Massless Particles and Fields*

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Free fields of massless particles transforming covariantly under the Poincaré group are constructed. The allowed infinite- and finite-dimensional representations of the Lorentz group are obtained. The wave functions are calculated in these representations in various bases. The commutation rules are computed, and turn out to be nonlocal for any infinite-dimensional fields. The transformation law of a certain irreducible infinite-dimensional representation is shown to coincide, for its lowest-spin component, with the usual, radiation-gauge, vector-potential transformation law, as already discovered by Bender.

I. INTRODUCTION

THIS paper is devoted to a general treatment of free zero-mass fields, transforming covariantly under the Poincaré group. The requirement of covariance is shown to impose restrictions on the transformation law for a free massless field. For a field transforming according to an allowed representation we construct the wave functions in various bases and study their properties. We also compute the explicit expression for a commutator or anticommutator of two fields. It follows that locality can be obtained only in the finite-dimensional case, and here only with the usual connection between spin and statistics. It is also demonstrated that in the spherical $j\sigma$ basis, the $j\pm 1$ components of the field in momentum space can be expressed in terms of the jth with coefficients linear in the components of the unit vector $\hat{p}=\mathbf{p}|\mathbf{p}|$ along the three-momentum $\mathbf{p}$. This implies that the result of a Lorentz transformation on a j component can be expressed in terms of the jth components itself. In particular, an infinitesimal Lorentz transformation can be so expressed, with coefficients linear in $\hat{p}_k$. As a special case, the transformation laws of the $j=1$ components for helicity ± 1 fields in special representations turn out to be those of the free electromagnetic vector potential in the radiation gauge. This result was obtained, using a somewhat less direct method, by Bender.[1]

As is shown in this paper, a free massless field can be incorporated in irreducible representations of the Lorentz group for which the lowest spin equals the absolute value of the helicity. This is no more true when interactions are introduced. A study of the electromagnetic potentials in the radiation gauge shows that a direct sum of a finite number of irreducible representations is not sufficient to describe the transformation law of these potentials. These facts and a study of the interaction case deserve further attention.

It was shown by Weinberg[2] that the requirement of covariance singles out, among the finite-dimensional representations of the Lorentz group, those for which the minimal spin equals to the helicity λ of the considered massless particle. The sign of λ is determined by the representation. In the representation $[j_a,j_b]$, with $\frac{1}{4}(\mathbf{J}-i\mathbf{K})^2=j_a(j_a+1)$ and $\frac{1}{4}(\mathbf{J}+i\mathbf{K})^2=j_b(j_b+1)$ ($\mathbf{J}$ and $\mathbf{K}$ are the generators of rotations and Lorentz transformations), only helicity $\lambda=j_a-j_b$ can be incorporated. We show that this result is general, and applies to infinite representations as well. The allowed representations are those for which the lowest sipn equals the absolute value of the helicity.

In Sec. II we summarize the properties of physical states for massless particles and establish our notation. In Sec. III we discuss the allowed representations for free massless fields and the appropriate wave functions in these representations. We also show there that, starting from a massive field and letting the mass go to zero, the only nonvanishing terms are those for which the absolute value of the helicity equals the minimal spin, as expected. We compute, in the same section, the various components and recursion relations (mentioned above) among them, in the $j\sigma$ basis and in a Cartesian basis. Finally, we give expressions for the irreducible massless fields and the relations among their various components which correspond to the relations found for the wave functions. In Sec. IV we express the Lorentz-transformed lowest-spin component in terms of the various components of the same spin, and discover, for $\lambda=\pm 1$, the connection with electromagnetism mentioned above. In Sec. V we discuss the commutation relations among the various massless fields.

The computations of the wave functions are performed in two ways. One uses generators and their matrix elements, and the other global methods. The first is summarized in Appendix A, and the second in Appendix B. The reader may thus choose, among the derivations in Sec. III, the one appealing to his taste. In the following sections only the global method is used, to obtain the simplest derivations for the purposes of the subjects discussed there. However, the persistent reader may still derive all results with the previous method, using the appropriate formulas of Sec. III.

Although the material on which our paper relies is quoted in our list of references, the latter is far from complete. We apologize to the authors of many papers

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[1] C. M. Bender, Phys. Rev. **168**, 1809 (1968).
[2] S. Weinberg, Phys. Rev. **134**, B882 (1964).

not mentioned here. The reader may find earlier references in the works of Bender[1] and Weinberg.[2]

II. PHYSICAL STATES AND POLARIZATION VECTORS

We start with the properties of physical states of massless particles. These were worked out by Wigner. For completeness we shall outline this construction and establish our notation.[2]

Let k be a standard four-vector of zero length with three-momentum along the z direction $k=(k^0=1, k^1=0, k^2=0, k^3=1)$. The subgroup of Lorentz transformations which leave this four-vector invariant (the little group) is obtained as follows. To each Lorentz transformation Λ (with $\det\Lambda=+1$, $\Lambda_0{}^0>0$) is associated a pair $\pm A$ of 2×2 matrices with $\det(\pm A)=1$ in such a way that

$$(\Lambda x^0)+(\Lambda x)^j\sigma^j=A(x^0+\boldsymbol{\sigma}\cdot\mathbf{x})A^\dagger, \qquad (2.1)$$

while to an infinitesimal Lorentz transformation $\Lambda\simeq I+i\boldsymbol{\varepsilon}\cdot\mathbf{J}+i\boldsymbol{\delta}\cdot\mathbf{K}$, with $\mathbf{J}$ and $\mathbf{K}$ the generators of rotations and pure Lorentz transformations (boosts), corresponds the 2×2 matrix $I+(i\boldsymbol{\varepsilon}-\boldsymbol{\delta})\cdot\frac{1}{2}\boldsymbol{\sigma}$. The generators satisfy the commutation rules

$$[J_l,J_m]=i\epsilon_{lmn}J_n\,,\quad [J_l,K_m]=i\epsilon_{lmn}K_n\,,\quad [K_m,K_n]=-i\epsilon_{lmn}J_n\,, \qquad (2.2)$$

The little group of k is then defined by the condition $\Lambda k=k$, or $E(k^0+\boldsymbol{\sigma}\cdot\mathbf{k})E^\dagger=k^0+\boldsymbol{\sigma}\cdot\mathbf{k}$, which requires E to be of the form

$$E=\begin{pmatrix} e^{i(\theta/2)} & -\alpha_1+i\alpha_2 \\ 0 & e^{-i(\theta/2)} \end{pmatrix}. \qquad (2.3)$$

In infinitesimal form the little group (an Euclidean group in two dimensions) is generated by J_3 (rotation) and $L_1=K_1-J_2$, $L_2=K_2+J_1$ ("translations") with

$$[J_3,L_1]=iL_2\,,\quad [J_3,L_2]=-iL_1\,,\quad [L_1,L_2]=0\,. \qquad (2.4)$$

In the language of 2×2 matrices, $J_3\to\frac{1}{2}\sigma_3$, $L_1\to\frac{1}{2}i\sigma_+$, $L_2\to\frac{1}{2}\sigma_+$, and

$$e^{i(\theta J_3+x_1L_1+x_2L_2)}\longrightarrow\begin{bmatrix} e^{i(\theta/2)} & \dfrac{\sin\frac{1}{2}\theta}{\frac{1}{2}\theta}(-x_1+ix_2) \\ 0 & e^{-i(\theta/2)} \end{bmatrix}. \qquad (2.5)$$

Let $|k,\lambda\rangle$ denote the various states of a particle of four-momentum k. We assume them to span a finite-dimensional vector space which is transformed into itself by the operations of the little group. Furthermore these operations are unitary as are all those of the Poincaré group. The concept of particle is then made precise by requiring the little group to act irreducibly. The Euclidean group has only one-dimensional irreducible unitary representations among its finite-dimensional ones. Hence the set $|k,\lambda\rangle$ is in fact one-dimensional. (The doubling of states necessary to implement discrete transformations will be discussed later.) Denoting the representatives of the generators of Lorentz transformations by the same symbols, one has

$$J_3|k,\lambda\rangle=\lambda|k,\lambda\rangle\,,\qquad L_1|k,\lambda\rangle=L_2|k,\lambda\rangle=0\,, \qquad (2.6)$$

where the helicity λ can take integer and half-integer values. (The question of representations "up to a phase" of the Poincaré group is well known to be solved by discussing the representations of its covering group, which amounts to replacing the 4×4 Lorentz matrices with their 2×2 counterparts introduced above.)

A physical state $|p,\lambda\rangle$ of the same massless particle, of three-momentum $\mathbf{p}$, positive energy $p^0=|\mathbf{p}|$, and helicity λ is obtained by applying a Lorentz transformation to the standard state $|k,\lambda\rangle$:

$$|p,\lambda\rangle=U[L(\mathbf{p})]|k,\lambda\rangle\,, \qquad (2.7)$$

where $L(\mathbf{p})$ is a Lorentz transformation which takes the four-vector k into p, and $U[L(\mathbf{p})]$ is its unitary representative acting in the space of physical states. The transformation $L(\mathbf{p})$ is in principle arbitrary to the extent of multiplication by the right by an element of the little group of k. Making a particular choice amounts then to define the phase of the state $|p,\lambda\rangle$. One convention which will sometimes be used below is the following[2]:

$$L(\mathbf{p})=R(\hat{p})B(|\mathbf{p}|)\,, \qquad (2.8)$$

where $B(|\mathbf{p}|)$ is a pure Lorentz transformation along the z direction taking the vector k into the vector $(|\mathbf{p}|,0,0,|\mathbf{p}|)$:

$$B(|\mathbf{p}|)=e^{-i\phi(|\mathbf{p}|)K_3}\,,\qquad \phi(|\mathbf{p}|)=\ln|p|\,; \qquad (2.9)$$

and $R(\hat{p})$ (with $\hat{p}=\mathbf{p}/|\mathbf{p}|$) is a rotation that brings $\mathbf{e}_3$, the unit vector along the z axis, into the unit vector $\hat{p}$. For all directions different from the z axis this rotation can be chosen to be around the axis defined by the unit vector $\mathbf{n}(\hat{p})=\mathbf{e}_3\times\hat{p}/|[\mathbf{e}_3\times\hat{p}]|$. Thus,

$$R(\hat{p})=\exp[-i\psi(\hat{p})\mathbf{J}\cdot\mathbf{n}(\hat{p})]\,,\qquad \cos\psi(\hat{p})=\mathbf{e}_3\cdot\hat{p}\,. \qquad (2.10)$$

If $\hat{p}$ is $+\mathbf{e}_3$ one can choose $R(\hat{p})=1$, while for $\hat{p}=-\mathbf{e}_3$ one has to define $R(\hat{p})$ as a rotation of π around some axis in the $x-y$ plane. In Eq. (2.10), the angle ψ is assumed to lie between 0 and π.

However, for most of the discussion it is immaterial to know the precise form of $L(\mathbf{p})$ provided one assumes that a definite choice has been made for all $\mathbf{p}\neq0$. To the transformation $L(\mathbf{p})$ corresponds the 2×2 matrix $A(\mathbf{p})$ such that

$$A(\mathbf{p})(k^0+\mathbf{k}\cdot\boldsymbol{\sigma})A^\dagger(\mathbf{p})=p^0+\mathbf{p}\cdot\boldsymbol{\sigma};\quad A(\mathbf{p})\equiv\begin{pmatrix}\alpha_p & \beta_p\\ \gamma_p & \delta_p\end{pmatrix}. \qquad (2.11)$$

This equation only determines the first column of $A(\mathbf{p})$

and only up to a phase:

$$\tfrac{1}{2}(p^0+\mathbf{p}\cdot\boldsymbol{\sigma})=\begin{pmatrix}\alpha_p\\ \gamma_p\end{pmatrix}\otimes(\bar{\alpha}_p\bar{\gamma}_p).$$

The choice of this phase amounts, as above, to choosing the phase of the state $|p,\lambda\rangle$ since

$$A(\mathbf{p})=\begin{pmatrix}\alpha_p & \beta_p\\ \gamma_p & \delta_p\end{pmatrix}=\begin{pmatrix}\alpha_p & -\bar{\gamma}_p/(|\alpha_p|^2+|\gamma_p|^2)\\ \gamma_p & \bar{\alpha}_p/(|\alpha_p|^2+|\gamma_p|^2)\end{pmatrix}\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix},$$

$$a=\frac{\bar{\alpha}_p\beta_p+\bar{\gamma}_p\delta_p}{|\alpha_p|^2+|\gamma_p|^2}.$$

This relation shows that $A(\mathbf{p})$ differs from a standard one [which depends only on the spinor $(\alpha_p\gamma_p)$ and is always well defined $(|\alpha|^2+|\gamma_p|^2=p_0>0)$] by an element of the little group which in the representations considered is mapped onto the identity. This form is well suited to describe the behavior of the states under arbitrary Lorentz transformations. Indeed, one has

$$U[\Lambda]|p,\lambda\rangle=e^{i\lambda\theta(\mathbf{p},\Lambda)}|\Lambda p,\lambda\rangle. \tag{2.12}$$

The angle $\theta(\mathbf{p},\Lambda)$ is given by

$$\exp\left[i\frac{\theta(\mathbf{p},\Lambda)}{2}\right]=\frac{a\alpha_p+b\gamma_p}{\alpha_{\Lambda p}}=\frac{c\alpha_\alpha+d\gamma_p}{\gamma_{\Lambda p}}$$

$$=\frac{\bar{\alpha}_{\Lambda p}(a\alpha_p+b\gamma_p)+\bar{\gamma}_{\Lambda p}(c\alpha_p+d\gamma_p)}{|\alpha_{\Lambda p}|^2+|\gamma_{\Lambda p}|^2}$$

$$\pm A=\pm\begin{pmatrix}a & b\\ c & d\end{pmatrix}\leftrightarrow\Lambda. \tag{2.13}$$

Note that at least one of the two quantities $\alpha_{\Lambda p}$ and $\gamma_{\Lambda p}$ is different from zero. In terms of the choice (2.8) one has

$$\begin{pmatrix}\alpha_p\\ \gamma_p\end{pmatrix}=\exp\left(-\frac{i\psi(\hat{p})}{2}\boldsymbol{\sigma}\cdot\mathbf{n}(\hat{p})\right)\begin{pmatrix}|\mathbf{p}|^{1/2}\\ 0\end{pmatrix}$$

$$=\begin{pmatrix}[\frac{1}{2}(p^0+p^3)]^{1/2}\\ (p^1+ip^2)/[2(p^0+p^3)]^{1/2}\end{pmatrix}. \tag{2.14}$$

Clearly, as was said before, this convention breaks down when $\hat{p}=-\mathbf{e}_3$ [one can set there $\alpha_p=0$, $\gamma_p=1$ for definiteness which amounts to $R(\hat{p})=e^{-i\pi J_2}$].

To complete this section we introduce two "polarization" four-vectors, both functions of p, $\epsilon^{(\pm)\mu}(p)$, which are defined as follows:

$$\epsilon^{(\pm)}(k)=[(\mp)/\sqrt{2}](0,1,\pm i,0),$$
$$\epsilon^{(\pm)}(p)=L(\mathbf{p})\epsilon^{(\pm)}(k)=R(\hat{p})\epsilon^{(\pm)}(k). \tag{2.15}$$

The fourth component of $\epsilon^{(\pm)}(p)$ is always zero and $\epsilon^{(\pm)}(p)$ depend on $\hat{p}$ only. These vectors can alternatively be defined by

$$\boldsymbol{\varepsilon}^{(\pm)}(\mathbf{p})\cdot\boldsymbol{\sigma}=(\pm)/[\sqrt{2}(p^1\mp ip^2)p^0]^{-1}\times(p^0\pm\boldsymbol{\sigma}\cdot\mathbf{p})\tfrac{1}{2}\sigma^3(p^0\mp\boldsymbol{\sigma}\cdot\mathbf{p}). \tag{2.16}$$

Consequently, they satisfy the "Maxwell" relations

$$p^0\boldsymbol{\varepsilon}^{(\pm)}(\mathbf{p})=\pm i\mathbf{p}\times\boldsymbol{\varepsilon}^{(\pm)}(\mathbf{p}); \tag{2.17a}$$

from which it follows that

$$\boldsymbol{\varepsilon}^{(\pm)}(\mathbf{p})\cdot\boldsymbol{\varepsilon}^{(\pm)}(\mathbf{p})=\mathbf{p}\cdot\boldsymbol{\varepsilon}^{(\pm)}(\mathbf{p})=0. \tag{2.17b}$$

The behavior of these vectors under Lorentz transformations is quite interesting. Under a transformation of the little group of k, written as

$$L^{-1}(\Lambda p)\Lambda L(p)=e^{i\theta J_3}e^{i(x_1L_1+x_2L_2)},$$

one has

$$\{[L^{-1}(\Lambda p)\Lambda L(p)]\epsilon^{(\pm)}(k)\}^\mu=e^{\pm i\theta}\epsilon^{(\pm)\mu}(k)-(1/\sqrt{2})(x_1\pm ix_2)k^\mu,$$

or

$$\{\Lambda\epsilon^{(\pm)}(p)\}^\mu=e^{\pm i\theta}\epsilon^{(\pm)}(\Lambda p)^\mu-(1/\sqrt{2})(x_1\pm ix_2)(\Lambda p)^\mu.$$

Since the fourth component of $\epsilon^{(\pm)}$ vanishes this can be rewritten as

$$[\Lambda\epsilon^{(\pm)}(p)]^l=e^{\pm i\theta}\epsilon^{(\pm)}(\Lambda p)^l+[\Lambda\epsilon^{(\pm)}(p)]^0\frac{(\Lambda p)^l}{(\Lambda p)^0}. \tag{2.18}$$

We note the appearance of the second term on the right-hand side, a "gauge term." We also remark that the little-group angle $\theta\equiv\theta(p,\Lambda)$ depends only on the direction of $\mathbf{p}$ and not on its magnitude, as was implicit in its expression (2.13) and is made clear by (2.18).

III. WAVE FUNCTIONS AND QUANTUM FIELDS

This section is devoted to the study of free fields describing the creation and annihilation of massless particles, and transforming according to an irreducible representation of the Lorentz group.

Let us introduce the operator $a^\dagger(p,\lambda)$, which creates a state $|p,\lambda\rangle$, with $p^0=|\mathbf{p}|$, from the vacuum state $|0\rangle$.

$$|p,\lambda\rangle=a^\dagger(p,\lambda)|0\rangle. \tag{3.1}$$

Note that the choice of phase of the state vector $|p,\lambda\rangle$ reflects in turn in the definition of $a^\dagger(p,\lambda)$. The corresponding destruction operator is $a(p,\lambda)$. Their commutation rules are

$$[a(p,\lambda),a^\dagger(p',\lambda')]_\delta=(2\pi)^3 2p^0\delta^{(3)}(\mathbf{p}-\mathbf{p}')\delta_{\lambda\lambda'}, \tag{3.2}$$

where $\delta=-1$ defines the commutator and $\delta=+1$ the anticommutator. The factor $2p^0$ on the right-hand side is dictated by the definition (2.7), which implies covariant normalization. Finally, we have left open the question of the existence of several states with the same helicity λ, to take into account possible discrete symmetries.

The transformation properties of the creation and annihilation operators under the Poincaré group follow from those of the states, Eq. (2.12), and the invariance of the vacuum:

$$U[\Lambda]a^{\dagger}(p,\lambda)U^{-1}[\Lambda]=e^{i\lambda\theta(p,\Lambda)}a^{\dagger}(\Lambda p,\lambda). \quad (3.3)$$

Since $U[\Lambda]$ is unitary, we also have

$$U[\Lambda]a(p,\lambda)U^{-1}[\Lambda]=e^{-i\lambda\theta(p,\Lambda)}a(\Lambda p,\lambda). \quad (3.4)$$

By linear superposition of these operators we look now for a quantum field that transforms irreducibly under the homogeneous Lorentz group. Let us first discuss the negative frequency or annihilation part of this field $\phi^{(-)}(x)$:

$$\phi^{(-)}(x,\lambda)=\int \frac{d^3p}{(2\pi)^3 2p^0}e^{-ip\cdot x}u(p,\lambda)a(p,\lambda). \quad (3.5)$$

The field $\phi^{(-)}$ is to be thought as a vector in a representation space of some irreducible representation of the Lorentz group (or rather its covering group) and the same is true for the wave function $u(p,\lambda)$. Once a basis in such a space has been chosen, we can as well discuss the components $\phi_\alpha^{(-)}(x,\lambda)$ and $u_\alpha(p,\lambda)$ of these vectors. The wave function is to be chosen in such a way that the field transforms covariantly, i.e.,

$$U[\Lambda]\phi^{(-)}(x,\lambda)U^{-1}[\Lambda]=T[\Lambda^{-1}]\phi^{(-)}(\Lambda x,\lambda). \quad (3.6)$$

In this equation $T[\Lambda]$ are the operators of the irreducible representations of the Lorentz group. A brief summary of their classification and main properties has been included in Appendices A and B. The reader is referred to them for the notations to be used below.

It immediately follows from Eq. (3.6) that the wave function has to obey

$$u(\Lambda p,\lambda)=e^{-i\lambda\theta(p,\Lambda)}T[\Lambda]u(p,\lambda), \quad (3.7)$$

where $\theta(p,\Lambda)=-\theta(\Lambda p,\Lambda^{-1})$ has been used.

By restricting (3.7) to $p=k$, where k is the standard momentum of Sec. II, and Λ to a transformation E of the Euclidean little group of k, we obtain a constraint equation for $u(k,\lambda)$, which reads

$$T[E]u(k,\lambda)=e^{i\lambda\theta(k,E)}u(k,\lambda). \quad (3.8)$$

Setting now $p=k$ and $\Lambda=L(p)$ in Eq. (3.7) yields

$$u(p,\lambda)=T[L(p)]u(k,\lambda), \quad (3.9)$$

where we have used $\theta(k,L(p))=0$. This relation, together with Eq. (3.8), ensures the validity of Eq. (3.7). The problem of solving for the wave function $u(p,\lambda)$ thus reduces to solving Eq. (3.8) for $u(k,\lambda)$.

We shall present two derivations for $u(k,\lambda)$. The first one uses the infinitesimal form of Eq. (3.8) and an expansion of $u(k,\lambda)$ in the basis $f_{j\sigma}$ which diagonalizes the rotation group. (See Appendix A.) The second uses the techniques and results of Appendix B, from which $u(k,\lambda)$ is obtained directly. The latter method also shows that in the case of infinite-dimensional representation Eqs. (3.6) and (3.7) are in fact improper in a sense to be discussed below. However, the discussion of finite- and infinite-dimensional cases proceeds formally in a similar way.

A. Generator Approach

Taking the infinitesimal form of (3.8), we obtain

$$\begin{aligned}J_3u(k,\lambda)&=\lambda u(k,\lambda),\\ L_1u(k,\lambda)&=L_2u(k,\lambda)=0,\end{aligned} \quad (3.10)$$

where J_3, $L_1=K_1-J_2$, $L_2=K_2+J_1$ are the generators of the little group of k [compare with Eq. (2.6) of the previous section]. We use the same symbol for a generator of the Lorentz group and its representative in the representation T.

To obtain the wave function $u(p,\lambda)$, we have only to solve Eq. (3.10). To achieve this goal we choose an irreducible representation characterized by a certain (j_0,c), and expand the vector $u(k,\lambda)$ in the basis $\{f_{j\sigma}\}$. From the first equation (3.10) it is clear that only the components with $\sigma=\lambda$ contribute to the expansion of $u(k,\lambda)$[3]:

$$\begin{aligned}u(p,\lambda)&=\sum_{j\sigma}u_{j\sigma}(p,\lambda)f_{j\sigma},\\ u(k,\lambda)&=\sum_j h(j)f_{j\lambda}.\end{aligned} \quad (3.11)$$

In other words, we have set $u_{j\sigma}(k,\lambda)=\delta_{\sigma,\lambda}h(j)$.

It is straightforward to show that the two remaining equations in (3.10) determine the coefficients $h(j)$ up to an over-all constant factor. The detailed calculation is performed in Appendix A. Let us bring the results here. It turns out that, given λ, the only representations allowed are those such that $\lambda=\epsilon j_0$ with $\epsilon=+1$ or $\epsilon=-1$. In these cases

$$h(j+1)=-ih(j)\left(\frac{(2j+3)}{(2j+1)}\frac{j+1-\epsilon c}{j+1+\epsilon c}\right)^{1/2}. \quad (3.12)$$

One also gets

$$K_3u(k,\lambda)=i(\epsilon c-1)u(k,\lambda). \quad (3.13)$$

It is not surprising that $K_3u(k,\lambda)$ is proportional to $u(k,\lambda)$, since $K_3u(k,\lambda)$ obeys Eq. (3.10) whenever $u(k,\lambda)$ does. It is clear from (3.12) that the finite-dimensional representations are obtained for $c=\epsilon(j_0+n+1)$, $n=0$, $1, \cdots$. In the notation $n_1=2j_1+1$, $n_2=2j_2+1$ (with $[\frac{1}{2}(J-iK)]^2=j_1(j_1+1)$, $[\frac{1}{2}(J+iK)]^2=j_2(j_2+1)$), one has

$$\begin{aligned}&j_0=|j_1-j_2|,\\ &c=[\mathrm{sgn}(j_1-j_2)](j_1+j_2+1), \quad \text{if } j_1\neq j_2;\\ &c=\pm(2j_1+1), \quad \text{if } j_1=j_2.\end{aligned} \quad (3.14)$$

Thus, the sign of the helicity λ is the one of j_1-j_2, i.e., $\lambda=j_1-j_2$ and its absolute value is given by the lowest "spin" contained in the representation of the

[3] We do not specify convergence properties here. We only require that $h(j)$ be finite. From the solution (3.12) it follows that $h(j)\rightarrow 1$ as $j\rightarrow\infty$.

Lorentz group, a well-known result for the case of finite-dimensional representations.[2]

Once $u(k,\lambda)$ is known, it is immediate to obtain $u(p,\lambda)$. Using the convention (2.8), for example, one has

$$u(p,\lambda)=T[L(\mathbf{p})]u(k,\lambda)=R(\hat{p})B(|\mathbf{p}|)u(k,\lambda) = (p^0)^{(\epsilon c-1)}R(\hat{p})u(k,\lambda), \tag{3.15}$$

where the Eqs. (3.9), (2.9), and (3.13) were used. Finally, when the wave function is expanded in the "$j\sigma$ basis" its components read

$$u_{j\sigma}(p,\lambda)=h(j)(p^0)^{(\epsilon c-1)}D_{\sigma,\lambda}{}^{j}[R(\hat{p})] \tag{3.16}$$

B. Global Approach

We repeat the previous calculation by using the explicit realization of the operators $T[\Lambda]$ of Appendix B. In other words, the vectors $\phi^{(-)}(x,\lambda)$, $u(p,\lambda)$ are exhibited as functions of two variables z and $\bar{z}$ [or rather two real variables $(z+\bar{z})/2$, $(z-\bar{z})/2i$]. We write $\phi^{(-)}(z;x,\lambda)$ and $u(z;p,\lambda)$. Setting

$$E=\begin{pmatrix} e^{i\theta/2} & \alpha_1-i\alpha_2 \\ 0 & e^{-i\theta/2}\end{pmatrix},$$

Eq. (3.8) takes the form

$$e^{i\lambda\theta}u(z;k,\lambda)=[(\alpha_1-i\alpha_2)z+e^{-i\theta/2}]^{n_1-1} \times[(\alpha_1+i\alpha_2)\bar{z}+e^{i\theta/2}]^{n_2-1}u\left(\frac{e^{i(\theta/2)}z}{(\alpha_1-i\alpha_2)z+e^{-i(\theta/2)}};k,\lambda\right).$$

It is elementary to solve this equation. With h a constant, we find

$$u(z;k,\lambda)=hz^{n_1-1}\bar{z}^{n_2-1}, \tag{3.17}$$

provided that

$$\lambda=\tfrac{1}{2}(n_1-n_2)=\epsilon j_0. \tag{3.18}$$

Equation (3.9) then enables one to find $u(z;p,\lambda)$, which reads

$$u(z;p,\lambda)=h(\alpha_p z+\gamma_p)^{n_1-1}(\bar{\alpha}_p\bar{z}+\bar{\gamma}_p)^{n_2-1}. \tag{3.19}$$

This compact expression for the wave function has still to be identified with the expression of its components in the $j\sigma$ basis. In this form it shows that, when n_1 and n_2 are positive integers, the wave function is a polynomial in z and $\bar{z}$, and hence belongs to a finite-dimensional representation of the Lorentz group. It also reveals the fact that for all other cases $u(z;p,\lambda)$, and hence $\phi(z;p,\lambda)$, does not really belong to the space $D_{(n_1,n_2)}$. (See Appendix B). Finally, as was expected, from (3.19) one sees that $u(z;p,\lambda)$ depends only on the spinor

$$\begin{pmatrix}\alpha_p\\ \gamma_p\end{pmatrix}$$

attached to p (see Sec. II), i.e., reflects the phase convention required to define the annihilation operator $a(p,\lambda)$. Note, however, that the product $u(z;p\lambda)a(p,\lambda)$ is independent of this phase convention. It is possible to expand (3.19) in the $j\sigma$ basis. The natural definition uses the scalar product (B22) in terms of which one has

$$u_{j\sigma}(p,\lambda)=\left|\frac{\Gamma(j+1+\frac{1}{2}(n_1+n_2))}{\Gamma(j+1-\frac{1}{2}(n_1+n_2))}\right|\langle f_{j\sigma}(z),u(z;p,\lambda)\rangle$$
$$=h\left|\frac{\Gamma(j+1+\frac{1}{2}(n_1+n_2))}{\Gamma(j+1-\frac{1}{2}(n_1+n_2))}\right|\left(\frac{2}{\pi}\right)\int\tfrac{1}{2}idzd\bar{z} \times(1+z\bar{z})^{-\mathrm{Re}(n_1+n_2)}f_{j\sigma}(z)(\alpha_p z+\gamma_p)^{n_1-1} \times(\bar{\alpha}_p\bar{z}+\bar{\gamma}_p)^{n_2-1}.$$

To evaluate the integral, it is useful to make the change of variables:

$$z=(\bar{\alpha}_p z'-\gamma_p)/(\bar{\gamma}_p z'+\alpha_p),$$

after which the integration is straightforward. One obtains

$$u_{j\sigma}(p,\lambda)=h(j)(p^0)^{\epsilon c-1}D_{\sigma\lambda}{}^{(j)}[R(\hat{p})], \tag{3.20}$$

where

$$R(\hat{p})=(|\alpha_p|^2+|\gamma_p|^2)^{-1/2}\begin{pmatrix}\alpha_p & -\bar{\gamma}_p\\ \gamma_p & \bar{\alpha}_p\end{pmatrix}. \tag{3.21}$$

and

$$h(j)=\frac{2h}{\pi}e^{i\pi j/2}\Gamma(n_1)\Gamma(n_2)\sin[\pi(\epsilon c-j)] \times\left[\left|(j+\tfrac{1}{2})\frac{\Gamma(j+1-\epsilon c)}{\Gamma(j+1+\epsilon c)}\right|\right]^{1/2}. \tag{3.22}$$

$R(\hat{p})$ is the rotation which brings the unit vector in the z direction to the direction $\hat{p}=\mathbf{p}/|\mathbf{p}|$. Equation (3.22) yields the result (3.12) for $h(j+1)/h(j)$, as expected. One also has $u_{j\sigma}(k,\lambda)=\delta_{\sigma\lambda}h(j)$, as before.

Finally, we write down a generating function for $u_{j\sigma}(p\lambda)$:

$$u_j(\mathbf{y};p\lambda)=\sum_{\sigma=-j}^{j}\frac{y_1{}^{j+\sigma}y_2{}^{j-\sigma}}{[(j+\sigma)!(j-\sigma)!]^{1/2}}u_{j\sigma}(p,\lambda)$$
$$=h(j)(p^0)^{\epsilon c-j-1} \times\frac{(\alpha_p y_1+\gamma_p y_2)^{j+\lambda}(-\bar{\gamma}_p y_1+\bar{\alpha}_p y_2)^{j-\lambda}}{[(j+\lambda)!(j-\lambda)!]^{1/2}}. \tag{3.23}$$

This generating function will appear to be useful later.

C. Connection with the Wave Functions for Nonvanishing Rest Mass

We include here a brief but instructive digression on the limit of massive-particle wave functions when the mass goes to zero. In particular, let us assume that we describe a particle of mass m and spin j by a field transforming according to a representation of the Lorentz group with lowest spin j_0 smaller than j.

Obviously, some singularity has to occur in the wave function when $m\to 0$, since only $\lambda=\pm j_0$ are allowed for massless particles. To avoid unnecessary complications we treat one example, where the field transforms according to the representation $n_1=n_2=2$ or $j_1=j_2=\frac{1}{2}$: to say it more plainly, a usual four vector field. The decomposition according to the rotation subgroup yields spin zero and spin one. Let $e^\mu(j,\sigma)$ $(j\leq\sigma\leq+j,\ j=0,1)$ be the wave function for a particle of spin j and angular momentum σ along the z axis, and vanishing three-momentum. Its wave function $e^\mu(\mathbf{p};J,\sigma)$ for three-momentum $\mathbf{p}$, energy $p^0=(|\mathbf{p}|^2+m^2)^{1/2}$ will be given by

$$e^\mu(\mathbf{p},J,\sigma)=L_\nu{}^\mu(\mathbf{p})e^\nu(J,\sigma)\,, \tag{3.24}$$

where the Lorentz transformation $L(\mathbf{p})$ transforms the time axis $n=(1,0,0,0)$ into p/m. Then

$$\begin{aligned} e^0(\mathbf{p},J,\sigma)&=[p^0e^0(J,\sigma)+\mathbf{p}\cdot\boldsymbol{\varepsilon}(J,\sigma)]/m\,,\\ \mathbf{e}(\mathbf{p},J,\sigma)&=\mathbf{e}(J,\sigma)+\frac{\mathbf{p}}{m}\left(e^0(J,\sigma)+\frac{\mathbf{p}\cdot\mathbf{e}(J,\sigma)}{p^0+m}\right). \end{aligned} \tag{3.25}$$

Suppose we describe a spin-zero particle; then one has $e^0(0,0)=1$, $\mathbf{e}(0,0)=0$, and Eq. (3.25) reduces to

$$e^\mu(\mathbf{p};0,0)=p^\mu/m\,. \tag{3.26}$$

We see that $me^\mu(\mathbf{p};0,0)=p^\mu$ has a smooth limit when $m\to 0$. On the other hand, from (3.19) it follows that for $n_1=n_2=2$,

$$u(z;p,\lambda=0)=h(\alpha_pz+\gamma_p)(\bar{\alpha}_p\bar{z}+\bar{\gamma}_p)=h(p\cdot A(z))\,,$$

where

$$A(z)=\left(\overset{(0)}{\frac{1+z\bar{z}}{2}},\ \overset{(1)}{-\frac{z+\bar{z}}{2}},\ \overset{(2)}{i\frac{z-\bar{z}}{2}},\ \overset{(3)}{\frac{1-z\bar{z}}{2}}\right)$$

transforms like a four-vector under the law

$$A(z)\to(bz+d)(\bar{b}\bar{z}+\bar{d})A\left(\frac{az+c}{bz+d}\right).$$

[This can be checked directly, by verifying that

$$\begin{pmatrix}a&b\\c&d\end{pmatrix}^{-1}[A^0(z)+\mathbf{A}(z)\cdot\boldsymbol{\sigma}]\begin{pmatrix}\bar{a}&\bar{c}\\\bar{b}&\bar{d}\end{pmatrix}^{-1}=(bz+d)(\bar{b}\bar{z}+\bar{d})\left\{A^0\left(\frac{az+c}{bz+d}\right)+\mathbf{A}\left(\frac{az+c}{bz+d}\right)\cdot\boldsymbol{\sigma}\right\}$$

with

$$A^0(z)+\mathbf{A}(z)\cdot\boldsymbol{\sigma}=\begin{pmatrix}1&-\bar{z}\\-z&z\bar{z}\end{pmatrix}=\begin{pmatrix}1\\-z\end{pmatrix}\otimes(1,-\bar{z})\,.]$$

Hence the projections of the "spin"-zero and "spin"-one parts of $u(z;p,\lambda=0)$ are proportional to p^0 and $\mathbf{p}$, respectively, in agreement with the previous limit.

On the other hand, if we start with a spin-one particle for which $e^0(1,\sigma)=0$ we get

$$\begin{aligned} e^0(\mathbf{p};1,\sigma)&=[\mathbf{p}\cdot\mathbf{e}(1,\sigma)/m]\,,\\ \mathbf{e}(\mathbf{p};1,\sigma)&=\mathbf{e}(1,\sigma)+\frac{\mathbf{p}\cdot\mathbf{e}(1,\sigma)}{p^0+m}\frac{\mathbf{p}}{m}\,. \end{aligned} \tag{3.27}$$

Obviously this four-vector has no limit as $m\to 0$. However, if we first multiply by m and then let m go to zero, we obtain

$$\lim_{m\to 0} me^\mu(\mathbf{p};1,\sigma)=[\mathbf{p}\cdot\mathbf{e}(1,\sigma)/p^0]p^\mu\,, \tag{3.28}$$

which is a zero-helicity wave function. There is no way to obtain the helicity-one wave function for massless particles starting from the $j_1=j_2=\frac{1}{2}$ representation of the Lorentz group, as we expected from the general considerations above. In fact this result holds in any spin case. Assume that one describes a massive particle of spin j by *a* wave function transforming as a finite-dimensional representations of the Lorentz group (n_1,n_2) with

$$j_0=\tfrac{1}{2}|n_1-n_2|\leq j\leq j_{\max}=\tfrac{1}{2}(n_1+n_2)-\tfrac{1}{2}=|c|-1$$

($j_{\max}$ is the highest spin in the representation). The only nonvanishing finite limit, when $m\to 0$, is obtained by multiplying the wave function by $m^{j_{\max}}$ and is proportional to the wave function for a massless particle of helicity $\lambda=\frac{1}{2}(n_1-n_2)=\epsilon j_0$, equal in absolute value to the lowest spin j_0.

D. Recursion Relations and Tensor Basis

From the explicit expression for $u_{j\sigma}(p,\lambda)$, Eq. (3.20), it follows that one can relate the various components to each other. Starting from the generating function Eq. (3.23), we observe that for any positive integer r one has

$$\begin{aligned} u_{j+r}(\mathbf{y};p\lambda)&=\frac{h(j+r)}{h(j)}\left[\frac{(j+\lambda)!(j-\lambda)!}{(j+r+\lambda)!(j+r-\lambda)!}\right]^{1/2}\times\left[\frac{(\alpha_py_1+\gamma_py_2)(-\bar{\gamma}_py_1+\bar{\alpha}_py_2)}{p^0}\right]^r u_j(\mathbf{y};p\lambda)\\ &=\frac{h(j+r)}{h(j)}\left[\frac{(j+\lambda)!(j-\lambda)!}{(j+r+\lambda)!(j+r-\lambda)!}\right]^{1/2}\left[\hat{p}^1\frac{y_2^2-y_1^2}{2}+\hat{p}^2\frac{i(y_1^2+y_2^2)}{2}+\hat{p}^3y^1y^2\right]^r u_j(\mathbf{y};p\lambda)\,, \end{aligned} \tag{3.29}$$

and

$$u_{j-r}(\mathbf{y};p\lambda)=\frac{h(j-r)}{h(j)}\left[\frac{(j-r+\lambda)!(j-r-\lambda)!}{(j+\lambda)!(j-\lambda)!}\right]^{1/2}\times\left[\left(\bar{\alpha}_p\frac{\partial}{\partial y_1}+\bar{\gamma}_p\frac{\partial}{\partial y_2}\right)\times\left(\alpha_p\frac{\partial}{\partial y_2}-\gamma_p\frac{\partial}{\partial y_1}\right)\Big/p^0\right]^r u_j(\mathbf{y};p\lambda)$$

$$=\frac{h(j-r)}{h(j)}\left[\frac{(j-r+\lambda)!(j-r-\lambda)!}{(j+\lambda)!(j-\lambda)!}\right]^{1/2}\times\left[\hat{p}^1\frac{1}{2}\left(\frac{\partial^2}{\partial y_2^2}-\frac{\partial^2}{\partial y_1^2}\right)+\hat{p}^2\frac{1}{2i}\left(\frac{\partial^2}{\partial y_1^2}+\frac{\partial^2}{\partial y_2^2}\right)+\hat{p}^3\frac{\partial^2}{\partial y_1\partial y_2}\right]^r u_j(\mathbf{y};p\lambda).\quad(3.30)$$

Thus, $u_{j\pm r,\sigma}(p,\lambda)$ is related to $u_{j\sigma'}(p,\lambda)$. For the case of adjacent j values, one obtains

$$u_{j+1,\sigma}(p,\lambda)=[h(j+1)/h(j)][(j+1)^2-j_0^2]^{-1/2}\times\{-[(j+\sigma)(j+\sigma+1)]^{1/2}\tfrac{1}{2}(\hat{p}^1-i\hat{p}^2)\times u_{j,\sigma-1}(p,\lambda)+[(j+\sigma+1)(j-\sigma+1)]^{1/2}\times\hat{p}^3u_{j\sigma}(p\lambda)+[(j-\sigma)(j-\sigma+1)]^{1/2}\times\tfrac{1}{2}(\hat{p}^1+i\hat{p}^2)u_{j,\sigma+1}(p,\lambda)\},\quad(3.31)$$

$$u_{j-1,\sigma}(p,\lambda)=[h(j-1)/h(j)][j^2-j_0^2]^{-1/2}\times\{[(j-\sigma)(j-\sigma+1)]^{1/2}\tfrac{1}{2}(\hat{p}^1-i\hat{p}^2)\times u_{j,\sigma-1}(p\lambda)+[(j+\sigma)(j-\sigma)]^{1/2}\times\hat{p}^3u_{j\sigma}(p,\lambda)-[(j+\sigma)(j+\sigma+1)]^{1/2}\times\tfrac{1}{2}(\hat{p}^1+i\hat{p}^2)u_{j,\sigma+1}(p,\lambda)\}.\quad(3.32)$$

A single equation, which combines both Eqs. (3.29) and (3.30), can be immediately derived from Eq.(3.20):

$$u_{j+n,\sigma}(p,\lambda)=[h(j+n)/h(j)]\times[\langle j\lambda,|n|0|j+n\,\lambda\rangle]^{-1}\sum_{mm'}D_{m'0}^{(|n|)}[R(\hat{p})]\times\langle jm,|n|m'|j+n\,\sigma\rangle u_{jm}(p,\lambda),\quad(3.33)$$

where $\langle j_1m_1,j_2m_2|jm\rangle$ are the usual Clebsch-Gordan coefficients and n is an integer. In deriving this relation, we made use of the identity

$$D_{\sigma\,\bar{m}+\bar{m}'}^{(j+n)}[R]\langle j\bar{m},|n|\bar{m}'|j+n\,\bar{m}+\bar{m}'\rangle=\sum_{mm'}D_{m\bar{m}}^{(j)}[R]D_{m'\bar{m}'}^{(|n|)}[R]\langle jm,|n|m'|j+n\,\sigma\rangle.$$

Using Eq. (3.21), which defines $R(\hat{p})$, and Eq. (A10), we can calculate $D_{m0}^{(j)}[R(\hat{p})]$. In fact,

$$\frac{1}{j!}\left[\frac{(\alpha_px_1+\gamma_px_2)(-\bar{\gamma}_px_1+\bar{\alpha}_px_2)}{p^0}\right]^j=\frac{1}{j!}\left[\hat{p}^1\frac{x_2^2-x_1^2}{2}+\hat{p}^2\frac{i(x_1^2+x_2^2)}{2}+\hat{p}^3x_1x_2\right]^j=\sum_{m=-j}^{j}\frac{x_1^{j+m}x_2^{j-m}}{[(j+m)!(j-m)!]^{1/2}}D_{m0}^{(j)}[R(\hat{p})],\quad(3.34)$$

which also shows that $D_{m0}^{(j)}[R(\hat{p})]$ is a polynomial of degree j in the components of $\hat{p}$.

The reader might notice that $p^1\mp ip^2$ carries angular momentum ±1 along the z axis. This corresponds to the fact that on the one hand $\frac{1}{2}\sigma_3$ corresponds to J_3, while for a rotation around the z axis of magnitude θ the behavior of the four-vector p as agreed in Sec. II is

$$e^{i(\theta/2)\sigma_3}(p^0+\mathbf{p}\cdot\boldsymbol{\sigma})e^{-i(\theta/2)\sigma_3}=\begin{pmatrix}p^0+p^3 & e^{i\theta}(p^1-ip^2)\\ e^{-i\theta}(p^1+ip^2) & p^0-p^3\end{pmatrix},$$

i.e., p^1-ip^2 carries one unit of angular momentum around the z axis while p^1+ip^2 carries the opposite amount. These relations are obviously preserved by the identities (3.31) and (3.32). These identities show that by applying suitable combinations of $\mathbf{p}/p^0$ to the $u_{j\sigma}$ components we generate the $j\pm1$ components of the wave function. Finally, one can write similar relations in configuration space for the field itself. This will be done in the next paragraph.

For later purposes it will become convenient to use a Cartesian basis instead of the $j\sigma$ basis of the rotation group. For integer λ this basis involves traceless symmetric tensors whose indices run from 1 to 3, while for the case of half-integer helicity an extra spinor index is involved. Let us bring here the explicit expressions for the integer λ case.

The required tensors can be computed in two ways. In the first, one introduces the transformation coefficients $t_{a_1\cdots a_j}{}^{\sigma}$ $(1\leq a_k\leq3)$,

$$t_{a_1\cdots a_j}{}^{\sigma}=\sum_{m_1\cdots m_j}\langle1,m_1;1,m_2|2,m_1+m_2\rangle\times\langle2,m_1+m_2;1,m_3|3,m_1+m_2+m_3\rangle\cdots\times\langle j-1\sum_{n=1}^{j-1}m_n;1,m_j|j,\sigma\rangle e_{a_1}{}^{(m_1)}\cdots e_{a_j}{}^{(m_j)},\quad(3.35)$$

where

$$\mathbf{e}^{(+1)}=-(\mathbf{e}^1+i\mathbf{e}^2)/\sqrt{2},\quad\mathbf{e}^{(-1)}-(\mathbf{e}^1=i\mathbf{e}^2)/\sqrt{2},\quad\mathbf{e}^{(0)}=\mathbf{e}^3.$$

The tensor $t_{a_1\cdots a_j}$ is thus symmetric in all its indices and traceless in any pair. It is easy to verify that

$$R_{a_1l_1}\ldots R_{a_jl_j}t_{l_1\cdots l_j}{}^{\sigma}=t_{a_1\cdots a_j}{}^{\sigma'}D_{\sigma'\sigma}^{(j)}[R],\quad(3.36)$$

where R_{ab} is the 3×3 matrix representation of a rotation R. One also has

$$\sum_{\{a\}} \bar{t}_{a_1\cdots a_j}{}^{\sigma} t_{a_1\cdots a_j}{}^{\sigma'}=\delta_{\sigma\sigma'}. \qquad (3.37)$$

The Cartesian basis wave functions are given by

$$U_{a_1\cdots a_j}(p\lambda)=\sum_{\sigma=-j}^{j} t_{a_1\cdots a_j}{}^{\sigma} u_{j\sigma}(p\lambda). \qquad (3.38)$$

The expression (3.20), when combined with Eq. (3.36), yields,

$$U_{a_1\cdots a_j}(p\lambda)=h(j)(p^0)^{(\epsilon c-1)} \times R_{a_1b_1}(\hat{p})\cdots R_{a_jb_j}(\hat{p})t_{b_1\cdots b_j}{}^{\lambda}. \qquad (3.39)$$

Since $\sum_a \hat{p}^a R^{al}(\hat{p})=\delta^{l3}$, and since $t_{l_1\cdots l_{j0}}{}^{\lambda}$ vanishes whenever one of $l_1\cdots l_{j0}$ equals 3, it follows that

$$p^{a_i}U_{a_1\cdots a_i\cdots a_{j0}}(p\lambda)=0. \qquad (3.40)$$

An alternative way of introducing the above tensors is to use the generating function (3.23) and to observe that for integer helicity λ it can be rewritten as

$$u_j(\mathbf{y};p\lambda)=\frac{h(j)}{[(j+\lambda)!(j-\lambda)!]^{1/2}}2^{|\lambda|/2}(p^0)^{\epsilon c-1}(\alpha_p/\bar{\alpha}_p)^{\lambda} \times(\hat{p}\cdot\mathbf{Y})^{j-|\lambda|}[\boldsymbol{\varepsilon}^{(\epsilon)}(\hat{p})\cdot\mathbf{Y}]^{|\lambda|}, \qquad (3.41)$$

with $\boldsymbol{\varepsilon}^{(\pm)}(\hat{p})$ defined through Eq. (2.14), and

$$\mathbf{Y}=(\tfrac{1}{2}(y_2^2-y_1^2),\tfrac{1}{2}i(y_1^2+y_2^2),y_1y_2). \qquad (3.42)$$

In deriving Eq. (3.41) we have used the relation

$$\begin{pmatrix} y_2 \\ -y_1 \end{pmatrix}\otimes(y_1y_2)=\mathbf{Y}\cdot\boldsymbol{\sigma}, \qquad (3.43a)$$

and the fact that

$$\boldsymbol{\varepsilon}^{(+)}(\hat{p})\cdot\boldsymbol{\sigma}=\frac{\sqrt{2}}{p^0}\frac{\bar{\alpha}_p}{\alpha_p}\begin{pmatrix}\alpha_p\\ \gamma_p\end{pmatrix}\otimes(\gamma_p-\alpha_p); \qquad (3.43b)$$

the latter may be derived from Eq. (2.16).

Notice that the factor $(\alpha_p/\bar{\alpha}_p)^{\lambda}$ carries all the ambuigity in phase associated with the wave function. The expansion of (3.41) as

$$u_j(\mathbf{y};p\lambda)=\sum_{a_1\cdots a_j} u^{a_1\cdots a_j}(p\lambda)Y_{a_1}\ldots Y_{a_j} \qquad (3.44)$$

allows one to define a tensor $u_{a_1\cdots a_j}(p\lambda)$ which is symmetric and traceless (since $\mathbf{Y}^2=0$). In the particular case of $j=j_0$, one obtains

$$u^{a_1\cdots a_{j0}}(p\lambda)=\frac{h(j_0)}{[(2j_0)!]^{1/2}}2^{j_0/2}(p^0)^{\epsilon c-1} \times(\alpha_p/\bar{\alpha}_p)^{\lambda}\epsilon_{a_1}{}^{(\epsilon)}(p)\cdots\epsilon_{a_{j0}}{}^{(\epsilon)}(p), \qquad (3.45)$$

which is proportional to the expression for $U_{a_1\cdots a_{j0}}(p\lambda)$ derived from Eq. (3.39), as expected [one uses $t_{a_1\cdots a_{j0}}{}^{\lambda}=\epsilon_{a_1}{}^{(\epsilon)}(p)\cdots\epsilon_{a_{j0}}{}^{(\epsilon)}(p)$].

One can also write recursion relations similar to Eqs. (3.31) and (3.32) in the Cartesian basis. Combining (3.41) and (3.44), we obtain, for instance,

$$u^{a_1\cdots a_{j+1}}(p\lambda) =\frac{h(j+1)}{h(j)}[(j+1)^2-j_0^2]^{-1/2}\frac{1}{j+1} \times\left[\sum_{k=1}^{j+1}\hat{p}^{a_k}u^{a_1\cdots\bar{a}_k\cdots a_{j+1}}(p\lambda)-\frac{2}{2j+1}\sum_{k<l}\delta_{\bar{a}_k a_l} \times\sum_{a=l}^{3}\hat{p}^{a}u^{aa_1\cdots\bar{a}_k\cdots\bar{a}_l\cdots a_{j+1}}(p\lambda)\right], \qquad (3.46)$$

$$\left(\frac{j-j_0}{j+j_0}\right)^{1/2}u^{a_1\cdots a_{j-1}}(p\lambda) =j\frac{h(j-1)}{h(j)}\sum_{a=1}^{3}\hat{p}^a u^{aa_1\cdots a_{j-1}}(p\lambda), \qquad (3.47)$$

where we have used

$$\left(\frac{j-j_0}{j+j_0}\right)^{1/2}u_{j-1}(\mathbf{y};p\lambda)=\frac{h(j-1)}{h(j)}\left(\hat{p}^a\frac{\partial}{\partial Y^a}\right)u_j(\mathbf{y};p\lambda).$$

The transversality (3.40) of the lowest component follows also from (3.47).

E. General Free Irreducible Fields for Massless Particles

We shall now complete the formulas pertaining to the quantized field. Up to now we have introduced the annihilation part of the field $\phi^{(-)}(x,\lambda)$, Eq. (3.5), and have obtained that if the field transforms according to the representation $T_{(n_1n_2)}$ then $\lambda=\frac{1}{2}(n_1-n_2)$. Similarly, we introduce a positive frequency or creation part $\phi^{(+)}(x,\lambda')$, defined as

$$\phi^{(+)}(x,\lambda')=\int\frac{d^3p}{(2\pi)^3(2p^0)}e^{ip\cdot x}v(p,\lambda')b^{\dagger}(p,\lambda'); \qquad (3.48)$$

$b^{\dagger}(p,\lambda')$ is a creation operator which, transforms according to (3.3) as:

$$U[\Lambda]b^{\dagger}(p,\lambda')U^{-1}[\Lambda]=e^{i\lambda'\theta(p,\Lambda)}b^{\dagger}(\Lambda p,\lambda'). \qquad (3.49)$$

Therefore, by arguments similar to the case of the annihilation part, the requirement that $\phi^{(+)}(x,\lambda')$ transforms according to (3.6), namely, by the same rule as $\phi^{(-)}(x,\lambda)$,

$$U[\Lambda]\phi^{(+)}(x,\lambda')U^{-1}[\Lambda]=T[\Lambda^{-1}]\phi^{(+)}(\Lambda x,\lambda'), \qquad (3.50)$$

yields that

$$\lambda' = \tfrac{1}{2}(n_2 - n_1) = -\lambda\,, \tag{3.51}$$

and

$$v(p,\lambda' = -\lambda) = c(\lambda) u(p,\lambda)\,, \tag{3.52}$$

where $c(\lambda)$ is a proportionality constant. At this point, we have to be slightly more specific about the physical meaning of the states. If $\lambda \neq 0$ then clearly we have to deal with two types of states, those with helicity λ and those with helicity $-\lambda$. At the level of Lorentz transformation properties (i.e., without including discrete operations like parity P or parity times charge conjugation PC) these states are distinct so that it is justifiable to use different symbols like $a(p,\lambda)$, $a^\dagger(p,\lambda)$, $b(p,\lambda)$, $b^\dagger(p,-\lambda)$ to describe their annihilation and creation. However, when one does not violate any principle by considering coherent superpositions of the type $\mu|p,\lambda\rangle + \nu|p,-\lambda\rangle$ (which is the case of photons but not of neutrino-antineutrino pairs, due to the lepton number superselection rule), it is possible to identify the operators a and b.

The full irreducible field now reads

$$\begin{aligned}\phi(x) &= \phi^{(-)}(x,\lambda) + \phi^{(+)}(x,-\lambda)\,,\\ &= \int \frac{d^3p}{(2\pi)^3(2p^0)}\big[e^{-ip\cdot x} u(p,\lambda) a(p,\lambda)\\ &\qquad + e^{ip\cdot x} v(p,-\lambda) b^\dagger(p,-\lambda)\big]\,.\end{aligned} \tag{3.53}$$

Its transformation law under translations and Lorentz transformations is

$$\begin{aligned} &U(a)\phi(x)U^\dagger(a) = \phi(x+a)\,,\\ &U[\Lambda]\phi(x)U^\dagger[\Lambda] = T_{(n_1,n_2)}[\Lambda^{-1}]\phi(\Lambda x)\,,\\ &\qquad\qquad \tfrac{1}{2}(n_1 - n_2) = \lambda\,.\end{aligned} \tag{3.54}$$

Finally, we can describe the vector character of $\phi(x)$ by introducing the variable z above. Or we can consider its components in either the tensor or "$j\sigma$" basis. We translate here the results, previously obtained for the wave function, to the field. We limit ourselves to the case of integer λ and tensor basis. Then the lowest component (for $\lambda \neq 0$) is divergenceless:

$$\sum_{a_r=1}^{3} \frac{\partial}{\partial x^{a_r}} \phi_{a_1,a_2\cdots a_r\cdots a_{j0}}(x) = 0\,. \tag{3.55}$$

More generally, introducing formally the nonlocal operator $1/\Delta$ such that

$$\frac{1}{\Delta}\phi(\mathbf{x},x^0) = \frac{1}{4\pi}\int \frac{d^3x'}{|\mathbf{x}'-\mathbf{x}|}\phi(\mathbf{x}',x^0)\,, \tag{3.56}$$

we can write the recursion relations as

$$\begin{aligned}\phi^{a_1\cdots a_{j-1}}(x) = &-\frac{h(j+1)}{h(j)} \frac{1}{[(j+1)^2 - j_0^2]^{1/2}} \frac{1}{(j+1)} \frac{\partial}{\partial x^0} \frac{1}{\Delta}\\ &\times\Bigg\{ \sum_{k=1}^{j+1} \frac{\partial}{\partial x^{a_k}} \phi^{a_1\cdots a_k\cdots a_{j+1}}(x) - \frac{2}{2j+1} \sum_{k<l} \delta_{a_k,a_l}\\ &\qquad \times \sum_{a=1}^{3} \frac{\partial}{\partial x^a} \phi^{a,a_1\cdots \bar{a}_k \cdots \bar{a}_l \cdots a_{j+1}}(x)\Bigg\}\,,\end{aligned} \tag{3.57a}$$

$$\begin{aligned}&\left(\frac{j-j_0}{j+j_0}\right)^{1/2} \phi^{a_1\cdots a_{j-1}}(x)\\ &\qquad = -j\frac{h(j-1)}{h(j)} \frac{\partial}{\partial x^0} \frac{1}{\Delta} \sum_{a=1}^{3} \frac{\partial}{\partial x^a} \phi^{a,a_1\cdots,a_{j-1}}(x)\,.\end{aligned} \tag{3.57b}$$

IV. RADIATION GAUGE

Up to now we have extensively discussed the wave functions suitable for describing massless particles. We have seen that the requirement of Lorentz invariance restricts the behavior of the wave function in such a way that it can only transform according to those representations $T_{(n_1 n_2)}$ for which $\lambda = \frac{1}{2}(n_1 - n_2)$. As a result it seems impossible to describe photons, for example, by the usual vector potential. Indeed, the usual four-vector A_μ corresponds to the representation with $n_1 = 2$, $n_2 = 2$, which accommodates only helicity-zero massless particles and is therefore unsuitable for the description of helicity ± 1 photons. On the other hand, the quantization procedure cannot be applied to the covariant four-vector potential without introducing extra unphysical states. The radiation gauge does not suffer from the latter defect, and is therefore used[4] in quantizing electromagnetism within the Hilbert space of physical states. However, this gauge appears to spoil manifest covariance.

In this section we shall demonstrate that the formalism developed in the previous section entails that the free electromagnetic field in the radiation gauge transforms covariantly under a certain infinite-dimensional representation of the Lorentz group. The potential is then the lowest-spin component in that representation (this result was derived before by Bender[1] using somewhat less direct methods). In fact we shall show that many apparently noncovariant transformation laws for potentials of massless particles are indeed covariant, when those potentials are incorporated in infinite-dimensional representations, in which the former are the lowest-spin components of the latter. All this will be done for the noninteracting case. The interacting case, which will add new structure to

[4] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill Book Co., New York, 1964).

the transformation laws, will be treated in another paper.

To achieve our aims we note that, when applying an infinitesimal boost to a lowest-spin component j_0, it acquires also a component with $j=j_0+1$. However, since all spin components of massless fields are related to each other, we can express the j_0+1 component in terms of the j_0 component. This yields a transformation law for the lowest component which appears noncovariant, and for the case $j_0=1$ and $c=\pm 1$ coincides with the law of radiation-gauge potentials. Note that a similar procedure may be applied to any component, and not necessarily to the lowest one.

Let us start by introducing the field

$$\phi^{(-)}(z;x)=h\int \frac{d^3p}{(2\pi)^3(2p^0)}e^{-ip\cdot x}(\alpha_p z+\gamma_p)^{n_1-1} \times(\bar{\alpha}_p\bar{z}+\bar{\gamma}_p)^{n_2-1}a(p,\lambda). \quad (4.1)$$

We treat the annihilation part only. (The creation part may be treated in exactly the same way.) Under a Lorentz transformation Λ corresponding to a 2×2 matrix

$$A=\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

the field $\phi^{(-)}(z;x)$ behaves as

$$\begin{aligned} U[\Lambda]\phi^{(-)}(z;x)U^{-1}[\Lambda] &= T[\Lambda^{-1}]\phi^{(-)}(z;\Lambda x) \\ &= h\int \frac{d^3p}{(2\pi)^3(2p^0)}e^{-ip\cdot\Lambda x} \times[(\alpha_p d-\gamma_p b)z+(-\alpha_p c+\gamma_p a)]^{n_1-1} \\ &\quad \times[(\bar{\alpha}_p\bar{d}-\bar{\gamma}_p\bar{b})\bar{z}+(-\bar{\alpha}_p\bar{c}+\bar{\gamma}_p\bar{a})]^{n_2-1}a(p,\lambda). \quad (4.2) \end{aligned}$$

The transformed field is given by the same formula as (4.1) with $x\rightarrow\Lambda x$ and

$$\begin{pmatrix}\alpha_p \\ \gamma_p\end{pmatrix}\rightarrow A^{-1}\begin{pmatrix}\alpha_p \\ \gamma_p\end{pmatrix}.$$

Defining the generating field,

$$\begin{aligned} \phi_j^{(-)}(\mathbf{y};x) &= \sum_{\sigma=-j}^{j}\frac{y_1^{j+\sigma}y_2^{j-\sigma}}{[(j+\sigma)!(j-\sigma)!]^{1/2}}\phi_j^{(-)}{}_\sigma(x) \\ &= \frac{h(j)}{[(j+\lambda)!(j-\lambda)!]^{1/2}}\int\frac{d^3p}{(2\pi)^3(2p^0)} \times e^{-ip\cdot x}(p^0)^{\epsilon c-j-1}(\alpha_p y_1+\gamma_p y_2)^{j+\lambda} \\ &\quad \times(-\bar{\gamma}_p y_1+\bar{\alpha}_p y_2)^{j-\lambda}a(p,\lambda), \quad (4.3) \end{aligned}$$

we get, combining formulas of the previous section with Eq. (4.2),

$$\begin{aligned} U[\Lambda]\phi_j^{(-)}(\mathbf{y};x)U^{-1}[\Lambda] &= h(j)[(j+\lambda)!(j-\lambda)!]^{-1/2}\int\frac{d^3p}{(2\pi)^3(2p^0)}e^{-ip\cdot\Lambda x} \\ &\quad \times[(\Lambda^{-1}p)^0]^{\epsilon c-j-1}\left[(y_1y_2)A^{-1}\begin{pmatrix}\alpha_p \\ \gamma_p\end{pmatrix}\right]^{j+\lambda} \\ &\quad \times\left[(\bar{\alpha}_p\bar{\gamma}_p)(A^{-1})^\dagger\begin{pmatrix}y_2 \\ -y_1\end{pmatrix}\right]^{j-\lambda}a(p,\lambda). \quad (4.4) \end{aligned}$$

Clearly if Λ is a rotation, we obtain the ordinary behavior corresponding to the representation j of the rotation group. On the other hand, for pure Lorentz transformations we find, for $j=j_0$,

$$\begin{aligned} i[K_a,\phi_{j_0}(\mathbf{y};x)] &= (x^0\partial^a-x^a\partial^a)\phi_{j_0}(\mathbf{y};x)+\tfrac{1}{2}\epsilon(\mathbf{y}^T\sigma_a\nabla_y) \\ &\quad \times\phi_{j_0}(\mathbf{y};x)-(j_0+1-\epsilon c)(\partial^0\partial^a/\Delta)\phi_{j_0}(\mathbf{y};x), \quad (4.5) \end{aligned}$$

where we remind the reader that $T(e^{-\frac{1}{2}\alpha\cdot\sigma})$ is identified with $e^{i\alpha\cdot K}$ (see Appendix B). We also note that when $\phi(x)$ transforms by $T_{n_1n_2}$, $\partial_0\phi(x)$ transforms by T_{n_1+1,n_2+1}, as is clear from (4.4).

For the case of integer j_0, combining (4.5) with (3.41) and (3.44) gives

$$\begin{aligned} i[K_a,\phi^{b_1b_2\cdots b_{j0}}(x)] &= (x^0\partial^a-x^a\partial^0)\phi^{b_1\cdots b_{j0}}(x) \\ &\quad -(j_0+1-\epsilon c)(\partial^0\partial^a/\Delta)\phi^{b_1\cdots b_{j0}}(x) \\ &\quad -i\epsilon\sum_{k=1}^{j0}\epsilon_{ab_kb'}\phi^{b_1\cdots b_{k-1}b'b_{k+1}\cdots b_{j0}}(x), \quad (4.6) \end{aligned}$$

where in the course of the calculation, it is useful to realize that

$$\begin{aligned} (\mathbf{y}^T\boldsymbol{\sigma}\nabla_\mathbf{y})(\boldsymbol{\varepsilon}(\hat{p})\cdot\mathbf{Y}) &= \tfrac{1}{2}(\mathbf{y}^T\boldsymbol{\sigma}\nabla_\mathbf{y})\,\mathrm{Tr}[(\boldsymbol{\varepsilon}(\hat{p})\cdot\boldsymbol{\sigma})(\mathbf{y}(-i\sigma_2)\mathbf{y}^T)] \\ &= 2i[\boldsymbol{\varepsilon}(\hat{p})\times\mathbf{Y}]. \quad (4.7) \end{aligned}$$

It can be verified, using Eq. (2.17a), that the right-hand side of Eq. (4.6) is transverse in $b_1\cdots b_{j_0}$, as it should be. Using the same equation, one also gets

$$\begin{aligned} (i\epsilon)\epsilon^{abc}\phi^{b_1\cdots b_{k-1}cb_{k+1}\cdots b_j0}(x) &= (\partial^0/\Delta) \\ &\quad \times[\partial^b\phi^{b_1\cdots b_{k-1}ab_{k+1}\cdots b_j0}(x)-\partial^a\phi^{b_1\cdots b_{k-1}bb_{k+1}\cdots b_j0}(x)]. \quad (4.8) \end{aligned}$$

Therefore, (4.6) turns into

$$\begin{aligned} i[K_a,\phi^{b_1\cdots b_{j0}}(x)] &= (x^0\partial^a-x^a\partial^0)\phi^{b_1\cdots b_{j0}}(x) \\ &\quad -(1-\epsilon c)(\partial^0\partial^a/\Delta)\phi^{b_1\cdots b_{j0}}(x) \\ &\quad -\sum_{k=1}^{j0}\frac{\partial^0\partial^{b_k}}{\Delta}\phi^{b_1\cdots b_{k-1}ab_{k-1}\cdots b_{j0}}(x). \quad (4.9) \end{aligned}$$

When $j_0=0$ the last term is absent. Since $\epsilon c=\frac{1}{2}(n_1+n_2)$, it follows that (4.9) is invariant under $n_1\leftrightarrow n_2$, which in turn implies that it is valid for both helicity λ and helicity $-\lambda$ and any combination of both. It therefore applies to the photon field as well. The choice

$$\tfrac{1}{2}(n_1+n_2)=1 \quad \text{or} \quad c=\pm 1\,, \tag{4.10}$$

with $c=+1$ for positive helicity and $c=-1$ for negative helicity, makes the transformation law (4.9) extremely simple. The lowest-spin component then transforms under pure Lorentz transformations in a sort of "minimal" way: Besides the orbital part one adds the simplest term needed to restore transversality. Finally, with $j_0=1$ and condition (4.10) fulfilled, one realizes that Eq. (4.9) represents the free-radiation-gauge potential transformation law. Thus helicity ± 1 radiation-gauge potentials transform covariantly under the infinite-dimensional representations $j_0=1$, $c=\pm 1$, respectively. Note that these representations are nonunitary.

V. COMMUTATION RELATIONS

In this section we investigate the commutation rules among the various field components. Let the field be expressed as

$$\phi_j(\mathbf{y};x)=\frac{h(j)}{[(j+\lambda)!(j-\lambda)!]^{1/2}}\int\frac{d^3p}{(2\pi)^3(2p^0)}(p^0)^{\epsilon c-j-1}\times(\alpha_p y_1+\gamma_p y_2)^{j+\lambda}(-\bar{\gamma}_p y_1+\bar{\alpha}_p y_2)^{j-\lambda}\times[e^{-ip\cdot x}a(p,\lambda)+c(\lambda)e^{ip\cdot x}b^\dagger(p,-\lambda)]\,. \tag{5.1}$$

The commutation rules among the creation and annihilation operators are

$$[a(p,\lambda),a^\dagger(p',\lambda')]_\delta=[b(p,\lambda),b^\dagger(p',\lambda')]_\delta=(2\pi)^3(2p^0)\delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{p}-\mathbf{p}')\,, \tag{5.2}$$

with $[A,B]_\delta=AB+\delta BA$ and $\delta=\pm 1$. All other commutation relations vanish. [The treatment of self-conjugate particles, namely $a(p,\lambda)=b(p,\lambda)$ does not yield any new results. In fact, there need not be a separate treatment for self-conjugate particles whenever $\lambda\neq 0$, since then $a(p,\lambda)$ and $a^\dagger(p,-\lambda)$ commute (or anticommute).] One therefore readily obtains

$$[\phi_j(\mathbf{y};x),\phi_j'(\mathbf{y}';x')]_\delta=[\phi_j^\dagger(y;x),\phi_{j'}^\dagger(y';x')]_\delta=0\,. \tag{5.3}$$

However, the commutator $[\phi_j(y;x),\ \phi_j^\dagger,\ (y';x')]_\delta$ does not vanish. It is

$$[\phi_j(y;x),\ \phi_{j'}^\dagger(y';x')]_\delta=h(j)h^*(j')[(j+\lambda)!(j-\lambda)!(j'+\lambda)!(j'-\lambda)!]^{-1/2}\times\delta_{\lambda\lambda'}\int\frac{d^3p}{(2\pi)^3(2p^0)}(p^0)^{[\epsilon(c+\bar{c}')-(j+j'+2)]}\times(\alpha_p y_1+\gamma_p y_2)^{j+\lambda}(-\bar{\gamma}_p y_1+\bar{\alpha}_p y_2)^{j-\lambda}\times(\bar{\alpha}_p y_1'+\bar{\gamma}_p y_2')^{j'+\lambda}(-\gamma_p y_1'+\alpha_p y_2')^{j'-\lambda}\times[e^{-ip\cdot(x-x')}+\delta|c(\lambda)|^2e^{ip(x-x')}]\,.$$

Let us now notice that

$$(\alpha_p y_1+\gamma_p y_2)^{j+\lambda}(-\bar{\gamma}_p y_1+\bar{\alpha}_p y_2)^{j-\lambda}(\bar{\alpha}_p y_1'+\bar{\gamma}_p y_2')^{j'+\lambda}\times(-\gamma_p y_1'+\alpha_p y_2')^{j'-\lambda}=(\mathbf{p}\cdot\mathbf{Y})^{j-|\lambda|}(\mathbf{p}\cdot\mathbf{Y}')^{j'-|\lambda|}\times(Z^0p^0+\epsilon\mathbf{Z}\cdot\mathbf{p})^{2|\lambda|}\,, \tag{5.4}$$

where we have defined

$$Z^0+\mathbf{Z}\cdot\boldsymbol{\sigma}=\begin{pmatrix}y_1'\\ y_2'\end{pmatrix}\otimes(y_1y_2)\,. \tag{5.5}$$

Therefore,

$$[\phi_j(\mathbf{y};x),\phi_{j'}^\dagger(y';x')]_\delta=\delta_{\lambda\lambda'}h(j)h^*(j')[(j+\lambda)!(j-\lambda)!(j'+\lambda)!(j'-\lambda)!]^{-1/2}\times\int\frac{d^3p}{(2\pi)^3(2p^0)}(p^0)^{[\epsilon(c+\bar{c}')-(j+j'+2)]}\times(\mathbf{p}\cdot\mathbf{Y})^{j-|\lambda|}(\mathbf{p}\cdot\mathbf{Y}')^{j'-|\lambda|}(Z^0p^0+\epsilon\mathbf{Z}\cdot\mathbf{p})^{2|\lambda|}\times[e^{-ip\cdot(x-x')}+\delta|c(\lambda)|^2e^{ip\cdot(x-x')}]\,. \tag{5.6}$$

These commutation relations are in general nonlocal, namely, the right-hand side does not vanish for spacelike separations $(x-x')^2<0$. It is easy to realize that local fields are obtained only for finite-dimensional representations, and then only with

$$|c(\lambda)|=1 \tag{5.7}$$

and

$$\begin{aligned}\delta&=-1 \quad \text{for} \quad j_{\max}+j_{\max}' \text{ even}\,,\\ \delta&=+1 \quad \text{for} \quad j_{\max}+j_{\max}' \text{ odd}.\end{aligned} \tag{5.8}$$

For fields within the same irreducible representation, the conditions (5.8) express nothing but the usual connection between spin and statistics.

For the free-radiation-gauge electromagnetic potentials $A_j(y;x)$ one has $\epsilon c=\epsilon c'=1$, $|\lambda|=1$, and hence,

$$[A_j(\mathbf{y};x),A_{j'}^\dagger(\mathbf{y};x')]_-=h(j)\bar{h}(j')[(j+1)!(j-1)!(j'+1)!(j'-1)!]^{-1/2}\times\frac{2}{i}\int\frac{d^3p}{(2\pi)^3(2p^0)}(p^0)^{-(j+j')}(\mathbf{p}\cdot\mathbf{Y})^{j-1}(\mathbf{p}\cdot\mathbf{Y}')^{j'-1}\times(Z^0p^0+\epsilon\mathbf{Y}\cdot\mathbf{p})^2\sin p\cdot(x-x')\,. \tag{5.9}$$

For the lowest components,

$$[A_1(\mathbf{y};x),A_1^\dagger(\mathbf{y}';x')]_-=\frac{4}{i}|h(1)|^2\int\frac{d^3p}{(2\pi)^3(2p^0)}\times\left(Z^0+\epsilon\frac{\mathbf{Z}\cdot\mathbf{p}}{p^0}\right)^2\sin p\cdot(x-x')\,. \tag{5.10}$$

The equal-time commutators between two fields or a field and its time derivative are then

$$[A_1(\mathbf{y};\mathbf{x}),A_1^\dagger(\mathbf{y}';\mathbf{x}')]_- = 4|h(1)|^2\epsilon(Z^0\mathbf{Z}\cdot\nabla/\Delta)\delta^{(3)}(\mathbf{x}-\mathbf{x}'), \quad (5.11a)$$

$$[A_1(\mathbf{y};\mathbf{x}),\dot{A}_1^\dagger(\mathbf{y}';\mathbf{x}')]_- = (4/i)|h(1)|^2[(Z^0)^2+(\mathbf{Z}\cdot\nabla)^2/\Delta]\delta^{(3)}(\mathbf{x}-\mathbf{y}), \quad (5.11b)$$

as expected. The first commutator vanishes when fields which include both helicities are used.

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APPENDIX A: LORENTZ GROUP—GENERATOR APPROACH

In this appendix we recall, for completeness, the action of the Lorentz generators in the $j\sigma$ basis, namely, the basis which diagonalizes rotations,[5] and then solve for the coefficients $h(j)$ of the wave function $u(k,\lambda)$ [see Eq. (3.11)]. A more formal treatment of the representation theory is given in the next appendix.

The generators of rotations $\mathbf{J}$ and Lorentz transformations (boosts) $\mathbf{K}$ obey the commutation rules

$$\begin{aligned}[][J_m,J_n]&=i\epsilon_{mnr}J_r,\\ [J_m,K_n]&=i\epsilon_{mnr}K_r,\\ [K_m,K_n]&=-i\epsilon_{mnr}J_r.\end{aligned} \quad (A1)$$

Let $f_{j\sigma}$ be basis "states" which diagonalize rotations, namely,

$$\mathbf{J}^2 f_{j\sigma}=j(j+1)f_{j\sigma} \quad (A2)$$

and

$$\begin{aligned}J_\pm f_{j\sigma}&=[j(j+1)-\sigma(\sigma\pm 1)]^{1/2}f_{j\,\sigma+1},\\ J_3 f_{j\sigma}&=\sigma f_{j\sigma}.\end{aligned} \quad (A3)$$

From the vector character of $\mathbf{K}$ under rotations and $[J_3,K_+]=K_+$ it follows that

$$K_+f_{j\sigma}=a_{j\sigma}f_{j-1\,\sigma+1}+b_{j\sigma}f_{j\,\sigma+1}+c_{j\sigma}f_{j+1\,\sigma+1}.$$

The dependence of the coefficients $a_{j\sigma}$, $b_{j\sigma}$, and $c_{j\sigma}$ on σ can be determined by a straightforward calculation using $[J_+,K_+]=0$. One then gets

$$\begin{aligned}K_+f_{j\sigma}=&a_j[(j-\sigma)(j-\sigma-1)]^{1/2}f_{j-1\,\sigma+1}\\ &+b_j[(j-\sigma)(j+\sigma+1)]^{1/2}f_{j\,\sigma+1}\\ &-c_j[(j+\sigma+1)(j+\sigma+2)]^{1/2}f_{j+1\,\sigma+1}.\end{aligned}$$

The action of K_3 is determined by that of K_+ and $[K_+,J_-]=2K_3$, and that of K_- from that of K_3 and $[J_-,K_3]=K_-$. We thus get

$$\begin{aligned}K_\pm f_{j\sigma}=&\pm a_j[(j\mp\sigma)(j\mp\sigma+1)]^{1/2}f_{j-1\,\sigma\pm1}\\ &+b_j[(j\mp\sigma)(j\pm\sigma+1)]^{1/2}f_{j\,\sigma\pm1}\\ &\mp c_j[(j\pm\sigma+1)(j\pm\sigma+2)]^{1/2}f_{j+1\,\sigma\pm1},\\ K_3f_{j\sigma}=&a_j[j^2-\sigma^2]^{1/2}f_{j-1\,\sigma}+b_j\sigma f_{j\sigma}\\ &+c_j[(j+1)^2-\sigma^2]^{1/2}f_{j+1\,\sigma}.\end{aligned} \quad (A4)$$

In a certain irreducible representation, $\mathbf{J}\cdot\mathbf{K}=p$ and $\mathbf{J}^2-\mathbf{K}^2=s$ are constants.

Thus,

$$(\mathbf{J}\cdot\mathbf{K})f_{jj}=pf_{jj}$$

implies

$$b_j=p/j(j+1),$$

while

$$(\mathbf{J}^2-\mathbf{K}^2)f_{jj}=sf_{jj}$$

implies

$$s=j(j+2)-p^2/(j+1)^2-N_j(2j+1)(2j+3),$$

where

$$N_j=a_{j+1}c_j.$$

Suppose j_0 is the lowest j in the representation. Then $N_{j_0-1}=0$, and hence

$$s=j_0^2-1-p^2/j_0^2.$$

Defining c through $p=ij_0c$, we thus get

$$\begin{aligned}\mathbf{J}\cdot\mathbf{K}&=ij_0c,\\ (\mathbf{J}^2-\mathbf{K}^2)&=j_0^2+c^2-1,\end{aligned} \quad (A5)$$

and

$$N_j=\frac{1}{(j+1)^2}\frac{[(j+1)^2-j_0^2][(j+1)^2-c^2]}{4(j+1)^2-1}.$$

Choosing $f_{j\sigma}$ in such a way that $a_{j+1}=c_j$, we thus finally obtain

$$\begin{aligned}a_j&=c_{j-1}=\frac{1}{j}\left[\frac{(j^2-j_0^2)(j^2-c^2)}{4j^2-1}\right]^{1/2},\\ b_j&=ij_0c/j(j+1).\end{aligned} \quad (A6)$$

Let us remark that for an orthonormal set $f_{j\sigma}$, the Hermiticity of $\mathbf{K}$, namely, a unitary representation, implies that

(a) $c=ir$, r real (principal series);
(b) $j_0=0$, c real, $0<c^2<1$ (supplementary series). (A7)

The finite-dimensional representations are obtained for $|c|=j_0+n+1$, $n=0, 1, 2,\cdots$. For those, denoting:

$$[\tfrac{1}{2}(\mathbf{J}-i\mathbf{K})]^2=j_1(j_1+1),\ [\tfrac{1}{2}(\mathbf{J}+i\mathbf{K})]^2=j_2(j_2+1), \quad (A8)$$

one obtains

$$\begin{aligned} j_0 &= |j_1 - j_2|, \\ c &= (j_1+j_2+1)\,\mathrm{sgn}(j_1-j_2), \quad \text{for} \quad j_1 \neq j_2; \\ c &= \pm(2j+1), \qquad \text{for} \quad j_1 = j_2. \end{aligned} \tag{A9}$$

For a more complete discussion, the reader is referred to the literature.[5] (See also Appendix B.)

We now proceed to solve the constraint Eqs. (3.10), which read

$$\begin{aligned} J_3 u(k,\lambda) &= \lambda u(k,\lambda), \\ (K_- - iJ_-)u(k,\lambda) &= 0, \\ (K_+ + iJ_+)u(k,\lambda) &= 0. \end{aligned} \tag{A10}$$

It is straightforward to show that Eqs. (A10) imply

$$\begin{aligned} (\mathbf{J}\cdot\mathbf{K})u(\lambda) &= \lambda(K_3+i)u(\lambda), \\ (\mathbf{J}^2-\mathbf{K}^2)u(\lambda) &= [\lambda^2 - 1 - (K_3+i)^2]u(\lambda). \end{aligned} \tag{A11}$$

Thus, in an irreducible representation (j_0,c),

$$\begin{aligned} \lambda(K_3+i)u(\lambda) &= (ij_0c)u(\lambda), \\ (K_3+i)^2 u(\lambda) &= (\lambda^2 - j_0^2 - c^2)u(\lambda). \end{aligned} \tag{A12}$$

It thus follows that one necessarily has

$$(\lambda^2 - j_0^2)(\lambda^2 - c^2) = 0. \tag{A13}$$

Hence either $\lambda = \epsilon j_0$ or $\lambda = \epsilon c$ (obviously, the latter is valid only for $j_0 - c$ integer). It turns out that $\lambda = \epsilon c$ does not give any solution not included already in the $\lambda = \epsilon j_0$ case. Thus one has

$$\begin{aligned} \lambda &= \epsilon j_0, \\ K_3 u(\lambda) &= i(\epsilon c - 1)u(\lambda), \quad \epsilon = \pm 1 \end{aligned} \tag{A14}$$

The derivation of (A14) from (A12) is not direct for $\lambda = 0$. However, Eqs. (A14) hold in general.

Using Eqs. (A3) and (A4) and the expansion (3.11), one can solve for the coefficients $h(j)$ from Eqs. (A10). The solution is

$$h(j+1) = -ih(j)\left(\frac{2j+3}{2j+1}\right)^{1/2}\left(\frac{j+1-\epsilon c}{j+1+\epsilon c}\right)^{1/2}. \tag{A15}$$

APPENDIX B: SUMMARY OF REPRESENTATION THEORY FOR SL(2C)

In this appendix we give a brief survey of the representation theory for the group SL(2C), the covering group of the homogeneous Lorentz group. This is mainly to define the notation and to derive some identities used in the text. We rely mainly on the classical reference texts from Naimark, Gelfand, and coworkers.[5] The results are first stated in global form; then, using a particular basis to diagonalize the $SU(2)$ subgroup, they are also given in the Lie-algebra or infinitesimal form.

To describe the representations one introduces function spaces $D_{(n_1,n_2)}$ with n_1, n_2 two-complex numbers, such that $n_1 - n_2$ is an integer. A function of two real variables x and y [which are conveniently grouped as $z = x+iy$, $\bar{z} = x - iy$ so that a function of the variables x, y is also written $f(z,\bar{z})$] belongs to $D_{(n_1,n_2)}$ if

(i) $f(z,\bar{z})$ is infinitely differentiable (abbreviated C^∞),
(ii) $\hat{f}(z,\bar{z}) = z^{n_1-1}\bar{z}^{n_2-1}f(-1/z,-1/\bar{z})$ is also C^∞.

One abbreviates $f(z,\bar{z})$ by $f(z)$. The topology on $D_{(n_1,n_2)}$, namely, uniform convergence of f and $\hat{f}$ and their derivatives on compact subsets of the z plane, will not be discussed here except to state some results.

In the space $D_{(n_1,n_2)}$, one defines a continuous representation of SL(2C) through the following relation. For any 2×2 matrix A with determinant 1, one sets up the mapping $A \to T(A)$, where the linear operator $T(A)$ acts in $D_{(n_1,n_2)}$ as follows:

$$f(z) \to [T(A)f](z) = (bz+d)^{n_1-1} \times (\bar{b}\bar{z}+\bar{d})^{n_2-1} f\left(\frac{az+c}{bz+d}\right), \tag{B1}$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

These representations exhaust in a certain sense all irreducible representations of SL(2C). We summarize irreducibility and equivalence by distinguishing the situation of "integer" and "noninteger" (n_1,n_2) points as follows: An index (n_1,n_2) is called integer if (n_1,n_2) are both nonzero integers of the same sign.

Noninteger points. The representation at a noninteger point $(n_1 n_2)$ is irreducible. Two representations (n_1,n_2) and (n_1',n_2') are equivalent if and only if $n_1'+n_1 = n_2'+n_2 = 0$. Equivalence means the existence of a continuous, invertible intertwining operator between between $D_{(n_1,n_2)}$ and $D_{(n_1',n_2')}$. Irreducibility is understood as

(i) subspace irreducible: no closed proper invariant subspace,
(ii) operator irreducible: all-continuous operators commuting with the $T(A)$ are multiples of the identity.

All these representations are infinite-dimensional. They contain in particular the important special case of unitary representation.

Unitary representations. They fall into two series:

(i) principal series characterized by $n_1 + \bar{n}_2 = 0$ or

$$\begin{aligned} n_1 &= \tfrac{1}{2}(n+i\rho), \qquad n \text{ integer}, \rho \text{ real} \\ n_2 &= \tfrac{1}{2}(-n+i\rho), \end{aligned} \tag{B2}$$

[5] For the theory of representations of the Lorentz group see M. A. Naimark, *Linear Representations of the Lorentz Group* (Pergamon Press Inc., New York, 1964); I. M. Gelfand, M. I. Graev, and N. Ya-Vilenkin, *Generalized Functions* (Academic Press Inc., New York, 1966), Vol. 5.

with scalar product

$$(f,g)=\frac{2}{\pi}\int \tfrac{1}{2}idzd\bar{z}\,\bar{f}(z)g(z). \tag{B3}$$

The measure is $\frac{1}{2}idzd\bar{z}=d\,\mathrm{Re}z\,d\mathrm{Im}z$;

(ii) complementary series characterized by

$$n_1=n_2=c,\quad -1<c<1,\quad c\neq 0, \tag{B4}$$

and scalar product (valid for $-1<c<0$; the representations with $0<c<1$ are equivalent to those with $-1<c<0$, since $n_1=n_2=c$ and $n_1=n_2=-c$ are equivalent).

$$(f,g)=\left(\frac{2}{\pi}\right)^2\int\int (\tfrac{1}{2}i)^2dz_1d\bar{z}_1dz_2d\bar{z}_2 \times|z_1-z_2|^{-2c-2}\bar{f}(z_1)g(z_2). \tag{B5}$$

Integer points. These representations are no-more subspace irreducibles though they are still operator irreducible. In fact, with n_1, n_2 both positive integers, the four representations T_{n_1,n_2}, T_{-n_1,n_2}, $T_{n_1,-n_2}$, and $T_{-n_1,-n_2}$, are related by various continuous mappings, which commute with the operations of SL(2*C*). Consequently, the kernels and images of these mappings are invariant subspaces. Denote by E_{n_1,n_2} the closed subspace of D_{n_1,n_2} of polynomials in z, $\bar{z}$ of degree at most n_1-1 in z, and n_2-1 in $\bar{z}$. This is an invariant subspace of D_{n_1,n_2}. Similarly, let $F_{-n_1,-n_2}$ be the subspace of $D_{-n_1,-n_2}$ of those functions f which satisfy

$$\int(\tfrac{1}{2}i)dzd\bar{z}z^j\bar{z}^k f(z)=0,\quad \text{for } 0\leq j\leq n_1-1,\quad 0\leq k\leq n_2-1.$$

This is a closed infinite-dimensional subspace of $D_{-n_1,-n_2}$. Note that $E_{n_1n_2}$ is finite-dimensional (dimension $n_1\times n_2$) and carries the usual finite-dimensional representations of SL(2*C*). The index (n_1,n_2) can be written $(2j_1+1,\ 2j_2+1)$ to make contact with the usual notation, where $\mathbf{J}-i\mathbf{K}$ is represented by spin j_1 and $\mathbf{J}+i\mathbf{K}$ by spin j_2 (see below).

Let the symbol $\sim$ denote isomorphism between spaces and equivalence between representations in the corresponding spaces. Then one has

$$\begin{aligned}&D_{n_1,n_2}/E_{n_1,n_2}\sim F_{-n_1,-n_2}\sim D_{-n_1,n_2}\sim D_{n_1,-n_2},\\ &D_{-n_1,-n_2}/F_{-n_1,-n_2}\sim E_{n_1,n_2}.\end{aligned} \tag{B6}$$

Infinitesimal Form: By identifying $T(e^{(i/2)(\epsilon+i\eta)\cdot\sigma})$ with $e^{i(\epsilon\cdot J+\eta\cdot K)}$ for $\boldsymbol{\epsilon}$ and η infinitesimal, we derive the expressions of the generators $\mathbf{J}$ and $\mathbf{K}$ in the space $D_{(n_1,n_2)}$. They read

$$\begin{aligned}
J_1&=\frac{1}{2}\left[(n_1-1)z-(n_2-1)\bar{z}+(1-z^2)\frac{\partial}{\partial z}-(1-\bar{z}^2)\frac{\partial}{\partial\bar{z}}\right],\\
J_2&=\frac{1}{2i}\left[(n_1-1)z+(n_2-1)\bar{z}-(1+z^2)\frac{\partial}{\partial z}-(1+\bar{z}^2)\frac{\partial}{\partial\bar{z}}\right],\\
J_3&=\left[\frac{n_2-n_1}{2}+z\frac{\partial}{\partial z}-\bar{z}\frac{\partial}{\partial z}\right],\\
K_1&=-\frac{1}{2i}\left[(n_1-1)z+(n_2-1)\bar{z}+(1-z^2)\frac{\partial}{\partial z}+(1-\bar{z}^2)\frac{\partial}{\partial\bar{z}}\right],\\
K_2&=-\frac{1}{2}\left[-(n_1-1)z+(n_2-1)\bar{z}+(1+z^2)\frac{\partial}{\partial z}-(1+\bar{z}^2)\frac{\partial}{\partial\bar{z}}\right],\\
K_3&=\frac{1}{i}\left[\frac{n_1+n_2}{2}-1-z\frac{\partial}{\partial z}-\bar{z}\frac{\partial}{\partial\bar{z}}\right].
\end{aligned} \tag{B7}$$

The usual invariants take the following values with $n_1=2j_1+1$, $n_2=2j_2+1$:

$$\begin{aligned}
\left(\frac{\mathbf{J}-i\mathbf{K}}{2}\right)^2&=\frac{n_1^2-1}{4}=j_1(j_1+1),\\
\left(\frac{\mathbf{J}+i\mathbf{K}}{2}\right)^2&=\frac{n_2^2-1}{4}=j_2(j_2+1),\\
\mathbf{J}^2-\mathbf{K}^2&=\frac{n_1^2+n_2^2}{2}-1\equiv\left(\frac{n_1-n_2}{2}\right)^2+\left(\frac{n_1+n_2}{2}\right)^2-1,\\
\mathbf{J}\cdot\mathbf{K}&=i\frac{n_1^2-n_2^2}{4}=i\left(\frac{n_1-n_2}{2}\right)\left(\frac{n_1+n_2}{2}\right).
\end{aligned} \tag{B8}$$

Rotation basis. Except for the cases corresponding to the unitary representations, the vector spaces $D_{(n_1,n_2)}$ are not naturally equipped with a bilinear form. However, they carry a reducible representation of the compact group $SU(2)$, which we expect to be equivalent to a unitary one-direct sum of the well-known representations of "spin" j. We shall, indeed, construct in $D_{(n_1,n_2)}$ a bilinear form invariant under $SU(2)$, which allows us to embed $D_{(n_1,n_2)}$ as a dense subspace of a Hilbert space, a basis of which diagonalizes the representation of this group.

We proceed by constructing sets of $2j+1$ functions $f_{j\sigma}(z)$ belonging to $D_{(n_1,n_2)}$, such that the subspace spanned by these functions is left invariant under the action of $T_{(n_1,n_2)}(A)$, for A restricted to $SU(2)$.

We are thus looking for $f_{j\sigma}(z)$, with the property that for A restricted to $SU(2)$,

$$[T_{(n_1,n_2)}(A)f_{j\sigma}](z)=f_{j\sigma'}(z)D_{\sigma'\sigma}{}^{(j)}(A)\,; \tag{B9}$$

where the Wigner functions $D_{\sigma'\sigma}{}^{(j)}(A)$ are defined by

$$\frac{x_1'^{j+\sigma}x_2'^{j-\sigma}}{[(j+\sigma)!(j-\sigma)!]^{1/2}}=\sum_{\sigma'=-j}^{j}\frac{x_1^{j+\sigma'}x_2^{j+\sigma'}}{[(j+\sigma')!(j-\sigma')!]^{1/2}}D_{\sigma'\sigma}{}^{(j)}(A)\,, \tag{B10}$$

$$\mathbf{x}'=A^T\mathbf{x}.$$

Combining (B9) and (B1) we get

$$(bz+d)^{n_1-1}(\bar{b}\bar{z}+\bar{d})^{n_2-1}f_{j\sigma}\left(\frac{az+c}{bz+d}\right)=f_{j\sigma'}(z)D_{\sigma'\sigma}{}^{(j)}(A)\,,$$
$$A=\begin{pmatrix} a & b\\ c & d\end{pmatrix}=\begin{pmatrix} \bar{d} & -\bar{c}\\ -\bar{d} & \bar{a}\end{pmatrix}. \tag{B11}$$

To solve for $f_{j\sigma}(z)$ from (B11), one simply chooses a matrix A such that $az+c=0$; for example:

$$A_z=(1+z\bar{z})^{-1/2}\begin{pmatrix} 1 & \bar{z}\\ -z & 1\end{pmatrix} \tag{B12}$$

thus obtaining

$$f_{j\sigma'}(z)D_{\sigma'\sigma}{}^{(j)}(A_z)=(1+z\bar{z})^{(n_1+n_2)/2-1}f_{j\sigma}(0)\,. \tag{B13}$$

It remains to compute $f_{j\sigma}(0)$. To this end we note that choosing $A=e^{(i/2)\theta\sigma_3}$ in (B11), and using $D_{\sigma'\sigma}{}^{(j)}(e^{(i/2)\theta\sigma_3})=\delta_{\sigma'\sigma}e^{i\theta\sigma}$, one gets

$$f_{j\sigma}(e^{i\theta}z)=f_{j\sigma}(z)e^{i\theta[\sigma+(n_1-n_2)/2]}\,,$$

and hence

$$f_{j\sigma}(0)=f_j\delta_{\sigma,(n_2-n_1)/2}. \tag{B14}$$

It thus follows that

$$f_{j\sigma}(z)=f_j(1+z\bar{z})^{(n_1+n_2)/2-1}D_{(n_2-n_1)/2}{}^{(j)}(A_z{}^{\dagger})\,. \tag{B15}$$

Defining the unitary matrix V_z by

$$\bar{A}_z=V_z(i\sigma_2)\,,$$

we obtain

$$f_{j\sigma}(z)=g_j(1+z\bar{z})^{(n_1+n_2)/2-1}D_{\sigma,(n_1-n_2)/2}{}^{(j)}(V_z)\,,$$
$$V_z=(1+z\bar{z})^{-1/2}\begin{pmatrix} z & -1\\ 1 & \bar{z}\end{pmatrix}, \tag{B16}$$

where $g_j=f_j(-)^{j+(n_1-n_2)/2}$ and where

$$D_{\sigma'\sigma}(i\sigma_2)=(-)^{j+\sigma'}\delta_{\sigma',-\sigma}$$

has been used.

As will become clear later, it is convenient to choose

$$g_j=e^{-(i/2)\pi j}\left[(j+\tfrac{1}{2})\frac{\Gamma(j+1-\frac{1}{2}(n_1+n_2))}{\Gamma(j+1+\frac{1}{2}(n_1+n_2))}\right]^{1/2}. \tag{B17}$$

Let us now introduce the following generating function:

$$\sum_{\sigma=-j}^{+j}\frac{x_1^{j+\sigma}x_2^{j-\sigma}}{[(j+\sigma)!(j-\sigma)!]^{1/2}}f_{j\sigma}(z)=f_j(\mathbf{x},z)=g_j\frac{(x_1z+x_2)^{j+(n_1-n_2)/2}(-x_1+x_2\bar{z})^{j-(n_1-n_2)/2}}{[(j+\frac{1}{2}(n_1-n_2))!(j-\frac{1}{2}(n_1-n_2))!]^{1/2}(1+z\bar{z})^{j+1-(n_1+n_2)/2}}. \tag{B18}$$

This definition requires of course (in order to have a homogeneous polynomial of degree $2j$ in $\mathbf{x}$) that

$$j-j_0\equiv j-|\tfrac{1}{2}(n_1-n_2)| \tag{B19}$$

be a non-negative integer. The lowest spin contained in the representation will thus be j_0. We now verify that $f_{j\sigma}$ have indeed the required properties.

(i) $f_{j\sigma}(z)$ is clearly C^∞, as is $\hat{f}_{j\sigma}(z)$, since $\hat{f}_{j\sigma}(z)=(-1)^{j+\sigma}f_{j\sigma}(z)$ or $\hat{f}_j(\mathbf{x},z)=f_j(i\sigma_2\mathbf{x},z)$.

(ii) If
$$V=\begin{pmatrix} a & b\\ c & d\end{pmatrix}$$
denotes an element of $SU(2)$ so that $V^{-1}=V^\dagger$, det $V=1$, or
$$\begin{pmatrix} a & c\\ b & d\end{pmatrix}=\begin{pmatrix} d & -b\\ -c & a\end{pmatrix},$$
then according to (B1),
$$T_{(n_1n_2)}(V)f_j(\mathbf{x},z)=f_j(V\mathbf{x},z)\,, \tag{B20}$$

where we have used the fact that unitarity of V implies that

$$\left(1+\left|\frac{az+c}{bz+d}\right|\right)^2=\frac{1+|z|^2}{|bz+d|^2}.$$

From this equation it immediately follows that the $f_{j\sigma}(z)$ obey Eq. (B9).

Apart from a factor, we observe that $f_{j\sigma}(z)$ is a D^j function. We can thus set up a Hermitian "scalar product" in $D_{(n_1,n_2)}$, invariant under $SU(2)$, such that with respect to this scalar product $f_{j\sigma}(z)$ and $f_{j'\sigma'}(z)$ will be orthogonal for $(j,\sigma)\neq(j',\sigma')$. We shall indeed show that one has

$$\frac{2}{\pi}\int\frac{i}{2}dzd\bar{z}\,(1+z\bar{z})^{-\mathrm{Re}(n_1+n_2)}f_{j\sigma}(z)f_{j'\sigma'}(z)=\delta_{jj'}\delta_{\sigma\sigma'}\left|\frac{\Gamma(j+1-\frac{1}{2}(n_1+n_2))}{\Gamma(j+1+\frac{1}{2}(n_1+n_2))}\right|. \tag{B21}$$

To prove this relation we set $z=e^{i\theta}\cot\frac{1}{2}\beta$, $0\leq\beta\leq\pi$, and remark that $f_{j\sigma}(\cot\frac{1}{2}\beta\, e^{i\theta})=e^{i\theta[\sigma+(n_1-n_2)/2]}f_{j\sigma}(\cot\frac{1}{2}\beta)$, and further that

$$V_{\cot\frac{1}{2}\beta}=\begin{pmatrix}\cos\frac{1}{2}\beta & -\sin\frac{1}{2}\beta\\ \sin\frac{1}{2}\beta & \cos\frac{1}{2}\beta\end{pmatrix}.$$

Hence, with the classical notations of quantum mechanics textbooks,

$$D_{\sigma,(n_1-n_2)/2}{}^j(V_{\cot\frac{1}{2}\beta})=d_{\sigma,(n_1-n_2)/2}{}^j(\beta),$$

the left-hand side of (B21) reads

$$(j+\tfrac{1}{2})\left|\frac{\Gamma(j+1-\frac{1}{2}(n_1+n_2))}{\Gamma(j+1+\frac{1}{2}(n_1+n_2))}\right|\delta_{\sigma\sigma'}$$
$$\times\int_0^\pi \sin\beta d\beta\; d_{\sigma,(n_1-n_2)/2}{}^j(\beta)d_{\sigma,(n_1-n_2)/2}{}^{j'}(\beta)$$
$$=\delta_{\sigma\sigma'}\delta_{jj'}\left|\frac{\Gamma(j+1-\frac{1}{2}(n_1+n_2))}{\Gamma(j+1+\frac{1}{2}(n_1+n_2))}\right|,$$

where we have used the orthogonality properties of the d^j functions.

Let us briefly comment on the scalar product derived from (B21). For any two f and g belonging to $D_{(n_1,n_2)}$ the following integral obviously exists:

$$\langle f,g\rangle=\frac{2}{\pi}\int\frac{i}{2}dzdz\;(1+z\bar{z})^{-\mathrm{Re}(n_1+n_2)}\bar{f}(z)g(z)$$
$$=\frac{2}{\pi}\int_{zz\leq 1}\frac{i}{2}dzd\bar{z}\;(1+z\bar{z})^{-\mathrm{Re}(n_1+n_2)}$$
$$\times[\bar{f}(z)g(z)+\hat{f}(z)\hat{g}(z)].\quad\text{(B22)}$$

(The use of the ⟨ , ⟩ notation is to distinguish this scalar product from that introduced before.) Clearly, $\langle f,f\rangle\geq 0$ and the equal sign only holds for $f=0$. $D_{(n_1,n_2)}$ is not complete with respect to this norm but is dense in its completion, a Hilbert space that we can denote $H_{(n_1,n_2)}$. It is easy to see that the set $f_{j\sigma}(z)$ is an orthogonal basis in this space ($j-j_0\geqslant 0$ non-negative integer). It is gratifying to observe that the scalar product (B22) is precisely the one corresponding to the principal series, since from $n_1+\bar{n}_2=0$ it follows that $\mathrm{Re}(n_1+n_2)=0$ [compare with (B3)] and in this case

$$\left|\frac{\Gamma(j+1-\frac{1}{2}(n_1+n_2))}{\Gamma(j+1+\frac{1}{2}(n_1+n_2))}\right|=1.\quad\text{(B23)}$$

However, in the case of the supplementary series, (B22) does not coincide with (B5). Indeed, one can easily verify that for $n_1=n_2=c$, $-1<c<0$, the following identity holds;

$$(f_{j'\sigma'},f_{j\sigma})=\left(\frac{2}{\pi}\right)^2\int\int(\tfrac{1}{2}i)^2dz_1d\bar{z}_1dz_2d\bar{z}_2$$
$$\times|z_1-z_2|^{-2c-2}f_{j'\sigma'}(z_1)f_{j\sigma}(z_2)$$
$$=2(-1)^{j+1}\frac{\Gamma(c-j)\Gamma(i+c+j)}{\Gamma(1+c)^2}\frac{2}{\pi}\int(\tfrac{1}{2}i)dz_1d\bar{z}_1$$
$$\times f_{j'\sigma'}(z_1)f_{j\sigma}(z_1)(1+z_1z_1)^{-2c},$$

and since $2c=n_1+n_2=\mathrm{Re}(n_1+n_2)$,

$$(f_{j'\sigma'},f_{j\sigma})=2(-1)^{j+1}\frac{\Gamma(c-j)\Gamma(1+c+j)}{[\Gamma(1+c)]^2}\langle f_{j'\sigma'},f_{j\sigma}\rangle$$
$$=\frac{2\Gamma(-c)}{\Gamma(1+c)}\delta_{\sigma\sigma}{}'\delta_{jj}{}'$$
$$n_1=n_2=c,\quad -1<c<0.\quad j,\,j'=0,1,\cdots.$$

Thus apart from an over-all constant factor the functions $f_{j\sigma}$ are again in that case an orthonormal basis for the unitary representations of the complementary series. This explains the particular normalization chosen above.

Let us exhibit the action of the generators **J**, **K** on the functions $f_{j\sigma}(z)$. With $J_\pm=J_1\pm iJ_2$, $K_\pm=K_1\pm iK_2$, they read

$$\begin{aligned}
&J_3f_{j\sigma}=\sigma f_{j\sigma},\quad J_\pm f_{j\sigma}=[(j\mp\sigma)(j\pm\sigma+1)]^{1/2}f_{j,\sigma\pm1},\\
&K_3f_{j\sigma}=a_j(j^2-\sigma^2)^{1/2}f_{j-1,\sigma}+b_j\sigma f_{j,\sigma}\\
&\qquad+a_{j+1}[(j+1)^2-\sigma^2]^{1/2}f_{j+1,\sigma},\\
&K_\pm f_{j\sigma}=\pm a_j[(j\mp\sigma)(j\mp\sigma-1)]^{1/2}f_{j-1,\sigma\pm1}\\
&\qquad+b_j[(j\mp\sigma)(j\mp\sigma+1)]^{1/2}f_{j,\sigma\pm1}\\
&\qquad\mp a_{j+1}[(j\pm\sigma+1)(j\pm\sigma+2)]^{1/2}f_{j+1,\sigma\pm1},
\end{aligned}\quad\text{(B24)}$$

with

$$a_j=\frac{1}{j}\left[\frac{(j^2-c^2)(j^2-j_0{}^2)}{4j^2-1}\right]^{1/2},\quad b_j=\frac{ij_0c}{j(j+1)};$$

where we have used the notation of Naimark;

$$j_0=\tfrac{1}{2}|n_1-n_2|,\quad c=\mathrm{sgn}[(n_1-n_2)]\tfrac{1}{2}(n_1+n_2),$$
$$\text{or}\quad\pm\tfrac{1}{2}(n_1+n_2)\quad\text{if}\quad n_1=n_2.\quad\text{(B25)}$$