi surface. We can divide $\vec{R}$ into a periodic and a secular part, $\vec{R}=\vec{R}_p+\vec{R}_s$, where

$$\vec{R}_p(p_z, s+T(p_z)) = \vec{R}_p(p_z, s) \quad , \tag{2.7}$$

and $$\vec{R}_s = \vec{v}_s(p_z)s \quad . \tag{2.8}$$

Here $T(p_z)$ is the period of an electron on the Fermi surface with wave number p_z, and

$$\vec{v}_s = T^{-1}\int_t^{t+T}\vec{v}(p_z, s)ds \quad . \tag{2.9}$$

The functions

$$f(p_z, s)\exp(i\vec{q}\cdot\vec{R}_p),\ \delta\mathcal{E}_1(p_z, s)\exp(i\vec{q}\cdot\vec{R}_p),\ \vec{v}(p_z, s)\exp(i\vec{q}\cdot\vec{R}_p),\ \text{and}\ \exp(i\vec{q}\cdot\vec{R}_p)$$

must all be periodic functions of s with period $T(p_z)$. Therefore, we can expand these functions in Fourier coefficients: f_l, $\mathcal{E}_l$, a_l, and $\vec{v}_l$

$$f(p_z, s)\exp(i\vec{q}\cdot\vec{R}_p) = \textstyle\sum_l f_l(p_z)\exp(2\pi ils/T),\quad \delta\mathcal{E}_1(p_z, s)\exp(i\vec{q}\cdot\vec{R}_p) = \sum_l \mathcal{E}_l(p_z)\exp(2\pi ils/T),$$

$$\exp(i\vec{q}\cdot\vec{R})_p = \textstyle\sum_l a_l(p_z)\exp(2\pi ils/T),\quad \text{and}\quad \vec{v}(p_z, s)\exp(i\vec{q}\cdot\vec{R})_p = \sum_l \vec{v}_l(p_z)\exp(2\pi ils/T). \tag{2.10}$$

By substituting (2.10) into (2.6) and performing a little algebraic manipulation, one can obtain the equation

$$f_l(p_z)=\left(-1-\frac{i\omega}{-i\omega+\tau^{-1}+i\vec{q}\cdot\vec{v}_s+2\pi il/T}\right)\mathcal{E}_l(p_z)+\frac{e\vec{E}\cdot\vec{v}_l+\tau^{-1}\langle f+\delta\mathcal{E}_1\rangle a_l}{-i\omega+\tau^{-1}+i\vec{q}\cdot\vec{v}_s+2\pi il/T} \quad . \tag{2.11}$$

We now assume that the interaction function $\phi(\vec{p},\vec{p}')$ is a constant for all $\vec{p},\vec{p}'$ on the Fermi surface and introduce a parameter A_0 defined by

$$A_0=(eBp_m\overline{T}/2\pi^3c)\phi_0 \quad . \tag{2.12}$$

In (2.12), p_m is the maximum value of p_z on the Fermi surface, and $\overline{T}$ is average period of electronic motion defined by

$$\overline{T}=(2p_m)^{-1}\int T(p_z)dp_z \quad . \tag{2.13}$$

We introduce further a function F defined by

$$F=\int dp_z\int_t^{t+T(p_z)}f \quad . \tag{2.14}$$

It can then easily be shown that the following equations hold

$$F=\sum_l\int dp_z T(p_z)f_l a_l^* \quad , \tag{2.15}$$

$$\langle f+\delta\mathcal{E}_1\rangle=[(1+A_0)/2p_m\overline{T}]F \quad , \tag{2.16}$$

$$\delta\mathcal{E}_1=(A_0F/2p_m\overline{T}) \quad , \tag{2.17}$$

and

$$\mathcal{E}_l=(A_0/2p_m\overline{T})Fa_l(p_z) \quad . \tag{2.18}$$

Multiplying (2.11) by $T(p_z)a_l^*(p_z)$, summing over l and integrating over p_z then gives a simple algebraic equation for F whose solution is

$$F=ie\vec{E}\cdot\vec{K}/\{1+A_0-[(1+A_0)i/\tau+A_0\omega]L/2p_m\overline{T}\} \quad , \tag{2.19}$$

where

$$\vec{K}=\sum_l\int\vec{v}_l a_l^*T(p_z)/[\tilde{\omega}-\vec{q}\cdot\vec{v}_s-2\pi l/T(p_z)]dp_z \quad , \tag{2.20}$$

and

$$L=\sum_l\int T(p_z)|a_l|^2/[\tilde{\omega}-\vec{q}\cdot\vec{v}_s-2\pi l/T(p_z)]dp_z \quad , \tag{2.21}$$

in (2.21), $\tilde{\omega}=\omega+i/\tau$. All the Fourier components of the distribution function can then be obtained by substituting (2.19) back into (2.11). The next step is to express the current density in terms of the distribution function. Nozieres[19] has shown that the current density due to the flow of quasiparticles can be written as

$$\vec{j}=2e(2\pi)^{-3}\int d^3p\delta f(\vec{p})[\vec{v}(p)+2\int d^3p'(2\pi)^{-3}\phi(\vec{p},\vec{p}')\vec{v}(\vec{p}')(-\partial\rho_0/\partial\mathcal{E}_{qp})] \quad . \tag{2.22}$$

The last term in (2.22) represents the backflow current density from the interaction of the moving wave packet with the surrounding fluid. By interchanging the order of integration $\vec{p}$ and $\vec{p}'$ in the second term, one can write (2.22) in the form

$$\vec{j}=2e(2\pi)^{-3}\int d^3p\vec{v}(p)(-\partial\rho_0/\partial\mathcal{E}_{qp})(f+\delta\mathcal{E}_1) \quad . \tag{2.23}$$

If we express the integration in terms of the coordinates $\mathcal{E}$, p_z, and s and then introduce the Fourier transform defined in (2.10), we can write $\vec{j}$ in the form

$$\vec{j}=[2e^2B/(2\pi)^3c]\int dp_z T(p_z)\sum_l\vec{v}_l^*(p_z)(f_l+\mathcal{E}_l) \quad . \tag{2.24}$$

Substituting the solution of the kinetic equation into (2.24), we finally obtain the result

$$\vec{j}=(\overleftrightarrow{\sigma}^0+\delta\overleftrightarrow{\sigma})\cdot\vec{E} \quad , \tag{2.25}$$

where

$$\overleftrightarrow{\sigma}^0=[2ie^3B/(2\pi)^3c]\int dp_z T(p_z)\sum_l\vec{v}_l^*\vec{v}_l/[\tilde{\omega}-\vec{q}\cdot\vec{v}_s-2\pi l/T] \quad , \tag{2.26}$$

is Eckstein's expression for the conductivity tensor, and

$$\delta\overleftrightarrow{\sigma}=[2e^3B/(2\pi)^3c](+iA_0\tilde{\omega}-\tau^{-1})\vec{K}'\vec{K}/[2p_m\overline{T}(1+A_0)+A_0\omega L-(1+A_0)iL/\tau] \quad . \tag{2.27}$$

Here $\vec{K}'(\tau)=\vec{K}^*(-\tau)$. In the absence of Fermi liquid effects $\overleftrightarrow{\sigma}$ reduces to

$$\overleftrightarrow{\sigma}^0+(i/\tau)[2e^3/(2\pi)^3c(2p_m\overline{T})]\vec{K}'\vec{K} \ .$$

The second term, which is proportional to $1/\tau$ gives rise to a diffusion current.

For the spherical Fermi surface, there is some simplification. We can choose q to lie in x-z plane without loss of generality. Then we have

$$p_m=p_f, \quad \overline{T}=T=2\pi/\omega_c \ , \quad v_s=v_z \ , \quad a_l=J_l(q_x v_\perp/\omega_c) \ . \tag{2.28}$$

Here p_f is the Fermi momentum, J_l is a Bessel function, and ω_c, the cyclotron resonance frequency, is equal to eB/m^*c. The Fourier components of the function $\vec{v}\exp(i\vec{q}\cdot\vec{R}_p)$ can easily be shown to be

$$\vec{v}_l=v_f\begin{pmatrix} \frac{1}{2}\sin\theta(J_{l-1}+J_{l+1}) \\ -(i/2)\sin\theta(J_{l-1}-J_{l+1}) \\ J_l\cos\theta \end{pmatrix} \ . \tag{2.29}$$

We have introduced $v_\perp=v_f\sin\theta$. Substitution of (2.28) and (2.29) into (2.26) gives the well-known result of Cohen, Harrison, and Harrison,[11] while $\delta\overleftrightarrow{\sigma}$ can be conveniently expressed as

$$\delta\overleftrightarrow{\sigma}=\tfrac{3}{4}(\sigma_0/\tau)[(iA_0\tilde{\omega}-\tau^{-1})/\{(1+A_0)-[A_0\omega+(1+A_0)i/\tau]M\}]\vec{N}'\vec{N} \ . \tag{2.30}$$

$$\text{Here} \quad M=\textstyle\sum_l\int_{-1}^{+1}d(\cos\theta)J_l^2(X\sin\theta)/(\tilde{\omega}-q_z v_f\cos\theta-l\omega_c) \ , \tag{2.31}$$

$$\text{and} \quad \vec{N}=\sum_l\int_{-1}^{+1}d(\cos\theta)\frac{J_l(X\sin\theta)}{\omega-q_z v_F\cos\theta-l\omega_c}\begin{pmatrix} (l/X)J_l \\ -i(\partial/\partial X)J_l \\ \cos\theta J_l \end{pmatrix} . \tag{2.32}$$

Again $N'(\tau)=N^*(-\tau)$. We have introduced $\sigma_0=ne^2\tau/m^*$, the dc conductivity, and $X=q_x v_f/\omega_c$. For a metal with a spherical Fermi surface, we can further generalize the above treatment of the collision term and the interaction function in the following way. If we consider scattering by impurities alone, then we can write[8]

$$I(n(\vec{p}))=-2\pi\textstyle\sum_{\vec{p}'}S_{\vec{p}\vec{p}'}\delta(\mathcal{E}_p-\mathcal{E}_{p'})\{n(\vec{p})[1-n(\vec{p}')]-n(\vec{p}')[1-n(\vec{p})]\} \ . \tag{2.33}$$

In (2.33), the factor $2\pi S_{pp'}\delta(\mathcal{E}_p-\mathcal{E}_{p'})$ is the transition probability for the scattering of a quasiparticle from state $\vec{p}$ to state $\vec{p}'$. The delta function corresponds to conservation of local energy $\mathcal{E}_p$. The determination of $S_{\vec{p},\vec{p}'}$ from first principle is outside the scope of Landau theory. Further, the spin dependence of S on $\vec{\sigma}$ and $\vec{\sigma}'$ varies with the particular kind of impurity. We therefore consider $S_{\vec{p},\vec{p}'}$ as a phenomenological parameter. Dropping terms of order δn^2 and $(\omega_c/\mathcal{E}_F)\delta n$, we can write (2.33) as

$$I(n)=-2\pi\textstyle\sum_{\vec{p}'}S_{\vec{p}\vec{p}'}\delta(\mathcal{E}_{qp}(\vec{p})-\mathcal{E}_{qp}(\vec{p}'))[\delta\tilde{n}(\vec{p})-\delta\tilde{n}(\vec{p}')] \ . \tag{2.34}$$

Now we assume that the transition probability has a spin dependence of the form

$$S_{\vec{p}\sigma,\vec{p}'\sigma'}=W_{\vec{p}\vec{p}'}+U_{\vec{p}\vec{p}'}\vec{\sigma}\cdot\vec{\sigma}' \ . \tag{2.35}$$

It then follows from (2.34) and (2.35) that

$$\tfrac{1}{2}\mathrm{Tr}I(n)=-4\pi\textstyle\sum_{\vec{p}'}W_{pp'}[\tilde{f}(\vec{p})-\tilde{f}(\vec{p}')]\delta(\mathcal{E}_{qp}-\mathcal{E}_{qp'}) \ . \tag{2.36}$$

With rotational symmetry, $W_{\vec{p},\vec{p}'}$ at the Fermi surface depends only on the angle between $\vec{p}$ and $\vec{p}'$. Hence, it can be expanded in a series of Legendre polynomials as

$$W_{\vec{p}\vec{p}'}=\textstyle\sum_l W_l P_l(\cos\xi) \tag{2.37}$$

Here ξ is the angle between $\vec{p}$ and $\vec{p}'$. Substitution of (2.37) into (2.36) gives

$$\tfrac{1}{2}\mathrm{Tr}I(n)=-\frac{1}{\tau}f+\sum_{l,\,|m|<l}\tilde{f}_l^{\,m}Y_l^{\,m}(\theta,\phi)/\tau_l \ , \tag{2.38}$$

where we have introduced the following definitions

$$1/\tau = (6\pi n/m^* v_f^2) W_0 \quad , \quad \text{and} \quad 1/\tilde{\tau}_l = (6\pi n/m^* v_f^2) W_l/(2l+1) \quad . \tag{2.39}$$

f_l^m denotes the (l, m)th coefficient when we expand f in a series of spherical harmonics Y_l^m. Obviously, the occurrence of a large number of relaxation times make comparison with experiment very difficult. Hopefully, only a finite number of terms are needed in the summation over l, m in (2.38). This is true if the angular dependence of $W(\vec{p}, \vec{p}')$ is small. In the literature on Fermi liquids, the usual procedure[20] is to drop the terms $\sum_{lm} \cdots$ completely, or to retain only the term f_0^0/τ_0. In the present treatment, we shall retain the same number of terms in the expansion of $W(\vec{p}, \vec{p}')$ as in the expansion of $\phi(\vec{p}, \vec{p}')$. The latter, when evaluated on the Fermi surface, is also a function of the angle ξ between $\vec{p}$ and $\vec{p}'$ only. Hence, it can be written as

$$\phi(\vec{p}, \vec{p}') = \sum_{l=0}^{\infty} \frac{(2l+1)\pi^2}{m^* p_F} A_l P_l(\cos\xi). \tag{2.40}$$

The A_l defined by (2.40) are the crucial parameters in the theory. The parameter A_0 defined by (2.12) reduces to that defined by (2.40) in the limit of a spherical Fermi surface. Substitution of (2.40) into (1.22) gives an expression for $\delta\mathcal{E}_1$ as

$$\delta\mathcal{E}_1 = \sum_{l|m|<l} A_l f_l^m Y_l^m(\theta, \phi). \tag{2.41}$$

In deriving (2.38) and (2.41), we have made use of the addition theorem

$$P_l(\cos\xi) = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y_l^{m*}(\theta, \phi) Y_l^m(\theta', \phi'), \tag{2.42}$$

and the convention we adopt for Y_l^m is

$$Y_l^m(\theta, \phi) = \theta_l^m(\theta) e^{im\phi} (2\pi)^{-\frac{1}{2}}. \tag{2.43}$$

Here $\theta_l^m = [\tfrac{1}{2}(2l+1)(l-|m|)!/(l+|m|)!\,]^{\frac{1}{2}} P_l^{|m|}(\cos\theta)\delta m$,

and $\delta_m = (-1)^m$, if $m \geq 0$; $\delta_m = 1$, if $m < 0$.

We now investigate the relation between the Fourier coefficient f_m defined in (2.10) and the spherical harmonic coefficient f_l^s. Substituting (2.10) and (2.28) into the definition of f_l^s gives

$$f_l^s = \int_{-1}^{+1} \int_0^{2\pi} Y_l^{s*}(\theta, \phi) f(\theta, \phi) d\phi d(\cos\theta) = \sum_m \int_{-1}^{+1} \theta_l^s(\theta) f_m(\theta) J_{m-s}(X \sin\theta) (2\pi)^{\frac{1}{2}} d(\cos\theta). \tag{2.44}$$

From (2.44), we see that a particular spherical harmonic component f_l^s is related to all the Fourier coefficients $f_m(\theta)$ with values of m running from $-\infty$ to $+\infty$. With the aid of (2.38), (2.41), (2.44), we can write the transport equation (1.30) as

$$(-i\omega + iq_z v_F \cos\theta + \tau^{-1} + im\omega_c) f_m + i[q_z v_F \cos\theta - (i/\tau) m\omega_c] Y_m(\theta) - \vec{E} \cdot \vec{v}_m$$

$$= \sum_{l, |s|<l} \frac{1}{\tau_l} \Gamma_l^s \theta_l(\theta) J_{m-s}(X \sin\theta). \tag{2.45}$$

In (2.45), we have introduced

$$Y_m(\theta) = (2\pi)^{-1} \iiint e^{i\vec{q} \cdot \vec{R}_p} e^{-im\phi_{\phi(\vec{p}, \vec{p}')}} (-\partial\rho_0/\partial\mathcal{E}_{qp'}) f(\vec{p}') d\phi d\phi' d(\cos\theta') p'^2 dp'. \tag{2.46}$$

Here $\Gamma_l^s \equiv (2\pi)^{-\frac{1}{2}} f_l^s$, $1/\tau_l = \alpha_l \tilde{\tau}_l$, and $\alpha_l = 1 + A_l$.

We now expand $\phi(\vec{p},\vec{p}')$ in the following way

$$\phi(\vec{p},\vec{p}')=\sum_{m=-\infty}^{\infty}\phi_m(\theta,\theta')e^{im(\phi-\phi')}. \tag{2.47}$$

Comparing with our previous expansion of $\phi(\vec{p},\vec{p}')$ in (2.40), and making use of the addition theorem yields

$$\phi_m(\theta,\theta')=\sum_{n\geq|m|}\phi_n\frac{2}{2n+1}\theta_n^{\,m}(\theta)\theta_n^{\,m}(\theta')=\frac{2\pi^2}{m^*p_f}\sum_{n\geq|m|}A_n\theta_n^{\,m}(\theta)\theta_n^{\,m}(\theta'). \tag{2.48}$$

It then follows from (2.46), (2.47), and (2.48) that

$$Y_m(\theta)=\sum_{l,|s|\leq l}A_l\theta_l^{\,s}(\theta)J_{m-s}(X\sin\theta)\Gamma_l^{\,s}. \tag{2.49}$$

The Fourier component f_m can now be written in the following way:

$$f_m=\frac{ie\vec{E}\cdot\vec{v}_m}{\tilde{\omega}-q_zv_f\cos\theta-m\omega_c}+\sum_{l,|s|\leq l}\left(-1+\frac{\tilde{\omega}_l}{\tilde{\omega}-q_zv_f\cos\theta-m\omega_c}\right)A_l\Gamma_l^{\,s}\theta_l^{\,s}(\theta)J_{m-s}(X\sin\theta), \tag{2.50}$$

where $\tilde{\omega}=\omega+i/\tau$, and $\omega_l=\omega+i/A_l\tau_l$. It is interesting to note that τ_l appears in the combination $1/A_l\tau_l$ which may be comparable with ω even when $\omega\tau_l\gg1$ for large l. Hence, it is desirable to keep as many of the higher-order collision terms as possible, at least to the same order as the Fermi-liquid interaction coefficients A_n. Now we multiply both sides of (2.50) by $\theta_{l'}^{\,s}J_{m-s'}(X\sin\theta)$, sum over m and integrate over $d(\cos\theta)$. Then (2.50) becomes an infinite set of linear simultaneous equations,

$$\sum_{l',|s'|<l'}(a_{ll'}^{\,ss'}-\delta_{ll'}\delta_{ss'})\Gamma_{l'}^{\,s'}=C_l^{\,s} \tag{2.51}$$

$$\text{with}\quad a_{ll'}^{\,ss'}=\sum_m\int_{-1}^{+1}d(\cos\theta)\left(-1+\frac{\tilde{\omega}_{l'}}{\tilde{\omega}-q_zv_f\cos\theta-m\omega_c}\right)\theta_l^{\,s}\theta_{l'}^{\,s'}A_{l'}J_{m-s}(X\sin\theta)J_{m-s'}(X\sin\theta) \tag{2.52}$$

$$\text{and}\quad C_l^{\,s}=-\sum_m\int_{-1}^{+1}[i\vec{E}\cdot\vec{v}_m/(\tilde{\omega}-q_zv_f\cos\theta-m\omega_c)]\,\theta_l^{\,s}J_{m-s}d(\cos\theta). \tag{2.53}$$

It is of course impossible to solve the infinite set of equations (2.51) exactly. Since theoretical estimates[14] for the interaction function indicate that it has only a slight angular dependence, it is reasonable to make the approximation of setting $A_i=0$ for i greater than a certain fixed number. By this procedure of truncating the ϕ function after one or two terms, one is able to obtain the dominant terms in $\delta\vec{\vec{\sigma}}$, the correction to the free electron expression for the conductivity tensor. However, the relative importance of the different terms in $\delta\vec{\vec{\sigma}}$ depends on how they are combined in the final expression. For example, there has been a great deal of confusion in the interpretation of the experimental results concerning plasma waves in potassium. If we try to explain the results with Landau-Silin theory, it turns out that A_0 and A_1 do not enter the final dispersion relation, and only for finite values of A_2 and A_3, etc., does the theory predict a deviation from the free electron behavior. We shall leave more detailed discussion on plasma wave in a later section. This serves to illustrate that the proper way of truncating the expansion of $\Phi(\vec{p},\vec{p}')$ in Legendre polynomials to get an approximate result really depends on the particular phenomena under consideration.

We shall now solve (2.51) assuming that $A_i=0$ for $i\geq N_0$. In principle, the method is applicable for arbitrary large value of N_0. However, the amount of numerical work required to obtain useful information from the final expression will be enormous for large N_0, since the number of simultaneous equations for a given N_0 will be N_0^2. In most ranges of values of q and ω, only terms linear in the A's are important. The final expression will then be fairly simple. A further simplification can be noted: either by employing the equation of continuity or by expanding the distribution function in spherical harmonics and substituting into the transport equation, it can be shown that

$$\Gamma_0^{\,0}=6^{-1/2}(q_xv_F/\omega)\alpha_1(\Gamma_1^{\,-1}-\Gamma_1^{\,+1})+3^{-1/2}(q_zv_F/\omega)\alpha_1\Gamma_1^{\,0}. \tag{2.54}$$

Except for the case where only A_0 is retained in the series (2.40), the number of simultaneous equations can be reduced to $N_0^2 - 1$ by making use of (2.54). From (2.51), it follows that

$$\sum_{l', |s'| \leq l'} (b_{ll'}^{ss'} - \delta_{ll'}\delta_{ss'})\Gamma_{l'}^{s'} = C_l^s \quad . \tag{2.55}$$

Here (l', s') runs through all pairs of values $(1,0)$, $(1,1)$, $(1,-1)\cdots$, etc., except $(0,0)$ and $b_{ll'}^{ss'} = a_{ll'}^{ss'} + \delta_{l's'}$. The function $\delta_{l's'}$ is defined by

$$\begin{aligned}
\delta_{l's'} &= 6^{-\frac{1}{2}}(q_x v_F/\omega)\alpha_1 a_{l0}^{s0}, \quad \text{for } (l', s') = (1,-1), \\
&= -6^{-\frac{1}{2}}(q_x v_F/\omega)\alpha_1 a_{l0}^{s0}, \quad \text{for } (l', s') = (1,1), \\
&= 3^{-\frac{1}{2}}(q_z v_F/\omega)\alpha_1 a_{l0}^{s0}, \quad \text{for } (l', s') = (1,0),
\end{aligned}$$

and $\delta_{l's'}$ is equal to zero otherwise. We can also represent (2.55) in a simpler matrix form

$$\overleftrightarrow{D}\,\vec{\Gamma} = \vec{C}. \tag{2.56}$$

The elements of the matrix $\overleftrightarrow{D}$ are

$$d_{ll'}^{ss'} = b_{ll'}^{ss'} - \delta_{ll'}^{ss'}, \tag{2.57}$$

and $\underline{\Gamma}$, $\underline{C}$ stand for the column vectors with $\Gamma_{l's'}$ and $C_{l's'}$ as components. The solution to (2.56) can then be easily written down as

$$\Gamma_l^s = X_l^s/\Delta, \quad \text{where } \Delta = |\overleftrightarrow{D}|, \quad \text{and } X_l^s = |\overleftrightarrow{D}^{ls}|. \tag{2.58}$$

In (2.58), $|\underline{D}|$ is the determinant of the matrix $\underline{D}$, and $\underline{D}^{ls}$ is the matrix obtained by replacing the (l,s)th column of $\underline{D}$ by the column vector $\underline{C}$. For simple metals with spherical Fermi surface, the expression (2.23) and (2.24) for the current density reduces to the following expressions:

$$\vec{j} = (em^* p_f/2\pi^2)\alpha_1 \textstyle\sum_m \int_{-1}^{+1} d(\cos\theta) f_m \vec{v}_m^* \tag{2.59}$$

and $$\vec{j} = \frac{ep_f^2}{4\pi}\alpha_1\left(\frac{4}{3}\right)^{\frac{1}{2}} \begin{pmatrix} \Gamma_1^{-1} - \Gamma_1^{1} \\ [\Gamma_1^{-1} + \Gamma_1^{1}]/i \\ \Gamma_1^{0}/\sqrt{2} \end{pmatrix} \quad . \tag{2.60}$$

Now it is easy to see that the solution for Γ_l^s of (2.58) is always in the form[21] $\Gamma_l^s = \vec{\Gamma}_l^s \cdot \vec{E}$, and if we write $C_l^s = \vec{C}_l^s \cdot \vec{E}$, the whole system of equations (2.55) still hold with C_l^s and Γ_l^s replaced by $\vec{C}_l^s$ and $\vec{\Gamma}_l^s$. It follows that X_l^s can also be expressed in the form

$$X_l^s = \vec{X}_l^s \cdot \vec{E} \quad . \tag{2.61}$$

Combining (2.58), (2.60), (2.61), we finally obtain an expression for the conductivity tensor as

$$\overleftrightarrow{\sigma} = \frac{ep_F^2\alpha_1}{\sqrt{3}\,2\pi^2\Delta} \begin{pmatrix} X_{1x}^{-1} - X_{1x}^{1} & X_{1y}^{-1} - X_{1y}^{1} & X_{1z}^{-1} - X_{1z}^{1} \\ (X_{1x}^{-1} + X_{1x}^{1})/i & (X_{1y}^{-1} + X_{1y}^{1})/i & (X_{1z}^{-1} + X_{1z}^{1})/i \\ X_{1x}^{0}/\sqrt{2} & X_{1y}^{0}/\sqrt{2} & X_{1z}^{0}/\sqrt{2} \end{pmatrix} \tag{2.62}$$

The form of the conductivity tensor as shown in (2.62) is complicated because each component (like X_{1x}^{-1}) is actually a $(N_0^2-1)\times(N_0^2-1)$ determinant. It simplifies considerably when we consider the particular geometry in which $q_z=0$. Then we have

$$a_{ll'}^{ss'}=0, \quad \text{if } (l+s) \text{ and } (l'+s') \text{ have different parity;}$$

and $C_{lz}^{s}=0$, for $(l+s)$ even; $C_{l(x,y)}^{s}=0$, for $(l+s)$ odd.

It follows readily from (2.62) that $\sigma_{xz}=\sigma_{zx}=0$, and the determinants for determining X_1^0 or X_1^{-1} and X_1^1 are separated. When we retain only terms linear in the A_n, we can write

$$\vec{X}_1^{\,0}=-\vec{C}_1^{\,0}-\sum{}'\vec{C}_l^{\,s}b_{1l}^{\,0s}, \quad \vec{X}_1^{\,1}=-\vec{C}_1^{\,1}-\sum{}'\vec{C}_l^{\,s}b_{1l}^{\,1s},$$

$$\vec{X}_1^{\,-1}=-\vec{C}_1^{\,-1}-\sum\vec{C}_l^{\,s}b_{1l}^{\,-1s}, \quad \Delta=1-\sum{}'b_{ll}^{\,ss}. \tag{2.63}$$

In (2.63), $\sum'$ in the equation for $\vec{X}_{l0}^{\,s_0}$ indicates that the summation excludes the term for which $l=l_0$ and $s=s_0$. Cohen, Harrison, and Harrison's result indicates that in the absence of Fermi-liquid interactions, the magnetoconductivity tensor has the following general property

$$\sigma_{xy}=-\sigma_{yx}. \tag{2.64}$$

It can be shown from the expression (2.62) or directly from the explicit form (2.30) and the result obtained by Hamilton and McWhorter[13] that (2.64) still hold when we retain A_0 only, or when we retain A_0 and A_1 in the geometry where $\vec{q}$ is perpendicular to $\vec{B}$. We have not been able to give a rigorous proof for (2.64) when the complete interaction function is taken into account. However, it is easy to show from (2.63) that if only the terms linear in the A_n are kept, (2.64) is still true.

Without actually solving for X_1^{-1}, X_1^1, and X_1^0 to obtain an explicit expression for σ, we can already obtain some general properties of σ in some limiting cases. These conclusions are independent of the manner in which we truncate the series (2.40). First, we note that at low-frequency and high-impurity concentration such that $\omega\tau\ll1$, (2.52) gives

$$a_{ll'}^{ss'}=\sum_m\int d(\cos\theta)(-A_{l'}+\tau/\tau_l')\theta_l^{\,s}\theta_{l'}^{\,s'}J_{m-s}J_{m-s'}. \tag{2.65}$$

With the aid of the mathematical identities[22]

$$\sum_{m=-\infty}^{\infty}J_{m-s}J_{m-s'}=\delta_{ss'}, \tag{2.66}$$

and $$\int_{-1}^{+1}d(\cos\theta)\theta_l^{\,s}(\theta)\theta_{l'}^{\,s}(\theta)=\delta_{ll'}, \tag{2.67}$$

(2.65) becomes $$a_{ll'}^{ss'}\approx\delta_{ll'}\delta_{ss'}(-A_l+\tau/\tau_l). \tag{2.68}$$

Hence it follows from (2.51) and (2.68) that

$$\Gamma_1^{\,m}=(\tau/\tau_1-\alpha_1)^{-1}C_1^{\,m}, \quad \text{for } m=\pm \text{ or } 0.$$

The conductivity tensor is therefore unchanged from the non-interacting expression $\vec{\vec{\sigma}}^{\,0}$ except for a prefactor $\alpha_1/(\alpha_1-\tau/\tau_1)$. Another situation in which the Fermi-liquid effect becomes very small is when the following criterion is satisfied

$$qv_F\gg|\tilde{\omega}|, \quad \text{and/or} \quad \omega_c\gg|\tilde{\omega}|. \tag{2.69}$$

In this case, we can drop terms of order ω/qv_F or ω/ω_c in (2.52). Then $a_{ll'}^{\,ss'}$ is approximately equal to $-A_l\delta_{ll'}\delta_{ss'}$, and it can be seen from (2.51) that $\Gamma_l^{\,s}$ is simply $-(1/\alpha_1)C_l^{\,s}$. The factor $1/\alpha_1$ cancels the factor α_1 appearing in (2.62) in the coefficient of $\vec{\vec{\sigma}}$, and the conductivity tensor then reduces exactly to the noninteracting expression. It is not difficult to see directly from the transport equation why the

Fermi-liquid effect disappears from the final result in the above limiting cases. In (2.6), if the time derivative term $-i\omega f$ is small compared with the term $(\tau^{-1}+i\vec{q}\cdot\vec{v}+\partial/\partial s)f$, then the function $f+\delta\epsilon_1$, which measures the departure from local equilibrium, obeys the same equation as the function f in the noninteracting case. Further, the expression for the current density (2.23) is simply the free electron expression with f replaced by $f+\delta\mathcal{E}_1$. Hence, the conductivity tensor is unaffected by the Fermi-liquid interaction when $\omega \ll |i/\tau+\vec{q}\cdot\vec{v}+\omega_c|$. It should be emphasized that this is a general result which is even true for metals with arbitrary Fermi surfaces. We would expect that the region where the Fermi-liquid effect is most dominant is that where the coefficient preceding f and $f+\delta\mathcal{E}_1$ are of the same order, i.e.,

$$qv_F \sim \omega, \quad \omega \sim \omega_c, \quad \text{and} \quad \omega\tau \gg 1. \tag{2.70}$$

The calculation of Platzman and Jacobs[23] on the Fermi-liquid effects in cyclotron resonance confirms the above criterion. These authors found by numerical calculations that considerable change results in the Doppler shifted cyclotron resonance when

$$(\omega_p/\omega)v_F/c \sim \omega/\omega_c \sim 1. \tag{2.71}$$

In the cyclotron-resonance experiment, the effective value of q inside the metal can be taken approximately as $1/\delta$ where δ is the skin depth given[24] by $\delta = c/(2\pi\sigma\omega)^{1/2}$. Since σ is of order ω_p^2/ω, q is then of order ω_p/c. Hence, (2.71) is just a special case of the general criterion (2.70).

Finally, we note that at $q=0$, i.e., for a uniform perturbing field, $J_l(X\sin\theta)$ is nonvanishing only for $l=0$, and (2.52), (2.53) reduces, respectively, to

$$a_{ll'}^{ss'} = [(-A_l + A_l\omega/(\omega - s\omega_c)]\,\delta_{ll'}\delta_{ss'}, \tag{2.72}$$

and

$$C_l^s = -\int_{-1}^{+1} [i\vec{E}\cdot\vec{v}_s/(\omega - s\omega_c)]\,\theta_l^s(\theta)\,d(\cos\theta). \tag{2.73}$$

In writing down (2.72) and (2.73), we have dropped the collision terms for simplicity. Γ_l^s is obtained by substituting (2.72) and (2.73) into (2.51) and is given by

$$\Gamma_l^s = -[(\omega - s\omega_c)/(\omega - \alpha_l\omega_c)]\,C_l^s. \tag{2.74}$$

It can then be easily seen that the conductivity tensor $\vec{\vec{\sigma}}$ in this limit can be obtained by simply replacing the effective mass m^* appearing in σ_0 by m^*/α_1 (both in the prefactor and in $\omega_c = eB/m^*c$). According to Pines and Nozieres,[8] m^*/α_1 is just the crystalline mass m_c of the electron.

III. SPIN-INDEPENDENT OSCILLATIONS OF THE ELECTRON LIQUID

A. General Theory

In the preceding section, we have demonstrated how the conductivity tensor $\vec{\vec{\sigma}}$ of a metal can be evaluated in the Fermi-liquid theory by solving the Landau-Silin equation. The wave number and frequency dependent spin susceptibility χ can be determined by an analogous procedure.[25-28] These two response functions are characteristic of the system, and they specify completely the behavior of the system under the influence of a perturbing field. Once the values of the electric field $\vec{E}(\vec{r},t)$ and the magnetic induction $\vec{b}(\vec{r},t)$ are given, the values of the current density $\vec{j}$ and the magnetization $\vec{M}$ are uniquely determined. However, the current or the magnetization will in turn influence the field. The fields $\vec{E}$ and $\vec{b}$ are coupled and have to be determined self-consistently from the Maxwell's equations. In general a propagating field of the plane wave form $\exp[i(\vec{q}\cdot\vec{r}-\omega t)]$ cannot exist in a metal because of the large value of the conductivity. A propagating field is only possible if some kind of collective oscillations like plasma waves or spin waves are excited. The investigation of these problems requires the simultaneous study of the kinetic equation and the Maxwell's equations. We write the four Maxwell's equations as[29]

$$\vec{\nabla}\times\vec{E} = -(1/c)\partial\vec{b}/\partial t, \tag{3.1}$$

$$\vec{\nabla}\cdot\vec{b} = 0, \tag{3.2}$$

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho \ , \tag{3.3}$$

$$\vec{\nabla} \times \vec{b} = (1/c)\, \partial \vec{E}/\partial t + (4\pi/c)\vec{j} + 4\pi \vec{\nabla} \times \vec{M} \ . \tag{3.4}$$

From the solution of kinetic equations, we have the two constitutive equations

$$\vec{j} = \overleftrightarrow{\sigma} \cdot \vec{E}, \tag{3.5}$$

and[30] $$\vec{M} = \overleftrightarrow{\alpha} \cdot \vec{b} \ . \tag{3.6}$$

Since we have used a semiclassical transport equation to solve for $\overleftrightarrow{\sigma}$ and $\overleftrightarrow{\alpha}$, quantum effects such as the Landau diamagnetism have been neglected. In discussing the electric field excitations, these diamagnetic terms are certainly small compared with the field $\vec{b}$ induced by the electric field from (3.1). In the case of spin wave excitations, we are interested in the pole of the susceptibility. Hence, the Landau diamagnetic terms can still be neglected even though it is of the same order as that of the spin susceptibility.

Let us investigate the condition for the existence of a self-sustaining field $\vec{E}(\vec{q}, \omega) \exp[i(\vec{q} \cdot \vec{r} - \omega t)]$ in a metal. The current density induced by such a field is given by (3.5). However, when we substitute (3.6) and (3.1) into (3.4), we see that there exists another relation between $\vec{j}$ and $\vec{E}$ which must also be satisfied by the actual value of the field and the current existing in the metal. After a little algebraic manipulation, this relation can be expressed as

$$\vec{j} = \overleftrightarrow{\Gamma} \cdot \vec{E} \ , \tag{3.7}$$

where $\overleftrightarrow{\Gamma}$ is given by

$$\overleftrightarrow{\Gamma} = (i\omega/4\pi)\, \overleftrightarrow{I} + (ic^2/4\pi\omega)(\overleftrightarrow{Q} - 4\pi \overleftrightarrow{Q}\, \overleftrightarrow{\alpha}) \cdot \overleftrightarrow{Q} \ , \tag{3.8}$$

in (3.8), $\overleftrightarrow{I}$ is the identity matrix, and $\overleftrightarrow{Q}$ is the tensor defined by

$$\overleftrightarrow{Q} = \begin{pmatrix} o & -q_z & 0 \\ q_z & 0 & -q_x \\ 0 & q_x & 0 \end{pmatrix}. \tag{3.9}$$

In writing (3.9) we have assumed that the wave vector $\vec{q}$ lies in the $y-z$ plane. Provided we are not near the pole of the spin susceptibility, elements[31] of $\overleftrightarrow{\alpha}$ are only of order 10^{-6}. Hence, the matrix $\overleftrightarrow{I} - 4\pi\overleftrightarrow{\alpha}$ can be simply replaced by the identity matrix $\overleftrightarrow{I}$. Then $\Gamma_{\mu\nu}$ reduces to the familiar expression

$$(ic^2q^2/4\pi\omega)(\omega^2/c^2q^2 - 1)\delta_{\mu\nu} + ic^2 q_\mu q_\nu/4\pi\omega \ .$$

The condition for the existence of a self-sustaining propagating field of the form $\vec{E}(\vec{q}\,\omega) \exp[i(\vec{q} \cdot \vec{r} - \omega t)]$ in a metal is equivalent to the condition that $\vec{E}$ satisfy both (3.7) and (3.5) simultaneously, i.e.,

$$| \overleftrightarrow{\sigma} - \overleftrightarrow{\Gamma} | = 0 \ . \tag{3.10}$$

The ratio of the elements of the two matrices $\sigma_{\alpha\beta}/\Gamma_{\alpha\beta}$ is of order $\omega_p^2\omega/c^2q^2\omega_c$. When the wavelength of the disturbance is not excessively small and the frequency is of order ω_c, then the factor $\omega_p^2\omega/c^2q^2\omega_c$ is much larger than unity. Hence, (4.10) reduces to the simpler equation

$$| \overleftrightarrow{\sigma} | = 0. \tag{3.11}$$

The propagation of plasma waves near the fundamental cyclotron resonance in potassium has been studied recently by Walsh and Platzman.[12,32] The experiments are carried out in the Azbel-Kaner geometry, with samples which are thin, plane-parallel slabs. The plasma waves propagate perpendicular to the dc magnetic field $\vec{B}$, and may be polarized either parallel or perpendicular to $\vec{B}$. In the former case,[32] the experimental data differs slightly from the prediction of the free electron model, while in the latter,[12] there appears to be considerable deviation. Platzman and Walsh attributed the deviation to Fermi-liquid effects, and present an analysis which involved only the Fermi-liquid interaction coefficients A_0 and A_1. This analysis contained some numerical errors, and the agreement of theory and experiment was fortuitous. Before studying this problem in detail, it is worth considering the condition for the self-sustaining magnetization of the plane wave form. By substituting (3.5) into (3.4) and making use of (3.1), we can obtain a relation between $\vec{M}$ and $\vec{b}$ as

$$\vec{b} = \overleftrightarrow{\Omega} \cdot \vec{M} , \tag{3.12}$$

where $\overleftrightarrow{\Omega}$ is the tensor defined by

$$\overleftrightarrow{\Omega} = (4\pi c/\omega)\overleftrightarrow{Q}\,[\overleftrightarrow{Q}^2 c/\omega + (\omega/c)\overleftrightarrow{I} + 4\pi i\overleftrightarrow{\sigma}/c]^{-1}\overleftrightarrow{Q} . \tag{3.13}$$

The constitutive equation (3.6) provides another relation between $\vec{b}$ and $\vec{M}$. The condition that a finite magnetization exists in the metal is then

$$|\overleftrightarrow{\alpha}^{-1} - \overleftrightarrow{\Omega}| = 0 . \tag{3.14}$$

The solution of (3.14) is quite complicated in general. However we see from (3.13) that the elements of $\overleftrightarrow{\Omega}$ are of order $c^2q^2\omega_c/\omega_p{}^2\omega$. Except near the region of frequency and wavelength such that $|\overleftrightarrow{\sigma}| = 0$, which is precisely the condition for the existence of plasma waves, (3.14) reduces to the simpler equation

$$|\overleftrightarrow{\alpha}^{-1}| = 0. \tag{3.15}$$

Taking into account the properties of the spin susceptibility tensor one can show that (3.15) factorize into three equations $\alpha_+{}^{-1} = 0$, $\alpha_-{}^{-1} = 0$, and $\alpha_{zz}{}^{-1} = 0$, corresponding to two waves circularly polarized in the plane perpendicular to $\vec{B}$, and another one polarized parallel to $\vec{B}$. It is important to note that the approximation of dropping the terms proportional to $\overleftrightarrow{\alpha}$ in (3.10) and the terms proportional to $\overleftrightarrow{\Omega}$ in (3.14) corresponds physically to an uncoupling of the plasma waves and spin waves. This will be a good approximation as long as the two dispersion curves do not cross each other. At the point of crossing of the curves however, the spin waves and plasma waves will interfere with each other and the resulting dispersion relation for the waves must be studied through the full equations (3.10) and (3.14) instead of the approximate ones $|\overleftrightarrow{\sigma}| = 0$ and $|\overleftrightarrow{\alpha}^{-1}| = 0$. Either from (3.8) and (3.13) or from the Maxwell's equations directly, it is clear that spin wave polarized parellel to $\vec{B}$ can only interact with plasma wave polarized perpendicular to $\vec{B}$ and vice versa.

We have seen that the dispersion relation for the plasma waves in a metal can be readily obtained by putting $|\overleftrightarrow{\sigma}| = 0$. However, this method of studying the collective oscillations proves to be very inconvenient for the following reasons. First, when we want to retain terms in the expansion of $\phi(\vec{p}, \vec{p}')$ that are proportional to A_n for $n > 2$, there exist more than one solution in the neighborhood of every integral multiple of ω_c. It is not easy to see which particular mode corresponds to the experimentally observed one and to understand the physical difference between the different modes. Secondly, if it is assumed that $A_n = 0$ for $n > 2$, then whatever the geometry is, A_0 and A_1 always drop out of the final dispersion relation. In the geometry with $q_z = 0$, a factorization always seems to result when we form the combination $\sigma_{xx}\sigma_{yy} - \sigma_{xy}\sigma_{yx}$. It is important to ascertain whether these results are just accidental, or really a general feature independent of the manner in which we truncate the interaction functions. All these questions are difficult to answer from a study of the equations (3.11) and (3.15). In this section we present[6] a different method for studying the plasma waves. This method is completely equivalent to that presented at the beginning of this section, but it provides the result in a more interpretable way. Instead of calculating the conductivity tensor from the transport equation, we use the Maxwell's equations to relate the electric field to the charge and current densities. We then express these in terms of the distribution function $f(\theta, \phi)$. The kinetic equation then becomes a homogeneous integral equation for the function $f(\theta, \phi)$. If we then use the technique developed in Sec. II, this integral equation can be converted into a set of homogeneous simultaneous linear equations. The condition for $f(\theta, \phi)$ to be finite, or equivalently the condition for the existence of a self-sustaining electric field, can then be obtained by setting the determinant of the coefficient matrix equal to zero. This method is very similar to that first introduced by Silin[16] in his classic paper on the oscillations of a degenerate electron fluid in the presence of a dc magnetic field. However, Silin did not go into a detailed study and has only obtained the result that for $n > |m| > 1$, the eigenfrequencies of the system are $\omega = m\omega_c(1 + A_n) + O(q^2)$. Recently, Mermin, and Cheng[33] have extended Silin's analysis to shorter wavelengths by evaluating the term of order q^2 for propagation perpendicular to the dc magnetic field. In their theory, the $\vec{q} \cdot \vec{v}$ term in the kinetic equation is regarded as a small perturbation and hence it cannot be extended to shorter wavelength region such that $q^2r_c{}^2$ is larger than a particular value of A_n. We shall be able to obtain a result which reduces to that of Silin in the long wavelength limit, but is moreover valid for arbitrary wavelength, provided of course that the condition for the validity of Fermi liquid theory, namely $qv_f \ll \mu$, is still satisfied.

B. Propagation Perpendicular to the dc Magnetic Field

From Maxwell's equations and the magnetic constitutive equation, the electric field is related to the current density by the equation

$$\vec{E} = \overleftrightarrow{\Gamma}^{-1} \cdot \vec{j} \tag{3.16}$$

Here $\overleftrightarrow{\Gamma}$ is the tensor defined in (3.7). When the term proportional to α is dropped in (3.8), the inverse of $\overleftrightarrow{\Gamma}$ can easily be obtained as

$$\overleftrightarrow{\Gamma}^{-1} = \frac{4\pi\omega}{i} \begin{pmatrix} \omega^{-2} & 0 & 0 \\ 0 & (\omega^2 - c^2q^2)^{-1} & 0 \\ 0 & 0 & (\omega^2 - c^2q^2)^{-1} \end{pmatrix}. \tag{3.17}$$

For simplicity in discussion, we have taken $\vec{q}$ to be in the x direction. Making use of (3.16), together with the expression for the current density (2.59), we can write the kinetic equation (2.50) in the form

$$\begin{aligned} f_m = {} & [\tfrac{1}{2}(\omega_p{}^2\alpha_1/\omega^2)\omega/(\omega - m\omega_c) + \tfrac{1}{2}A_1 m\omega_c/(\omega - m\omega_c)](J_{m+1}\theta_1{}^{-1} - J_{m-1}\theta_1{}^{1})G_1{}^{1} \\ & + [-(i/2)[\omega_p{}^2\alpha_1/(\omega^2 - c^2q^2)]\omega/(\omega - m\omega_c) + \tfrac{1}{2}A_1 m\omega_c/(\omega - m\omega_c)](J_{m+1}\theta_1{}^{-1} + J_{m-1}\theta_1{}^{1})G_1{}^{-1} \\ & + \{[\omega_p{}^2\alpha_1/(\omega^2 - c^2q^2)]\omega/(\omega - m\omega_c) + \tfrac{1}{2}A_1 m\omega_c/(\omega - m\omega_c)\}J_m\theta_1{}^{0}G_1{}^{0} \\ & + A_0 \frac{m\omega_c}{\omega - m\omega_c}\left[\left(\frac{1}{6}\right)^{1/2}\frac{q_z v_F}{\omega}\alpha_1 J_m G_1{}^{0} + \left(\frac{1}{12}\right)^{1/2}\frac{q_x v_F}{\omega}\alpha_1 J_m G_1{}^{+1}\right] + \sum_{l,\,|s|<l}\left(-1 + \frac{\omega}{\omega - m\omega_c}\right) \\ & \times A_l G_l{}^{s}\theta_l{}^{s} J_{m-s}. \end{aligned} \tag{3.18}$$

In writing down (3.18), we have made use of the equation of continuity (2.54) to eliminate $G_0{}^0$. The sum over l runs through all integral values starting from $l = 2$. The $G_l{}^s$ are defined in the following manner:

$$G_l{}^{s} \equiv \Gamma_l{}^{s} \equiv (2\pi)^{-1/2} f_l{}^{s} = \sum_m \int_{-1}^{+1} \theta_l{}^{s}(\theta) f_m(\theta) J_{m-s}(X\sin\theta)\, d(\cos\theta),$$

for $(l, s) \neq (1, 1)$ or $(1, -1)$, while

$$\begin{aligned} G_1{}^{1} &= \sum_m \int_{-1}^{+1} d(\cos\theta)(\theta_1{}^{-1}J_{m+1} - \theta_1{}^{1}J_{m-1}) f_m = \Gamma_1{}^{-1} - \Gamma_1{}^{1}, \\ G_1{}^{-1} &= \sum_m \int_{-1}^{+1} d(\cos\theta)(\theta_1{}^{1}J_{m-1} + \theta_1{}^{-1}J_{m+1}) f_m = \Gamma_1{}^{1} + \Gamma_1{}^{-1}. \end{aligned} \tag{3.19}$$

We have repeatedly emphasized that in the region of frequency and wavelength under consideration, terms of order (ω^2_p/c^2q^2) (ω/ω_c) is much larger than unity. It is then obvious from (3.18) that in the coefficient $G_1{}^1$, $G_1{}^{-1}$, and $G_1{}^0$ all terms involving A_0 and A_1 can be dropped with no appreciable error. This explains why A_0 and A_1 do not enter the final dispersion relation for the plasma waves. The approximation made here is equivalent to that of dropping the term $\overleftrightarrow{\Gamma}$ compared with $\overleftrightarrow{\sigma}$ in the dispersion relation $|\overleftrightarrow{\sigma} - \overleftrightarrow{\Gamma}| = 0$. The difference in form arises because we are working here with the scalar distribution directly instead of the electric field vector. Multiplying (3.18) by $\theta_l{}^s(\theta)J_{m-s}(X\sin\theta)$, summing over m and integrating over $d(\cos\theta)$ gives an infinite set of homogeneous simultaneous equations for the functions $G_l{}^s$. The condition that a non-zero distribution exists is obtained by setting the determinant of the coefficient matrix equal to zero. We find it most convenient to write the infinite determinantal equation as

$$\left| a_{nn'}{}^{mm'} - \delta_{nn'}{}^{mm'} \right| = 0, \tag{3.20}$$

where for $n', n \neq 1$,

$$a_{nn'}{}^{mm'} = -A_n\delta_{nn'}{}^{mm'} + \sum_s \frac{\omega A_{n'}}{\omega - s\omega_c}\int_{-1}^{+1} d(\cos\theta)\theta_n{}^{m}\theta_{n'}{}^{m'}J_{s-m}J_{s-m'}, \tag{3.21}$$

for $n'=1,\ n\neq 1,$ $$a_{n1}^{mm'}=\sum_s \frac{\omega}{\omega-s\omega_c}\Lambda_1^{m'}\int_{-1}^{+1}d(\cos\theta)u_s^{m'}\theta_n^{m}J_{s-m}(X\sin\theta), \tag{3.22}$$

for $n=1,\ n'\neq 1,$ $$a_{1n'}^{mm'}=\sum_s A_{n'}\left(-1+\frac{\omega}{\omega-s\omega_c}\right)\int_{-1}^{+1}d(\cos\theta)\theta_{n'}^{m'}u_s^{m}J_{s-m'}(X\sin\theta) \tag{3.23}$$

and for $n=n'=1,$ $$a_{11}^{mm'}=\sum_s \frac{\omega}{\omega-s\omega_c}\Lambda_1^{m'}\int_{-1}^{+1}d(\cos\theta)u_s^{m}u_s^{m'}. \tag{3.24}$$

In these equations, we have introduced

$$u_s^{1}=J_{s+1}(X\sin\theta)\theta_1^{-1}-J_{s-1}(X\sin\theta)\theta_1^{1},\ u_s^{-1}=J_{s+1}(X\sin\theta)\theta_1^{-1}+J_{s-1}(X\sin\theta)\theta_1^{1},\ u_s^{0}=J_s(X\sin\theta)\theta_1^{0}, \tag{3.25}$$

and $\Lambda_1^{1}=\frac{1}{2}\omega_p^{2}\alpha_1/\omega^2,\quad \Lambda_1^{-1}=-\frac{1}{2}i\omega_p^{2}\alpha_1(\omega^2-c^2q^2)^{-1},\quad \Lambda_1^{0}=\omega_p^{2}\alpha_1/(\omega^2-c^2q^2).$

We have assumed the condition $\omega\tau_l\gg 1$ for all l and dropped the collision terms. This is just for the sake of brevity in writing. In any case, our discussion in Sec. II indicates that this is the region where Fermi liquid effects become appreciable. Hence, experimental investigation of plasma waves should be performed in the region $\omega\tau_l\gg 1$.

In the geometry under consideration with $\vec{q}$ perpendicular to $\vec{B}$, it is not difficult to see that $a_{nn'}^{mm'}$ vanishes if $(n-m)$ is of different parity from $(n'-m')$. Thus the secular equation reduces to one for odd $(n-m)$ and one for even $(n-m)$. They contain modes polarized perpendicular and parallel to $\vec{B}$ respectively. Before going to a detailed study of the solutions of equations (3.20), it is interesting to demonstrate that a similar infinite determinantal equation can be obtained by starting from the kinetic equation (2.3) and expanding the distribution function f directly in terms of the spherical harmonics. When this is done, an infinite set of equation results in the form[34]

$$\begin{aligned}\{\omega-\alpha_n m\omega_c-(\delta_{n1}^{m1}+\delta_{n1}^{m-1})[\omega^{-2}-(\omega^2-c^2q^2)^{-1}]\omega_p^{2}\alpha_1\omega-\delta_{n1}^{m0}\omega_p^{2}\alpha_1/(\omega^2-c^2q^2)\}f_n^{m}\\ -\alpha_{n-1}\left(\left\langle\begin{matrix}m\\n\end{matrix}\middle|\vec{q}\cdot\vec{v}\middle|\begin{matrix}m-1\\n-1\end{matrix}\right\rangle f_{n-1}^{m-1}+\left\langle\begin{matrix}m\\n\end{matrix}\middle|\vec{q}\cdot\vec{v}\middle|\begin{matrix}m+1\\n-1\end{matrix}\right\rangle f_{n-1}^{m+1}\right)\\ -\alpha_{n+1}\left(\left\langle\begin{matrix}m\\n\end{matrix}\middle|\vec{q}\cdot\vec{v}\middle|\begin{matrix}m-1\\n+1\end{matrix}\right\rangle f_{n+1}^{m-1}+\left\langle\begin{matrix}m\\n\end{matrix}\middle|\vec{q}\cdot\vec{v}\middle|\begin{matrix}m+1\\n+1\end{matrix}\right\rangle f_{n+1}^{m+1}\right)\\ +\omega_p^{2}\alpha_1[\omega^{-2}+(\omega^2-c^2q^2)^{-1}](\delta_{n1}^{m1}f_1^{-1}+\delta_{n1}^{m-1}f_1^{1})=0.\end{aligned} \tag{3.26}$$

If we proceed as before and set the determinant of the coefficient matrix equal to zero, we obtain again two secular equations, one for $n-m$ odd and one for $n-m$ even. Except for $n=1$ or 2, a typical block in the determinant would appear as

$$\begin{vmatrix}\omega-\alpha_{n-1}(m-1)\omega_c & 0 & \left\langle\begin{matrix}m-1\\n-1\end{matrix}\middle|\vec{q}\cdot\vec{v}\middle|\begin{matrix}m\\n\end{matrix}\right\rangle & 0 & 0\\ 0 & \omega-\alpha_{n-1}(m+1)\omega_c & \left\langle\begin{matrix}m+1\\n-1\end{matrix}\middle|\vec{q}\cdot\vec{v}\middle|\begin{matrix}m\\n\end{matrix}\right\rangle & 0 & 0\\ \left\langle\begin{matrix}m\\n\end{matrix}\middle|\vec{q}\cdot\vec{v}\middle|\begin{matrix}m-1\\n-1\end{matrix}\right\rangle & \left\langle\begin{matrix}m\\n\end{matrix}\middle|\vec{q}\cdot\vec{v}\middle|\begin{matrix}m+1\\n-1\end{matrix}\right\rangle & \omega-\alpha_n m\omega_c & \left\langle\begin{matrix}m\\n\end{matrix}\middle|\vec{q}\cdot\vec{v}\middle|\begin{matrix}m-1\\n+1\end{matrix}\right\rangle & \left\langle\begin{matrix}m\\n\end{matrix}\middle|\vec{q}\cdot\vec{v}\middle|\begin{matrix}m+1\\n+1\end{matrix}\right\rangle\\ 0 & 0 & \left\langle\begin{matrix}m-1\\n+1\end{matrix}\middle|\vec{q}\cdot\vec{v}\middle|\begin{matrix}m\\n\end{matrix}\right\rangle & \omega-\alpha_{n+1}(m-1)\omega_c & 0\\ 0 & 0 & \left\langle\begin{matrix}m+1\\n+1\end{matrix}\middle|\vec{q}\cdot\vec{v}\middle|\begin{matrix}m\\n\end{matrix}\right\rangle & 0 & \omega-\alpha_{n+1}(m+1)\omega_c\end{vmatrix}=0. \tag{3.27}$$

The block for $n=1$, $m=1, 0$, etc., is similar with extra terms proportional to ω_p^2 in both the diagonal and the off-diagonal terms. It can be easily seen this method is exactly equivalent to that of Mermin and Cheng's perturbation theory. The form of (3.27) reminds one of the usual eigenvalue equation $|H_0+H'-I|=0$ in the representation with the eigenstates of H_0, i.e., Y_n^m, as the basis. H' is just the perturbation $\vec{q}\cdot\vec{v}$. For $n>1$, the eigenvalue of the system is

$$\omega_{nm}=\alpha_n m\omega_c+\gamma_{nm}X^2, \tag{3.28}$$

while for $n=1$, $\omega=\omega_p+O(q^2)$. (3.29)

The modes with $n=1$ are just the "ordinary" plasmons discussed by many authors before.[17] The coefficient of X^2 in (3.29) cannot be taken too seriously because for metals, ω_p is comparable with the Fermi energy and the validity of the Fermi-liquid theory is then doubtful. The coefficient γ_{nm} can be obtained by using standard second-order perturbation theory as done by Mermin and Cheng.

We now go back to (3.20) and show that we can obtain the result (3.28) and an explicit expression for γ_{nm} as well. In addition, the result is valid for arbitrary value of qr_c. The method is particularly simple when we can approximate the interaction function $\phi(\vec{p},\vec{p}')$ by a finite number of terms in its expansion in Legendre polynomials. In that case, (3.20) reduces to a finite size determinantal equation. First, we note that whenever ω is not exactly an integral multiple of ω_c, the diagonal element in the matrix $a_{nn'}^{mm'}-\delta_{nn'}^{mm'}$ is

$$a_{nn}^{mm}-1=(\alpha_n m\omega_c-\omega)/(\omega-m\omega_c)+O(X^2), \tag{3.30}$$

when we are considering the (n,m) mode corresponding to that given by (3.28), then the diagonal element $a_{nn}^{mm}-1$ as given by (3.30) is of order X^2. In order to get the dispersion relation correct to order X^2, we need only retain the term independent of X in all diagonal elements and the terms proportional to X in the (n,m)th column and row, γ_{nm} is then easily evaluated as

$$\gamma_{nm}=\frac{A_n\alpha_n m\omega_c\left\langle{m\atop n}\middle|\sin^2\theta\middle|{m\atop n}\right\rangle}{2(m^2A_n^2-1)}+\frac{m\omega_c\alpha_n^2}{4}\sum_{n',m}{}'\frac{A_{n'}\left|\left\langle{m'\atop n'}\middle|\sin\theta\middle|{m\atop n}\right\rangle\right|^2}{(m\alpha_n-m')(m\alpha_n-m'\alpha_{n'})}, \tag{3.31}$$

where the sum on n', m' includes the terms $m'=m\pm1$, $n'=n\pm1$ only. For $n=2$, the terms involving A_{n-1} must be replaced by

$$-\frac{\omega_c}{4}\left|\left\langle{0\atop 1}\middle|\sin\theta\middle|{1\atop 2}\right\rangle\right|^2 \text{ for } m=1, \quad \text{and by} \quad -\frac{1}{4}\frac{\alpha_2\omega_c}{2\alpha_2-1}\left|\left\langle{1\atop 1}\middle|\sin\theta\middle|{2\atop 2}\right\rangle\right|^2 \text{ for } m=2.$$

In these equations $\left\langle{m\atop n}\middle|\sin\theta\middle|{m'\atop n'}\right\rangle=\int_{-1}^{+1}d(\cos\theta)\theta_n^m\theta_{n'}^{m'}\sin\theta$. (3.32)

It should be emphasized that the relation $\omega_{nm}=\alpha_n m\omega_c+\gamma_{nm}X^2$ is only valid in the region $\gamma_{nm}X^2\ll mA_n\omega_c$ since we have treated quantities like $X^2/(\omega-m\omega_c)$ as of order X^2 in arriving at (3.31). The result here is equivalent to that of Mermin and Cheng. The first point to be noted from (3.31) is that except for $n=2$, γ_{nm} vanishes when all A_n are put to zero. This indicates that at least in the long wavelength limit, Fermi-liquid interaction does introduce drastic changes in the propagation of wave in simple metals because the dispersion relation becomes quadratic in powers of X when the A_n are finite, instead of quartic or higher powers of X in the absence of electron correlations. Secondly, the question arises as to how each mode is related to the plasma waves observed experimentally. To answer this question, it is easier to look at (3.26). The modes with $n=1$ are clearly the transverse or longitudinal plasmons, and we shall not discuss them further. For $n>2$, in the (n,m) mode, ω tends to $\alpha_n m\omega_c$ as X tends to zero, and only f_n^m can remain finite, all other spherical-harmonic components of the distribution function goes to zero as different powers of X. In particular, the components f_1^1, f_1^{-1}, f_1^0 and hence the electric field associated with them all tend to zero in the infinite wavelength limit for all modes with $n\geq 2$. The plasma waves observed by Platzman and Walsh are predominantly zero sound waves. Equation (3.26) also indicates that if f_n^m is independent of X, f_1^m varies with X as

$$f_1^m\alpha X^{n-1}. \tag{3.33}$$

Hence, the electric field associated with the oscillations is strongest in the (2,1) mode. If we are considering the plasma waves in the neighborhood of $\omega = \omega_c$, then the mode easiest to excite is the (2,1) mode for polarization parallel to $\vec{B}$ and (3,1) mode for the perpendicular polarization. For propagation near the higher harmonics $\omega = m\omega_c$ with $m > 1$, it is readily seen that the electric field is strongest in the (n,m) mode for which $n = |m|$ or $n = |m|+1$ in the perpendicular and parallel polarization, respectively. Hence, as X tends to zero, the plasma wave frequency tends to $m(1+A_{|m|})\omega_c$ or $m(1+A_{|m|+1})\omega_c$ depending on the polarization. This fact enables us to determine, in principle at least, the Fermi-liquid interaction coefficients A_n for $n > 1$. An alternative method of obtaining values of the interaction coefficients A_n is to study the coefficient of X^2 term in the dispersion relation of the plasma wave in the mth harmonic. This furnishes information concerning A_m, A_{m-1}, and A_{m+1}. Except for the (2,1) mode, the Fermi liquid effects reveal themselves eminently since the coefficients of X^2 vanish in the absence of correlations. If we are able to study experimentally plasma waves with $\omega \sim \omega_c$ and $\omega \sim 3\omega_c$ for the perpendicular polarization and $\omega \sim 2\omega_c$ for the parallel case in the long wavelength limit, A_2, A_3, and A_4 can then be determined. Let us illustrate here, for example, the dispersion relation for the (3,1) mode and the (2,1) mode to terms of order X^2. These results are obtained by evaluating γ_{nm} from (3.31). For the (2,1) mode, we have

$$\omega_{21} = \omega_c(1+A_2) - \omega_c X^2(\tfrac{1}{10} + \tfrac{2}{7}A_2 - \tfrac{13}{70}A_3), \tag{3.34}$$

while for the (3,1) mode,

$$\omega_{31} = \omega_c(1+A_3) - \omega_c X^2(-\tfrac{1}{10}A_2 + \tfrac{2}{15}A_3 - \tfrac{1}{6}A_4). \tag{3.35}$$

In writing down (3.34) and (3.35), we have only kept terms linear in the A_n.

We now study the validity of the model in which we truncate the series (2.40) in an arbitrary manner. If the interaction coefficients A_n decrease rapidly with increasing n, naturally we would expect that retaining only a few terms in the expansion should be sufficient. This may lead to erroneous conclusions in some circumstances. For example, in the extreme case when the Fermi-liquid interaction is completely neglected, all different modes with the same value of m starts at the frequency $\omega = m\omega_c$ in the infinite wavelength limit. Using degenerate perturbation theory, Mermin and Cheng have shown that each of these modes then has a dispersion relation as $\omega_{nm} = m\omega_c + X^{2(n-1)}$. If we study the plasma wave through (3.20) in this limit, the dispersion relation becomes

$$a_{11}{}^{00} = 0, \tag{3.36}$$

and $$a_{11}{}^{11}a_{11}{}^{-1-1} - a_{11}{}^{-11}a_{11}{}^{1-1} = 0, \tag{3.37}$$

for the two polarizations, respectively. These are identical to the equations $\sigma_{zz}{}^0 = 0$ and $\sigma_{xx}{}^0\sigma_{yy}{}^0 + \sigma_{xz}{}^{02} = 0$, where $\bar{\sigma}^0$ is the conductivity tensor for the noninteracting electron gas. The solutions of (3.36) and (3.37), however, contain only one branch near every harmonics and many modes have apparently disappeared. The reason for this can be seen from the expression (3.24) for $a_{11}{}^{00}$, which is

$$a_{11}{}^{00} = \Lambda_1{}^0 \textstyle\sum_s [\omega/(\omega - s\omega_c)] \int_{-1}^{+1} d(\cos\theta) J_s{}^2(X\sin\theta). \tag{3.38}$$

The factor $\omega - s\omega_c$ in the denominator originates from the process of dividing through the kinetic equation by this term. For the (l,s) mode in the long-wavelength limit, $\omega - s\omega_c$ varies with $X^{2(l-1)}$, whereas the numerator for this term varies as $X^{2|s|}$ for $X \ll 1$. Hence, the solution of (3.36) in the neighborhood of $\omega \sim s\omega_c$ and $X \ll 1$ is only valid if the following inequality is satisfied

$$2(l-1) \leq 2|s|, \tag{3.39}$$

s, being the azimuthal index, always satisfy the relation

$$|s| \leq l. \tag{3.40}$$

It follows from (3.39) and (3.40) then that for each given value of s, the solution of (3.36) correspond to the mode $(|s|+1, s)$. We have already seen that this is exactly the mode which is easiest to excite for ω near $s\omega_c$. A similar argument holds for the other polarization, and the mode determined from (3.37) in the neighborhood of $\omega \sim s\omega_c$ is the $(|s|, s)$ mode. When we truncate the series for $\phi(\vec{p},\vec{p}')$ at successively larger values of n, more and more modes will appear. In general, to study a particular (n,m) mode, it

is clear from the above discussion that all A_n should be retained up to $n'=n+1$ in the long-wavelength limit. As X increases, more and more modes will be mixed and the full determinantal equation (3.20) has to be solved. An exact solution with all the A_n finite would be extremely complicated and impractical. A considerable simplification results when only terms linear in the A_n are kept in (3.2). This can be easily seen to be a good approximation when the following inequality is satisfied:

$$\omega_c A_n \ll \omega - s\omega_c \ , \tag{3.41}$$

for all allowed integer values of n and s. The form of the dispersion relation (3.34) and (3.35) suggests that, in general, (3.41) would be satisfied for values of X greater than unity. With this linearization, (3.20) becomes

$$a_{11}^{\ 00}\left(1-\sum_{n,|m|<n} a_{nn}^{\ mm}\right)+\sum_{n,|m|<n} a_{1n}^{\ 0m}a_{n1}^{\ m0}=0 \ , \tag{3.42}$$

for polarization parallel to the dc field, and

$$K_0\left(1-\sum_{n,m} a_{nn}^{\ mm}\right)+a_{11}^{\ 11}\sum_{n,m} a_{n1}^{\ m-1}a_{1n}^{\ -1m}+a_{11}^{\ -1-1}\sum_{n,m} a_{n1}^{\ m1}a_{1n}^{\ 1m}$$

$$-a_{11}^{\ 1-1}\sum_{n,m} a_{n1}^{\ m1}a_{1n}^{\ -1m}-a_{11}^{\ -11}\sum_{n,m} a_{n1}^{\ m-1}a_{1n}^{\ 1m}=0 . \tag{3.43}$$

In these equations the summations are performed over all $n>1$ and all m such that $|m|<n$; in addition $n-m$ must be odd in (3.42) and even in (3.43). The function K_0 is equal to $a_{11}^{\ 11}a_{11}^{\ -1-1}-a_{11}^{\ -11}a_{11}^{\ 1-1}$.

We now examine the dispersion relation in the short-wavelength limit but with values of X such that the criterion $qv_f \ll \epsilon_f$ is still satisfied. The asymptotic form of the Bessel function is[35]

$$J_l(X\sin\theta)=(2/X\sin\theta)^{1/2}\cos(X\sin\theta-\tfrac{1}{2}l\pi-\tfrac{1}{4}\pi) \quad \text{for } X\gg 1 . \tag{3.44}$$

Making use of (3.44), a general matrix element appearing in (3.20) can be written as

$$a_{nn'}^{\ mm'}=-A_{n'}\delta_{nn'}^{\ mm'}+(a/X)A_{n'}R_{nn'}^{\ mm'}\cot\pi a+(a/X)A_n S_{nn'}^{\ mm'}\csc\pi a \quad \text{for } n'\neq 1, \tag{3.45}$$

$$\text{while} \quad a_{n1}^{\ m0}=\binom{0}{1}[(a/X)R_{n1}^{\ m0}\cot\pi a+(a/X)S_{n1}^{\ m0}\csc\pi a] . \tag{3.46}$$

In (3.45) and (3.46) $a=\omega/\omega_c$, and we have made use of the identities[36]

$$\sum_{s=-\infty}^{\infty}\frac{1}{a-s}=\pi\cot\pi a , \tag{3.47}$$

$$\text{and} \quad \sum_{s=-\infty}^{\infty}\frac{(-1)^s}{a-s}=\pi\csc\pi a , \tag{3.48}$$

which is valid for a not equal to any integral value. The functions R and S appearing in (3.45) and (3.46) are defined as

$$R_{nn'}^{\ mm'}=\int_0^{\pi}\theta_n^{\ m}\theta_{n'}^{\ m'}\cos\tfrac{1}{2}(m+m')\pi\,d\theta , \tag{3.49}$$

$$\text{and} \quad S_{nn'}^{\ mm'}=\int_0^{\pi}\theta_n^{\ m}\theta_{n'}^{\ m'}\cos[2X\sin\theta+\tfrac{1}{2}(m+m'-1)\pi]\,d\theta . \tag{3.50}$$

Substituting (3.45) and (3.46) into (3.42), multiplying[37] through by $\sin^2\pi a\cdot(x/a)$ gives the following expression for the dispersion relation

$$(1+\textstyle\sum_n A_n)(\cos\pi a+S_{11}^{\ 00})-(a/X)(\cos\pi a+S_{11}^{\ 00})(\sum_n A_n R_{nn}^{\ mm}\cos\pi a+\sum_n A_n S_{nn}^{\ mm})$$

$$+(a/X)[\sum_{n,m}(R_{n1}^{m0})^2 A_n \cos^2\pi a + \sum_{n,m} A_n (S_{n1}^{m0})^2 + 2\sum_{n,m} A_n R_{n1}^{m0} S_{n1}^{m0} \cos\pi a] = 0. \quad (3.51)$$

If we put all A_n equal to zero in (3.51), the solution corresponds to that of a free electron gas. It is

$$a = \pm\pi^{-1}\cos^{-1}[-S_{11}^{00}(x)] + 2m, \quad (3.52)$$

with m being any integral value, and $\cos^{-1}[-S_{11}^{00}]$ is the principal branch of inverse cosine such that

$$0 \leq \cos^{-1}[-S_{11}^{00}] \leq \tfrac{1}{2}\pi, \quad \text{if } -S_{11}^{00} \geq 0; \quad \text{and } \tfrac{1}{2}\pi \leq \cos^{-1}[-S_{11}^{00}] \leq \pi, \quad \text{if } -S_{11}^{00} < 0.$$

There are an infinite number of branches corresponding to values of $m = 0, \pm 1 \cdots$, etc., in (3.37) with the plus or minus sign before $\cos^{-1}[-S_{11}^{00}]$. Since R_{nn}^{mm} and S_{nn}^{mm} are at most of order unity, it can be seen from the form of (3.51) that unlike the situation at the long-wavelength limit, the inclusion of all or a finite number of A_n would only introduce a small correction for all branches. To terms linear in the A_n (3.52) can be written

$$a = \pm\pi^{-1}\cos^{-1}(-S_{11}^{00}) + 2m + \Delta_m^{(\pm)}, \quad (3.53)$$

$\Delta_m^{(\pm)}$ is a small quantity depending on the parameters A_n and vanishes in the absence of electron correlation. To terms linear in $\Delta_m^{(\pm)}$, (3.53) can be written as

$$\cos\pi a = -S_{11}^{00} \mp \Delta_m^{(\pm)}[1-(S_{11}^{00})^2]^{\frac{1}{2}}. \quad (3.54)$$

Substitution of (3.54) into (3.36) then gives the expression for $\Delta_m^{(\pm)}$ as

$$\Delta_m^{(\pm)} = X^{-1}[\pi^{-1}\cos^{-1}(-S_{11}^{00}) + 2m] \times \left(\sum_{n'm'} A_{n'}(R_{n'1}^{m'0})^2 (S_{11}^{00})^2 + \sum_{n'm'} A_{n'}(S_{n'1}^{m'0})^2 - 2\sum_{n'm'} A_{n'} R_{n'1}^{m'0} S_{n'1}^{m'0} S_{11}^{00} \right). \quad (3.55)$$

The summation in (3.55) is carried out over all integral values of n' and m' such that $n' > 2$, $|m'| \leq n'$. In addition, $n' + m'$ must be odd.

We have discussed in the above paragraph the modes of oscillation of an electron gas both in the long wavelength limit and the short-wavelength limit. In the intermediate wavelength region where $X \approx 1$, an analytic solution of (3.42) or (3.43) is difficult to obtain, and a detailed study must rely on numerical method. We present here a numerical study[5] of (3.42) retaining only A_2 in the expansion series of $\phi(\vec{p}, \vec{p}')$. It is clear from the above discussion that the solution near $\omega = \omega_c$ in this model corresponds to the (2,1) mode which is precisely the one observed experimentally.[32] As can be seen from (3.34), A_3 only introduces a small correction in the coefficients of X^2. The dominant term which shows the Fermi-liquid effect in this mode is the factor $(1 + A_2)\omega_c$. Hence, the procedure of fitting the theoretical curve to the experimentally observed one should give a good estimate of the magnitude of A_2. At larger values of X, our discussion in Sec. II shows that Fermi-liquid effects start to disappear and the resulting dispersion curves should be close to the one for a free electron gas. All these points are clearly borne out in our calculation. When $A_n = 0$ for $n > 2$, the dispersion relation (3.42) can be written in the form

$$X^2 L(1 + 2A_2 - \tfrac{15}{2}A_2 a^2 M) + \tfrac{5}{3}A_2[(3a^2 L - 1)^2 + (\tfrac{3}{2}aL')^2] = 0, \quad (3.56)$$

$$\text{where} \quad L' = \partial L/\partial X, \quad L = \sum_{m=0}^{\infty} (1 + \delta_{m0})^{-1} S_m (a^2 - m^2)^{-1}, \quad (3.57)$$

$$\text{and} \quad M = \sum_{m=0}^{\infty} (1 + \delta_{m0})^{-1}(T_{m+1} + T_{m-1})(a^2 - m^2)^{-1}. \quad (3.58)$$

In (3.57) and (3.58)

$$S_m = \int_{-1}^{+1} d(\cos\theta)\cos^2\theta J_m{}^2(X\sin\theta), \quad T_m = \int_{-1}^{+1} d(\cos\theta)\sin^2\theta\cos^2\theta J_m{}^2(X\sin\theta). \tag{3.59}$$

In Fig. 1 we display the experimental data of Walsh and Platzman[32] together with the solution of (3.56) in the neighborhood of $\omega \sim \omega_c$ for $A_2 = -0.026$, -0.036 and zero. A choice of -0.026 for the value of the parameter A_2 gives a theoretical curve which fits very well with the experimental in the long-wavelength region $X \leq 2$. As X increases, the experimental points begin to fall on the free-electron curve as expected. The slight deviation of the theoretical curve with $A_2 = -0.026$ from the experimental one can be attributed to the omission of all the terms proportional to A_n for $n > 2$ in (3.42). For the other polarization perpendicular to B, plasma waves near $\omega = \omega_c$ has also been observed experimentally.[12] This corresponds to the (3, 1) mode in the long-wavelength limit. However, owing to a wrong assignment of the q values, the first published data was erroneous.[38] Recently, Walsh[39] informed us that a reassignment of the q values for the experimental points would yield a dispersion curve very close to the free-electron curve for $X > 1$. For $X < 1$, there are derivations from the noninteracting picture. This is precisely the kind of qualitative picture that we have obtained. For small values of X, our theory predicts a X^2 dependence with a coefficient depending on A_2, A_3, and A_4. Detailed comparison between theory and experiment has yet to wait until further experimental results are obtained.

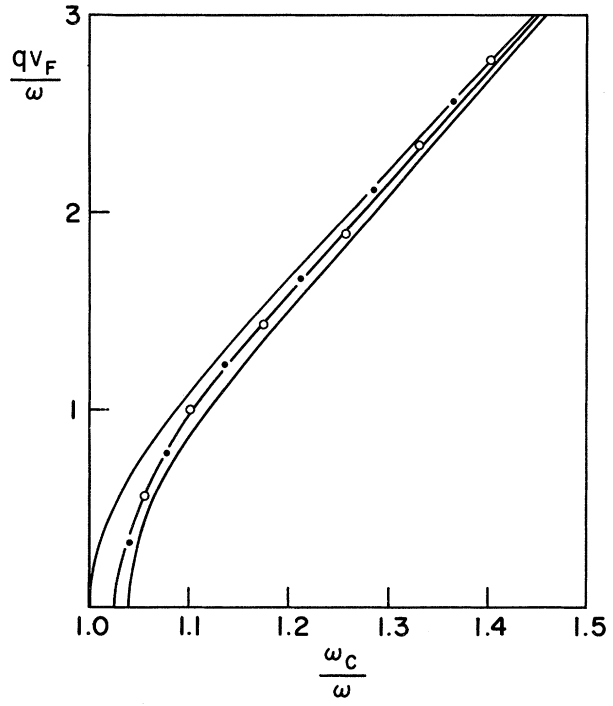

FIG. 1. Plot of ω_c/ω verses $q_x v_F/\omega$ for plasma waves polarized parallel to the dc magnetic field. The open and solid circles represent the experimental minima and maxima respectively of the derivative with respect to B of the power absorbed.[32] The three curves are results of the present calculation for values of A_2 of zero, -0.026, and -0.036, going from left to right.

C. Propagation Parallel to the dc Magnetic Field

When the wave vector q of the plasma wave is not exactly perpendicular to the dc field, modes with even $n-m$ and odd $n-m$ are mixed and the situation is more complicated. The extreme case where q is parallel to B is rather interesting. In the collisionless limit, the kinetic equation then becomes

$$(\omega - q_z v_z - m\omega_c) f_m(\theta) - ie\vec{E}\cdot\vec{v}_m(\theta) - (q_z v_z + m\omega_c) \sum_{n \geq |m|} \int_{-1}^{+1} A_n \theta_n{}^m(\theta)\theta_n{}^m(\theta) f_m(\theta')\, d(\cos\theta') = 0 \tag{3.60}$$

In (3.60), f_m and $\vec{v}_m$ are just the usual Fourier components defined by

$$f = \sum_{m=-\infty}^{\infty} f_m e^{im\varphi}, \quad \vec{v} = \sum_{m=-\infty}^{\infty} \vec{v}_m e^{im\varphi} \tag{3.61}$$

Since $\vec{v}_m$ has nonvanishing value only for m equal to zero or ± 1, all the Fourier components with $|m| > 1$ are completely decoupled from the electric field. Moreover, Fourier components with different values of m are independent of each other. Each of them obeys an integral equation

$$(\omega - q_z v_z - m\omega_c) f_m(\theta) - (q_z v_z + m_c) \sum_{n > |m|} \int A_n \theta_n^m(\theta) \theta_n^m(\theta') f_m(\theta') d(\cos\theta') = 0, \tag{3.62}$$

for $|m| > 1$. These equations can be studied by the same technique as described for the other geometry. For $|m| > 1$, the oscillations correspond to pure zero sound modes – the Fermi surface of the metal undergoes a periodic distortion without inducing any electric field, current density or charge density. The modes with $|m| = 1$ are the plasma waves. As before, those modes with $n = 1$ are just the ordinary plasmons and we shall not discuss them any further. To obtain the dispersion relation of all the other modes, we can first express the term $i\vec{E} \cdot \vec{v}_m$ in terms of the distribution as

$$ie\vec{E} \cdot \vec{v}_m = \delta_m \theta_1^m \int_{-1}^{+1} d(\cos\theta)\, \theta_1^m(\theta) f_m(\theta)\,, \tag{3.63}$$

where $\delta_m = \omega_p^2 \omega/(\omega^2 - c^2q^2)$, for $m = \pm 1$, and $\delta_m = \omega_p^2/\omega$, for $m = 0$.

Substituting (3.63) into (3.60), multiplying by θ_n^m and integrating over $d(\cos\theta)$ gives a system of equations

$$\sum_{n' \geq |m|} (a_{nn'}^{(m)} - \delta_{nn'}) u_{n'}^{(m)} = 0\,. \tag{3.64}$$

In (3.64), $u_n^{(m)}$ is similar to the Γ_n^m defined previously except that m is fixed in the present case, i.e.,

$$u_n^{(m)} = \int_{-1}^{+1} d(\cos\theta) \theta_n^m f_m, \tag{3.65}$$

and $$a_{nn'}^{(m)} = \int_{-1}^{+1} d(\cos\theta) \theta_n^m \theta_{n'}^m A_{n'} (q_z v_z + m\omega_c)/(\omega - q_z v_z - m\omega_c), \text{ for } n' > 1, \tag{3.66}$$

while $$a_{n1}^{(m)} = \int_{-1}^{+1} d(\cos\theta) \theta_n^m \theta_1^m (\omega - q_z v_z - m\omega_c)^{-1} \delta_m\,. \tag{3.67}$$

The dispersion relations of the various modes are then the solution of the determinantal equation

$$|a_{nn'}^{(m)} - \delta_{nn'}| = 0\,. \tag{3.68}$$

In the long-wavelength limit, we can again distinguish various modes with different values of n and m. The dispersion relation for the (n, m) mode to order Z^2 is

$$\omega_{nm} = \alpha_n m\omega_c + Z^2\omega_c \frac{\alpha_n \left\langle {m \atop n} \middle| \cos^2\theta \middle| {m \atop n} \right\rangle}{mA_n} + \frac{A_{n-1} Z^2 \left\langle {m \atop n} \middle| \cos\theta \middle| {m \atop n-1} \right\rangle^2 \alpha_n^2}{mA_n(A_n - A_{n-1})} + \frac{A_{n+1} Z^2 \left\langle {m \atop n} \middle| \cos\theta \middle| {m \atop n+1} \right\rangle^2 \alpha_n^2}{mA_n(A_n - A_{n+1})}, \tag{3.69}$$

for $n > |m| \geq 1$, and $n > 2$. For $n = |m|$, the term proportional to A_{n-1} should be set to zero. Among the plasma wave modes, the (2, 1) mode is most strongly coupled to the electric field and corresponds to that which would be observed experimentally. Its dispersion relation in the long-wavelength limit is given by

$$\omega = (1 + A_2)\omega_c + \tfrac{8}{35} \alpha_2 \alpha_3 \omega_c (A_2 - A_3)^{-1} Z^2\,. \tag{3.70}$$

This expression is valid in the region $Z^2 < |A_2 - A_3|(35A_2/8)$ and $Z < A_2$. Because of the restriction $Z < A_2$, (3.70) is only valid over a rather small range of values of wavelength. The general dispersion relation for the plasma wave can be written as

$$(1+A_2)I_1 - \tfrac{15}{4}A_2 a I_1 I_2 (1-a)/X - \tfrac{15}{4}A_2 a I_2^2 = 0 \ . \tag{3.71}$$

Here $$I_1 = 2(1-a)/Z^2 + [1/Z - (1-a)^2/Z^3]\ln[(1-a+Z)/(1-a-Z)] \ , \tag{3.72}$$

$$I_2 = \frac{4}{3Z} - \frac{2(1-a)^2}{Z^3} + \left[\frac{(1-a)^3}{Z^4} - \frac{(1-a)}{Z^2}\right]\ln[(1-a+Z)/(1-a-Z)] . \tag{3.73}$$

In these equations, $Z = q_z v_f/\omega_c$, and $a = \omega/\omega_c$. Equation (3.71) can be solved by numerical methods. As the wavelength decreases, we will arrive at a point such that $q_z v_f > |\omega - \omega_c|$. Then there exists a value of θ for which $q_z v_f \cos\theta$ is equal to $\omega - \omega_c$, and the waves are severely damped by those electrons traveling in phase with the wave. Owing to this Landau damping in this geometry, plasma waves cannot be observed experimentally beyond a critical value of q determined by the solution of (3.71).

IV. SUMMARY

We have presented a rather general method of evaluating the electrical conductivity tensor of a degenerate electron liquid in the presence of a dc magnetic field. The results are valid for arbitrarily short wavelengths, provided only that the condition $q \ll k_F$ is satisfied. The method of solution is demonstrated quite simply by the case in which the interaction function $\phi(\vec{p}, \vec{p}')$ can be approximated by a constant. This case is studied for a Fermi surface of arbitrary shape, and the results may be a useful first approximation for studying some effects in materials with complicated Fermi surfaces. For a more general interaction function we limit our consideration to a spherical Fermi surface. The expression for the conductivity involves the solutions of an infinite matrix equation. Only for the case in which the interaction function is approximated by a finite number of terms in the expansion in Legendre polynomials, can numerical results be obtained. We have evaluated the conductivity for the case in which the first three terms in the expansion are nonzero.

The self-sustaining oscillations of the electron liquid are obtained by making the constitutive equations (the solutions of the kinetic equations) consistent with Maxwell's equations. Except in the vicinity of the crossing points of the dispersion relations, plasma waves and spin waves are independent to a very high degree of approximation. Then, the dispersion relations are given by (3.11) and (3.15). These equations are rather cumbersome to use because they require the complete solutions of the kinetic equations. For the study of plasma wave propagation, we find it simpler to use Maxwell's equations from the start. We use them to express the electric field $\vec{E}$ in terms of the current density $\vec{j}$ and then write $\vec{j}$ in terms of an integral over the distribution function. This procedure reduces the kinetic equation for the spin independent oscillation to a homogeneous integral equation. The solutions are determined by setting an infinite determinant equal to zero. We have explicitly evaluated the long-wavelength limit for propagation perpendicular to and parallel to the dc magnetic field. We have also studied the short wavelength limit (e.g., $qr_c \gg 1$) and found that Fermi-liquid corrections to the free-electron model are small. The intermediate wavelengths can be studied by approximating the interaction function by a finite number of terms in its expansion, and investigating numerically the solutions of the resulting finite sized determinantal equation.

In principle the slopes and intercepts of the plasma wave modes in the long wavelength limit contain more than enough information for the accurate determination of m^*, and of all the A_n coefficients except A_0 and A_1. There is need however of considerably more experimental data than is available in the literature at present. In fact, our estimate of $A_2 \simeq -0.026$ cannot be taken too seriously because the cyclotron effective mass is not known with sufficient accuracy. This emphasizes the need for more and better experimental data.

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