width than the neutral state. This is in direct contrast to an $SU(3)$ $\bar{1}\bar{0}$ assignment for this resonance.

One can also take the matrix elements of the photon interaction operator between nucleon states. This gives the result[28] that the magnetic moments of the proton and neutron are in the ratio $-\frac{3}{2}$. Also, the quantity $\mu=eg/2M$ for the quark should be equal to the magnetic moment of the proton. The experimental value of the latter is 0.13 GeV^{-1}, which is somewhat less[29] than the value of μ deduced in Sec. IV from the physical radiative process $\Delta^+(1236)\to p\gamma$, viz., $\mu=0.18$ GeV^{-1}. The situation is similar to one encountered in I, where the value of the pion-nucleon coupling constant $f_q/\sqrt{(4\pi)}$ as determined[30] by simply taking the matrix element of the pion emission operator between nucleon states is smaller than the value obtained by fitting the physical decay processes [$f_q/\sqrt{(4\pi)}$ equal to 0.17, compared to 0.24]. These two facts serve to remind us that the quark model is not yet quite numerically self-consistent.

However, our results for the $N\pi$ widths turned out to be reasonably good when compared with the experimental numbers. In the present absence of experimental data on the $N\gamma$ decay widths, we therefore believe that the results of the present investigation warrant consideration in any future analysis of pion photoproduction, proton Compton scattering, or inelastic electron scattering. The recent data[27] on the latter process certainly seem to be in general agreement with some of our expectations.

[28] M. A. B. Bég, B. W. Lee, and A. Pais, Phys. Rev. Letters **13**, 514 (1964); G. Morpurgo, Physics (N.Y.) **2**, 95 (1965).
[29] A similar discrepancy has been noted previously in the literature; see Refs. 12 and 16.
[30] C. Becchi and G. Morpurgo, Phys. Rev. **149**, 1284 (1966).

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Multiperipheral Dynamics at Zero Momentum Transfer*

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Following a suggestion by Goldberger and Low, the crude multi-Regge bootstrap model of Chew and Pignotti is reformulated through a generalization of the physical-region integral equation discovered in 1962 by Fubini and collaborators. When consideration is restricted to zero momentum transfer, Lorentz symmetry permits almost complete diagonalization of the kernel, Lorentz poles corresponding to eigenvalues thereof. Cuts also appear but in a manner dynamically and unambiguously related to the poles. Being an expression of unitarity, the equation encompasses "absorptive" effects.

1. INTRODUCTION

THEORETICAL study of strong-interaction dynamics heretofore has concentrated on reactions between two-particle channels, human capacities still not having mastered the combined requirements of Lorentz invariance, analyticity, and unitarity for this simplest reaction type. The time is nonetheless ripe for serious study of multihadron systems. It has long been recognized (a) that unitarity precludes dynamical isolation of two-particle from multiparticle channels, and (b) that indefinite proliferation of particle production characterizes any relativistic process. Theoretical attention to such questions has been inhibited not by belief in their unimportance but by the technical difficulties attendant on an indefinitely increasing number of spin-momentum variables. Recent experimental and theoretical developments, however, have suggested a general kinematical technique for decomposing arbitrarily large particle systems into finite subunits of manageable proportions; the approach may loosely be described as "multiperipheral." In this paper we propose a physically plausible and theoretically tractable dynamical equation suggested by multiperipheral kinematics.

The physical content of our equation is equivalent to that presented by Chew, Goldberger, and Low,[1] our work being stimulated by theirs. The difference between the two papers lies in the kinematical techniques employed. The principal advantage in the techniques of

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[1] G. F. Chew, M. L. Goldberger, and F. Low, Phys. Rev. Letters **22**, 208 (1969).

this paper is the simplicity achieved through almost complete diagonalization of the kernel of the integral equation. Both the inhomogeneous and homogeneous problems then become tractable. We began this work motivated by the desire to clarify the crude multi-Regge bootstrap model of Chew and Pignotti[2] (CP), and the ensuing equation amounts to a generalization of that proposed in 1962 by Fubini and collaborators.[3]

After diagonalization our multiperipheral equation is of the Fredholm type in a single variable, with both kernel and inhomogeneous term fixed by "input" Regge poles. The derivation employs forward-direction unitarity in two-particle elastic scattering, but applicability of the underlying principles to broader situations will be apparent. The kernel of the equation, in particular, is independent of the amplitude being unitarized, so the determination of Regge poles as eigenvalues of the kernel is correspondingly channel-independent.

An important aspect of multiperipheral dynamics is the broad basis that it provides for Regge asymptotic behavior. It will be seen that any finite number of "input" Regge poles lead to "output" Regge poles.[4] Regge cuts are also to be expected, but these are dynamically and unambiguously related to the poles. (Being an expression of unitarity, our equation encompasses the effects often described as "absorptive.")

A second important feature of the multiperipheral equation is that it never strays outside the physical region. The kernel correspondingly has direct physical meaning and there can be no divergence difficulties.

Of great potential importance is the bootstrap application, in which the multiperipheral kernel is related to the equation's solution. It is straightforward to implement versions of the CP proposal based on duality, and improvements of the CP model quickly come to mind. In this paper, however, we do not venture into such questions. A separate paper now in preparation deals with speculations concerning the Pomeranchuk trajectory that are motivated by the multiperipheral equation.

2. KINEMATICS

To describe the multiparticle production amplitude we shall use variables of the type introduced by Bali, Chew, and Pignotti[5] (BCP). The process is

$$a+b\rightarrow 0+1+2+\cdots+(n+1)\,, \tag{2.1}$$

where the numbers identify the $n+2$ outgoing particles. Preliminary to the definition of our variables, recall that Toller[6] has suggested describing such a process through an amplitude

$$M(b_a,b_0,b_1,\cdots,b_{n+1},b_b)\,, \tag{2.2}$$

where b_k denotes an element of the six-parameter homogeneous Lorentz group $SL(2,C)$. Physical meaning attaches to Toller's variables through the decomposition

$$b_k=w_k u_k\,, \tag{2.3}$$

where u_k is an element of the three-parameter rotation group which constitutes the little group of the kth particle momentum p_k, and $w(p_k)$ is the three-parameter transformation connecting an arbitrary reference frame to the rest frame of particle k.[7] The four-vector particle momentum p_k is related to w_k by

$$p_k=L(w_k)\hat{p}m_k\,, \tag{2.4}$$

the unit vector $\hat{p}$ having only an energy component, while u_k acquires significance by expanding the amplitude into representations of the rotation group:

$$M(\cdots b_k\cdots)=\sum_{m_k,m} M_{m_k}(\cdots p_k\cdots)D_{m_k,m}{}^{s_k}(u_k)\,. \tag{2.5}$$

The expansion coefficient $M_{m_k}(\cdots p_k\cdots)$ can be interpreted as the amplitude for finding m_k to be the z component of the spin of particle k in some arbitrarily oriented rest frame, s_k being the magnitude of the particle spin.

Conservation of energy momentum,

$$p_a+p_b=p_0+\cdots+p_{n+1}\,, \tag{2.6}$$

must be remembered as placing a constraint on the set of w_k's, while Lorentz invariance implies that

$$M(b_a,b_0,\cdots,b_b)=M(bb_a,bb_0,\cdots,bb_b)\,. \tag{2.7}$$

The set of elements $(b_a\cdots b_b)$ corresponds to the association with each particle of a conventional rest frame. The kinematic description proposed by BCP is similar in spirit to that of Toller but selects a set of conventional frames in which momentum transfers play the role occupied above by the particle momenta. This momentum-transfer emphasis is better suited to multiperipheralism.

Let Q_i denote a four-momentum transfer, such that

$$Q_i=-p_a+\sum_{j=0}^{i-1} p_j\,,\quad i=1,\cdots,n+1 \tag{2.8}$$

corresponding to Fig. 1. Now in the rest frame of the ith outgoing particle the three-momenta of Q_i and Q_{i+1}

[2] G. F. Chew and A. Pignotti, Phys. Rev. **176**, 2112 (1968).

[3] L. Bertocchi, S. Fubini, and M. Tonin, Nuovo Cimento **25**, 626 (1962); D. Amati, A. Stanghellini, and S. Fubini, *ibid.* **26**, 896 (1962); L. Bertocchi, E. Predazzi, A. Stanghellini, and M. Tonin, *ibid.* **27**, 913 (1963).

[4] We confine ourselves here to forward-direction unitarity, and the corresponding "output" zero-momentum-transfer Regge poles automatically fall into families corresponding to Lorentz poles. A subsequent paper will deal with nonforward unitarity.

[5] N. F. Bali, G. F. Chew, and A. Pignotti, Phys. Rev. **163**, 1572 (1967).

[6] M. Toller, Nuovo Cimento **54**, 295 (1968).

[7] We may associate an explicit set of six parameters with b_k as follows: $u_k=R_z(\psi_k)R_y(\theta_k)R_z(\phi_k)$, $w(p_k)=R_z(\Phi_k)R_y(\Theta_k)B_z(\xi_k)$, where R_z and R_y are rotations about the indicated axes and B_z is a boost. The two initial rotations in u_k, characterized by θ_k and ϕ_k, merely serve to define the direction about which particle spin is to be measured. Thus, there are really only four degrees of freedom per particle.

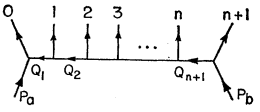

FIG. 1. Kinematical diagram defining the momentum transfers $Q_1 \cdots Q_{n+1}$.

are collinear, since

$$p_i = Q_{i+1} - Q_i. \tag{2.9}$$

We now adopt the convention that in the special rest frame associated with b_i these three-momenta lie along the z axis. If we assume spacelike momentum transfers, a boost along the z axis can bring Q_i to the form

$$Q_i^{(i,r)} = (0,0,0,(-t_i)^{1/2}). \tag{2.10}$$

The required boost is uniquely determined by t_i and t_{i+1}, and the frame (i,r) defined in this way is seen to be the same as that designated with a similar but slightly different notation by BCP. The reason for the alternative approach here is to amplify the significance of the Lorentz transformation associated with the frame (i,r).

Let us designate by a_i the Lorentz transformation connecting the frame (i,r) to our fixed reference frame. Still following BCP, we introduce the z boost q_i which carries the frame (i,r) to a frame $(i+1,l)$ in which Q_{i+1} has the form

$$Q_{i+1}^{(i+1,l)} = (0,0,0,(-t_{i+1})^{1/2}). \tag{2.11}$$

The boost required here is given by

$$\cosh q_i = \frac{-Q_i \cdot Q_{i+1}}{(-t_i)^{1/2}(-t_{i+1})^{1/2}} = \frac{m_i^2 - t_i - t_{i+1}}{2(-t_i)^{1/2}(-t_{i+1})^{1/2}}. \tag{2.12}$$

By construction, however, recall that the frame $(i+1,r)$ also gives to Q_{i+1} the form (2.11). Thus the two frames $(i+1,l)$ and $(i+1,r)$ must be related by an element of the little group of Q_{i+1}—a three-parameter transformation in $SU(1,1)$ which we designate

$$g_{i+1} = R_z(\mu_{i+1}) B_x(\xi_{i+1}) R_z(\nu_{i+1}). \tag{2.13}$$

The a_i are thus successively connected by the formula

$$a_{i+1} = a_i q_i g_{i+1}, \quad i = 1 \cdots n \tag{2.14}$$

where q_i is shorthand for $B_z(q_i)$, a recursion relation fundamental to the BCP kinematical analysis. Note that the parameters in each g_i have been uniquely defined.

The two ends of the chain in Fig. 1 require separate consideration. Starting with the b_0 frame, a z boost defines a special rest frame of p_a which we may denote $(0,r)$ and associate with the transformation a_0. We then z-boost from $(0,r)$ to $(1,l)$, where Q_1 has only a z component, thereby defining q_0. The frame $(1,r)$ has already been defined, so we achieve a meaning for g_1. In particular,

$$a_1 = a_0 q_0 g_1. \tag{2.15}$$

At the other end of the chain we define a boost q_{n+1} by an analogous procedure, and adopt the convention that the special rest frame associated with b_b corresponds to

$$b_b = a_{n+1} q_{n+1} r_b, \tag{2.16}$$

where r_b is a rotation still to be specified. Similarly, we define r_a by $a_0 = b_a r_a$. The upshot of all the above analysis is that the amplitude may be regarded a function of $g_1 \cdots g_{n+1}$ and $t_1 \cdots t_{n+1}$, together with r_b and r_a. Energy-momentum conservation and Lorentz invariance are then implicit. This was the BCP result. What has been added here is a more explicit description of the physical meaning of the BCP variables. In particular, our approach has called attention to the Lorentz transformations

$$a_i = b_a r_a q_0 g_1 q_1 g_2 \cdots q_{i-1} g_i \tag{2.17}$$

with the recursive property (2.14). Even though the a_i are not fully independent of each other (as are the g_i), they will turn out to be extremely convenient for the formulation of multiperipheral dynamics.

3. PHASE SPACE

The chief technical difficulty in multiperipheral dynamics is the treatment of phase space. The multiperipheral amplitude factorizes in its dependence on the successive g_i, so one desires a corresponding factorization of phase space. BCP found a phase-space expression that factored to a considerable extent, but the over-all constraint of energy conservation was handled in such a way as to impose an awkward condition on the g_i's. The constraint treated all g_i's symmetrically by requiring that in the rest frame of particle b,

$$p_a = L^{-1}(r_a q_0 g_1 q_1 \cdots g_{n+1} q_{n+1} r_b) \hat{p} m_a. \tag{3.1a}$$

In the present approach this constraint is satisfied by an inductive process. Energy and momentum are conserved at the leftmost "vertex" in the BCP chain and the phase space is so constructed that the addition of each new vertex automatically satisfies energy-momentum conservation. Thus if we require that

$$p_a = L(a_0) \hat{p} m_a \tag{3.1b}$$

and

$$a_{i+1} = a_i q_i g_{i+1}, \quad i = 0, 1, \cdots, n+1 \tag{3.1c}$$

then the over-all constraint (3.1a) will have been fulfilled.

What is potentially confusing about the inductive approach is that in the end $b_a^{-1} a_{n+2}$ is to be set equal to $b_a^{-1} b_b$, which is fixed during the integration over the phase space. One then works backwards through the inductive chain. If the g_i are chosen outside the phase space, then constraint (3.1b) cannot be satisfied. This constraint appears as a δ function in the phase space and so ensures over-all energy-momentum conservation. The inductive approach to energy conservation is the

crucial first step in formulating a recursive phase space. That this approach is not the same as the BCP approach will become clear when it is noted that in the BCP approach (3.1a) is used to eliminate r_b, whereas in the present approach r_b is a variable of the phase space and g_1 will be eliminated by using (3.1b).

Let us begin with the momentum phase space for $n+2$ particles:

$$d\Phi^{(n)}=d^4p_0\delta^+(p_0^2-m_0^2)\cdots d^4p_{n+1}\delta^+(p_{n+1}^2-m_{n+1}^2)\delta^4(\sum_{i=0}^{n+1}p_i-p_a-p_b). \quad (3.2a)$$

Eliminating p_0 via the energy-momentum δ function and successively converting from p_i to Q_i according to Fig. 1, we find

$$d\Phi^{(n)}=\delta^+[(p_a+Q_1)^2-m_0^2]d^4Q_1\times\delta^+[(Q_2-Q_1)^2-m_1^2]d^4Q_2\cdots d^4Q_{n+1}\times\delta^+[(p_b-Q_{n+1})^2-m_{n+1}^2]. \quad (3.2b)$$

Consider the invariant volume element d^4Q_1. The four-momentum transfer Q_1 may be written in terms of the Lorentz transformation a_1 as

$$Q_1=L(a_1)\hat{Q}(-t_1)^{1/2}, \quad (3.3)$$

where $\hat{Q}$ is a unit four-vector in the positive z direction. Keeping fixed $a_2, a_3, \cdots$ (and thus $Q_2, Q_3, \cdots$), and remembering from formula (2.14) that

$$a_1=a_2(q_1g_2)^{-1}, \quad (3.4)$$

we may replace d^4Q_1 by d^4Q_1', where

$$Q_1'=L(g_2^{-1}q_1^{-1})\hat{Q}(-t_1)^{1/2}, \quad (3.5)$$

whence, by straightforward calculation from formula (2.13),

$$d^4Q_1'=-\tfrac{1}{2}t_1dt_1\sinh q_1\,d\cosh q_1\,d\cosh\xi_2\,dv_2. \quad (3.6)$$

A similar change $d^4Q_i\to d^4Q_i'$ may be applied in turn to each invariant volume element, provided the order of integration is maintained. The last integration requires special attention. Here we have

$$Q_{n+1}=L(a_{n+1})\hat{Q}(-t_{n+1})^{1/2}, \quad (3.7)$$

but now

$$a_{n+1}=b_b(q_{n+1}r_b)^{-1}, \quad (3.8)$$

where r_b is a rotation, rather than an element of the form (2.13). If we parametrize r_b as

$$r_b=R_z(\psi_b)R_y(\theta_b)R_z(\phi_b), \quad (3.9)$$

it follows that

$$d^4Q_{n+1}'=-\tfrac{1}{2}t_{n+1}dt_{n+1}\times\cosh q_{n+1}\,d\sinh q_{n+1}\,d\cos\theta_b\,d\phi_b. \quad (3.10)$$

Next we eliminate the $d\cosh q_i$, $i=1\cdots n+1$, and $d\sinh q_{n+1}$, using formula (2.12) and the mass-shell δ functions:

$$\begin{aligned} &d\cosh q_i\,\delta^+((Q_{i+1}-Q_i)^2-m_i^2)\\ &\qquad\to 1/2(-t_i)^{1/2}(-t_{i+1})^{1/2},\quad i=1\cdots n;\\ &d\sinh q_{n+1}\,\delta^+((p_b-Q_{n+1})^2-m_{n+1}^2)\\ &\qquad\to 1/2m_b(-t_{n+1})^{1/2}. \end{aligned} \quad (3.11)$$

Putting all factors together, we finally have

$$d\Phi^{(n)}=\frac{\delta^+[\sinh q_0-((m_0^2-m_a^2-t_1)/2m_a(-t_1)^{1/2})]}{2^{2n+3}m_am_b}\times dt_1\cdots dt_{n+1}\sinh q_1\cdots\sinh q_n\cosh q_{n+1}\times d\cosh\xi_2\,dv_2\cdots d\cosh\xi_{n+1}\,dv_{n+1}d\cos\theta_b\,d\phi_b. \quad (3.12)$$

The expression becomes more concise if we remember to add helicity phase space. Since the frame b_i is related to the frame $(i+1, l)$ by a z boost, the angle μ_{i+1} represents rest-frame rotations about the direction of the particle momentum p_i. The sum over helicities m_i then becomes an integral over μ_{i+1}, and the full phase space is

$$d\Phi^{(n)}=\text{const}\,\delta\left(\sinh q_0-\frac{m_0^2-m_a^2-t_1}{2m_a(-t_1)^{1/2}}\right)dt_1\cdots dt_{n+1}\times\sinh q_1\cdots\sinh q_n\cosh q_{n+1}\times d\mu_1d^3g_2d^3g_3\cdots d^3r_b, \quad (3.13)$$

where

$$d^3g_i=d\mu_i\,d\cosh\xi_i\,dv_i, \quad (3.14)$$

$$d^3r_b=d\psi_b\,d\cos\theta_b\,d\phi_b. \quad (3.15)$$

The quantity q_0, appearing in the δ function, is to be regarded as a function of $t_1\cdots t_{n+1}$, $g_2\cdots g_{n+1}$, and r_b, determined for given b_a and b_b by the constraint

$$b_a^{-1}b_b=(r_aq_0g_1)(q_1g_1\cdots g_{n+1}q_{n+1}r_b). \quad (3.16)$$

The essential point here is to realize that *each* of the three transformations r_a, q_0, and g_1 is *separately* determined by (3.16) (apart from the usual ambiguity that only the *sum* of the final z rotation in g_1 and the initial z rotation in r_a is determined). The z boost q_0 is thus expressible in terms of the variables employed in (3.13).

4. DYNAMICS

The defining characteristic of multiperipheral models is the factorization of the amplitude into a product of functions that each depends on only a finite number of variables, the functional form of an individual factor being independent of the total number of particles. Motivation for assuming localized particle correlation comes from the experimental observation that the mean transverse momentum of any produced particle is small and independent of total energy. If produced particles are sequentially arranged according to longitudinal momenta, defining a definite set of momentum transfers Q_i, it follows that the average magnitude of any t_i is

small and independent of the chain length. Furthermore, the *relative* momentum of a particle *pair* increases with the separation between pair members in the sequence. Adjacent particles in the sequence tend to lie closest to each other in phase space.[8] It thus seems natural to assume "short-range order" along the BCP chain.

The simplest multiperipheral model is of the type proposed by Fubini and collaborators,[3] where each factor depends on a single t_i. There is minimal interparticle correlation here, dependence on the g_i being totally absent. A more realistic model allows each factor to depend on a finite number of "adjacent" t_i's and g_i's. The dynamical equation associated with such a model shares many characteristics with the equation that will be developed below from a slightly different type of short-range correlation. The particular model used here to illustrate multiperipheral dynamics has been selected with an eye toward bootstrap applications.

Let us make a multiple $O(2,1)$ decomposition of the absolute square of the amplitude (summed over final-particle helicities),

$$\begin{aligned}|M^{(n)}(r_a,g_1,\cdots,g_{n+1},r_b;t_1\cdots t_{n+1})|^2 \\ \approx\sum_{s_a,s_b}\int[ds_1][ds_2]\cdots[ds_{n+1}] \\ \times\sum_{m_a,m_0\cdots m_{n+1},m_b}M_{m_0\cdots m_{n+1}}{}^{s_a,s_1\cdots s_{n+1},s_b} \\ \times D_{m_am_0}{}^{s_a}(r_a)E_{m_0m_1}{}^{s_1}(g_1)\cdots E_{m_nm_{n+1}}{}^{s_{n+1}}(g_{n+1}) \\ \times D_{m_{n+1}m_b}{}^{s_b}(r_b).\end{aligned}\tag{4.1}$$

This expansion is completely general, $[ds_i]$ denoting the appropriate measure for the $O(2,1)$ group. Multiperipheralism is injected by assuming that

$$M^{s_a\cdots s_i\cdots s_b}$$

can be approximated as an analytic function of s_i containing only simple poles with factorizable residues. The integral $\int[ds_i]$ may then be replaced by a sum over these poles. Making this pole approximation in succession for $i=1\cdots n+1$, we find

$$\begin{aligned}|M^{(n)}|^2\approx\sum_{\substack{s_a,\gamma_1\cdots\gamma_{n+1},s_b;\\ m_a,m_0\cdots m_{n+1},m_b}}R_{m_0}{}^{a\gamma_1}(t_1)R_{m_1}{}^{\gamma_1\gamma_2}(t_1,t_2)\cdots \\ R_{m_{n+1}}{}^{\gamma_{n+1}b}(t_{n+1})D_{m_am_0}{}^{s_a}(r_a)E_{m_0m_1}{}^{\alpha_{\gamma_1}(t_1)}(g_1)\cdots \\ E_{m_nm_{n+1}}{}^{\alpha_{\gamma_{n+1}}(t_{n+1})}(g_{n+1})D_{m_{n+1}m_b}{}^{s_b}(r_b),\end{aligned}\tag{4.2}$$

where γ_i labels the different poles in s_i, the symbol $\alpha_{\gamma_i}(t_i)$ denoting the position of a pole.[9] We furthermore assume that the residue factor $R^{\gamma\gamma'}(t,t')$ is large only for $|t|$ and $|t'|$ both small.[10]

Unitarity gives for the absorptive part of the elastic $(ab\to ab)$ forward amplitude the expression

$$A(b_a^{-1}b_b)=\sum_{n=0}^{\infty}\int d\Phi^{(n)}|M^{(n)}(\Phi^n)|^2,\tag{4.3}$$

where it is understood that

$$b_a^{-1}b_b=r_aq_0g_1q_1\cdots g_{n+1}q_{n+1}r_b$$

is held fixed in the integration. Designating by ${}_{(n)}A(b_a^{-1}b_b)$ the contribution to A from n-particle production, we now introduce an auxiliary function

$$\begin{aligned}{}_{(n)}B_{m_{n+1}}{}^{\gamma_{n+1}}(b_a^{-1}a_{n+1},t_{n+1})\equiv\sum_{\substack{s_a,\gamma_1\cdots\gamma_n;\\ m_a,m_0\cdots m_n}}\int d\mu_1 \\ \times d^3g_2\cdots d^3g_{n+1}dt_1\cdots dt_n\sinh q_1\cdots\sinh q_n \\ \times\delta\left(\sinh q_0-\frac{m_0^2-m_a^2-t_1}{2m_a(-t_1)^{1/2}}\right)D_{m_am_0}{}^{s_a}(r_a) \\ \times G_{m_0m_1}{}^{a\gamma_1}(t_1,g_1)G_{m_1m_2}{}^{\gamma_1\gamma_2}(t_1,t_2,g_2)\cdots \\ G_{m_nm_{n+1}}{}^{\gamma_n\gamma_{n+1}}(t_n,t_{n+1},g_{n+1}),\end{aligned}\tag{4.4}$$

with

$$\begin{aligned}G_{m_im_{i+1}}{}^{\gamma_i\gamma_{i+1}}(t_i,t_{i+1},g_{i+1}) \\ \equiv R_{m_i}{}^{\gamma_i\gamma_{i+1}}(t_i,t_{i+1})E_{m_im_{i+1}}{}^{\alpha_{\gamma_{i+1}}(t_{i+1})}(g_{i+1}).\end{aligned}\tag{4.5}$$

The relation between ${}_{(n)}A$ and ${}_{(n)}B$ is then

$$\begin{aligned}{}_{(n)}A(b_a^{-1}b_b)=\sum_{\substack{m_{n+1},m_b;\\ \gamma_{n+1},s_b}}\int d^3r_bdt_{n+1}\cosh q_{n+1} \\ \times{}_{(n)}B_{m_{n+1}}{}^{\gamma_{n+1}}(b_a^{-1}a_{n+1},t_{n+1}) \\ \times R_{m_{n+1}}{}^{\gamma_{n+1}b}(t_{n+1},m_b^2)D_{m_{n+1}m_b}{}^{s_b}(r_b),\end{aligned}\tag{4.6}$$

with

$$a_{n+1}=b_br_b^{-1}q_{n+1}^{-1}.\tag{4.7}$$

The heart of multiperipheral dynamics lies in the recursion relation that can be read off from the definition (4.4):

$$\begin{aligned}{}_{(n+1)}B_{m'}{}^{\gamma'}(a',t')=\sum_{\gamma,m}\int d^3g'dt\sinh q \\ \times{}_{(n)}B_m{}^{\gamma}(a,t)G_{mm'}{}^{\gamma\gamma'}(t,t',g'),\end{aligned}\tag{4.8}$$

where $a'=aqg'$ and

$$\cosh q=(m_\gamma{}^2-t-t')/2(tt')^{1/2}.\tag{4.9}$$

[8] "Distance" between b_i and b_j may be defined as the boost in $b_i^{-1}b_j$. This boost is equivalent to the "relative momentum."

[9] Notice that $\alpha_{\gamma_i}(t_i)$ is the position of a pole in the absolute square of the amplitude, not in the amplitude itself.

[10] The pair of superscripts $\gamma\gamma'$ on the residue R can be used to specify the type of particle produced at the vertex γ of the BCP chain. [In fact, for a given pair of adjoining poles at $\alpha_\gamma(t)$ and $\alpha_{\gamma'}(t')$, there is rarely more than one possible stable particle that can be emitted from the intervening vertex.] The sum over γ and γ' thus includes all possible arrangements of particle types along the chain if we understand that the vertex boost q depends on these indices. Because of the emphasis on small values of $|t_i|$, double counting is expected to be unimportant.

If we define

$$B \equiv \sum_{n=0}^{\infty} {}_{(n)}B, \tag{4.10}$$

it follows that

$$B = {}_{(0)}B + \int BG, \tag{4.11}$$

where the variables and integration are the same as in (4.8). Performing the linear operation (4.6) on B rather than on ${}_{(n)}B$ evidently produces A. Thus if the integral equation (4.11) can be solved for B in terms of ${}_{(0)}B$, we can find the complete absorptive part in terms of

$${}_{(0)}B_{m'}{}^{\gamma'}(a',t') = \sum_{s_a,m_a,m} \delta\left(\sinh q_0 - \frac{m_0{}^2 - m_a{}^2 - t'}{2m_a(-t')^{1/2}}\right) \times D_{m_a m}{}^{s_a}(r_a) R_m{}^{a\gamma'}(t') E_{mm'}{}^{a\gamma'(t')}(g'),$$

with $a' = r_a q_0 g'$.

5. LORENTZ SYMMETRY AND PARTIAL-WAVE ANALYSIS

The integral Eq. (4.11), which written out is

$$B_{m'}{}^{\gamma'}(a',t') = {}_{(0)}B_{m'}{}^{\gamma'}(a',t') + \sum_{\gamma,m} \int d^3g'dt \sinh q \times B_m{}^{\gamma}(a,t) G_{mm'}{}^{\gamma\gamma'}(t,t',g'), \tag{5.1}$$

possesses a fundamental symmetry that (a) facilitates its solution through diagonalization, and (b) leads to Regge asymptotic behavior. The kernel is invariant under the transformation $a \to ca$, $a' \to ca'$, where c is an arbitrary Lorentz transformation. The volume element in (5.1) evidently possesses this same invariance. The symmetry operation in question does not involve any transformation of b_a and therefore is more than a statement of over-all Lorentz invariance.[11] It is a dynamical symmetry arising from the basic multiperipheral assumption that only a *finite* number of particles are correlated. It would not matter if N particles, rather than 2, were correlated, so long as N is independent of the chain "length" $b_a{}^{-1}b_b$. The function B and the kernel would then have more variables but there would be invariance of kernel and volume element under a common Lorentz transformation of all N of the correlated a's, keeping b_a fixed.

An analogous symmetry is present in the Bethe-Salpeter equation, a circumstance which encourages a tendency to equate the content of multiperipheral dynamics with that of this celebrated off-shell equation. Without prejudging the question of whether an off-shell kernel can be found that corresponds to an arbitrary on-shell kernel, we stress that the advantages gained by attempting to go off shell are obscure. A tremendous asset of multiperipheral dynamics is that everything takes place not only on shell but *in the physical region.* The kernel G has a direct physical significance, subject to experimental check.

Because of the Lorentz symmetry of the kernel it is natural to expand the function $B(a')$ into its irreducible components with respect to the Lorentz group. The consequent diagonalization of Eq. (5.1) will be explored in detail in a subsequent paper. We note briefly, however, that after projection onto representations of the Lorentz group with Toller quantum numbers M and λ, Eq. (5.1) will have the structure

$$B^{M\lambda}(t) = {}_{(0)}B^{M\lambda}(t) + \int B^{M\lambda}(t) G^{M\lambda}(t,t')dt. \tag{5.2}$$

The symbolic solution of this Fredholm equation,

$$B^{M\lambda} = {}_{(0)}B^{M\lambda}/(I - G^{M\lambda}), \tag{5.3}$$

will contain t'-dependent Lorentz poles arising from input Regge poles in ${}_{(0)}B^{M\lambda}$ and $G^{M\lambda}$, together with t'-independent Lorentz poles wherever $G^{M\lambda}$ has the eigenvalue 1. The latter will propagate essentially unchanged into the corresponding Lorentz projection of the absorptive part $A^{M\lambda}$, while the former will become branch cuts. Inverting this projection to achieve the absorptive part itself, $A(b_a,b_b)$, leads to asymptotic behavior in $b_a{}^{-1}b_b$ that is controlled in the familiar way by the leading singularities in λ. Our dynamics, of course, yields the complete absorptive part, not simply an asymptotic representation thereof.

[11] Of course, the invariance of the kernel guarantees Lorentz invariance of the final absorptive part $A(b_a{}^{-1}b_b)$.

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