Current Algebras, Sum Rules, and Canonical Field Theories*

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A very large class of c-number Lagrangian field theories, called canonical field theories—being all theories whose field equations have sufficiently many solutions—satisfy a number of imporant completeness relations. These relations ensure the existence of Gell-Mann current algebras and Fubini sum rules, and guarantee that canonical field quantization is possible. We have studied finite-component, first- and second-order Lagrangians in general, and a class of fourth-order Lagrangians. The work is preliminary to an investigation of infinite-component field theories.

I. INTRODUCTION

THIS report is a study of canonical field theories of a finite number of local fields. Under certain conditions these provide model saturations of Gell-Mann current algebras and Fubini sum rules.

Physically interesting models must involve an infinite number of states, with all values of spin, otherwise the currents are polynomials in momentum transfer. It has therefore been suggested[1] that the most convenient framework for current algebra is infinite-component field theory, and in fact it has been found[2] that the currents of all known saturations with nontrivial mass spectra are the canonical currents of Lagrangian field theories of infinitely many fields. However, some infinite-component wave equations have solutions with spacelike momentum, and these unphysical solutions must be included to obtain saturation, even in the infinite-momentum frame. Wave equations without spacelike solutions can easily be written down,[3] but such theories seem not to provide a current algebra.

We have studied finite-component theories in order to understand what are the special conditions that must be satisfied in order that the canonical current provide a current algebra. To learn something for possible application to infinite-component theories, it is important to choose models that can easily be generalized from the finite- to the infinite-component case. On the other hand, it should not be necessary to eliminate from them those unphysical features that are peculiar to finite-component field theories. Positivity of the physical probability metric is an important physical requirement. In finite-component field theories it is very difficult to satisfy except for the lowest values of spin; it usually requires a complicated structure of subsidiary conditions. In infinite-component theories this particular problem is quite trivial, and subsidiary conditions play no role there. We have therefore ignored the positivity requirements in our choice of models.

The results are as follows: There is a very large class of c-number Lagrangian field theories that we may call canonical field theories; they are theories whose field equations have a sufficient number of independent solutions. The c-number solutions of canonical field theories satisfy a set of sum rules—one sum rule for first-order theories, and three sum rules for second-order theories (Sec. II). These sum rules are very closely related to current commutation relations (Sec. III), to Fubini sum rules (Sec. IV) and to the possibility of constructing local quantized fields by canonical field quantization (Sec. VI). These results are quite general for first- and second-order Lagrangians.[4] No complete investigation of higher-order theories has been carried out, but a class of fourth-order Lagrangians is examined in detail (Sec. VII) and no hint of special difficulties has come to light.

The conclusion is that the only requirement necessary to ensure that a Lagrangian field theory provide a model current algebra is that it be "canonical"; i.e., that the field equations have enough solutions. If the field has d components and the Lagrangian is of the nth order in the momentum, then "enough" solutions means $\frac{1}{2}nd$ independent solutions for each sign of the energy. The extension of these results to the case of infinite-component fields is not completely obvious. It turns out that the most convenient tool for studying the important completeness relations is a set of meromorphic functions

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[3] An infinite-component field equation that has no spacelike solutions and no negative probabilities was given by C. Fronsdal, Phys. Rev. **171**, 1811 (1968).
[4] The restriction to currents of special types ("constant" plus "convective") introduced by Hamprecht and Kleinert (Ref. 2) is quite unnecessary. All the solutions found by Chang *et al.* (Ref. 2) are also of this type.

that can be interpreted as scattering amplitudes. The study of such functions in infinite-component theories will be reported separately.

II. COMPLETENESS

Completeness of the set of c-number solutions of the classical field theory is basic to current algebra, sum rules, and canonical quantization. We investigate completeness in successively larger classes of field theories.

Klein-Gordon Theory

This is a theory of a single scalar complex field $\psi(x)$. The Lagrangian is

$$\mathcal{L}=\int d^4x\,\psi^*(x)L\left(i\frac{\partial}{\partial x}\right)\psi(x) \tag{1}$$

with

$$L(p)=p^2-m^2. \tag{2}$$

The conserved canonical current density is

$$J_\mu(x)=i\psi^*(x)\overleftrightarrow{\partial}_\mu\psi(x) \tag{3}$$

or, in momentum space

$$J_\mu(p,q)=\psi^*(p)(p+q)_\mu\psi(q). \tag{4}$$

Let the three-momentum $\mathbf{p}$ be fixed. Then the time component $J_0(p,q)$, $\mathbf{p}=\mathbf{q}$, defines the orthogonality and normalization properties of the c-number solutions of the Lagrangian field equation. Indeed, since $p_0=\pm q_0$,

$$\begin{aligned}J_0(p,q)&=\psi^*(p)(p+q)_0\psi(q)\\&=\psi^*(q)2q_0\psi(q), \qquad p_0=q_0;\\&=0 \qquad\qquad\qquad\quad p_0=-q_0.\end{aligned} \tag{5}$$

The quantity

$$J_0(p,q)=\psi^*(q)2q_0\psi(q) \tag{6}$$

is the number of particles per unit volume.[5] We shall use a covariant normalization, setting

$$\psi^*(q)\psi(q)=1. \tag{7}$$

Then, obviously, if $\mathbf{q}=\mathbf{p}$,

$$\sum_{\pm}\frac{\psi(q)\psi^*(q)}{2q_0}(q_0+p_0)\psi(p)=\psi(p), \tag{8}$$

where the sum is over the positive- and negative-energy solutions. That is, in the space of fixed three-momentum,

$$\sum_{\pm}\frac{\psi(q)\psi^*(q)}{2q_0}(q_0+p_0)=1. \tag{9}$$

A more subtle result is that (8) remains true when $\mathbf{p}\neq\mathbf{q}$. This is because both of the following sum rules are satisfied:

$$\sum_{\pm}\frac{\psi(q)\psi^*(q)}{2q_0}=0, \quad \text{(first sum rule)}; \tag{10}$$

$$\sum_{\pm}\frac{\psi(q)\psi^*(q)}{2q_0}q_0=1, \quad \text{(second sum rule)}. \tag{11}$$

$U(3,1)$ Theory

The preceding analysis can be generalized to a wide class of theories. Here we consider an example of field theories that contain many fields with different spins. Instead of a single scalar field we consider a vector field $\psi(x)=\{\psi_\mu(x),\ \mu=0,1,2,3\}$. For clarity, components will be labeled with Greek indices, solutions with Latin indices; the Greek indices will be suppressed whenever possible. The Lagrangian studied is

$$\mathcal{L}=\int d^4x\,\psi^\dagger(x)L\left(i\frac{\partial}{\partial x}\right)\psi(x) \tag{12}$$

with $\psi^\dagger=\{\psi^{*\mu}\}$ and

$$L(p)=pCp+Ap^2-m^2, \tag{13}$$

$$pCp=p^\mu C_{\mu\nu}p^\nu, \quad (C_\nu{}^\mu)_\lambda{}^\rho=\delta_\lambda{}^\mu\delta_\nu{}^\rho. \tag{14}$$

The 16 matrices $C_\mu{}^\nu$ form a four-dimensional representation of the algebra $U(3,1)$. All the results of this section can be generalized to an infinite family of canonical theories in which the $C_\mu{}^\nu$ are replaced by any one of a set of finite-dimensional representations of $U(3,1)$. This is done in the Appendix.

To solve the field equations we diagonalize the matrix pCp. We choose A and m^2 positive so that all four solutions have positive p^2; then p_μ can be transformed to rest, and in this frame[6]

$$pCp=p^2(1-n), \tag{15}$$

where n is the diagonal matrix with matrix elements 0, 1, 1, 1. In an arbitrary frame we define the operator $n=n(p)$ by this equation; then the eigenvalues are always 0 (singlet) and 1 (triplet). The corresponding mass values, obtained by diagonalizing $L(p)$, are

$$L(p)_{nn'}=L_n(p^2)\delta_{nn'}, \tag{16}$$

$$L_n(p^2)=p^2(1-n+A)-m^2, \tag{17}$$

and, putting $L_n(p^2)=0$, are

$$m_0{}^2=m^2/(A+1), \quad m_1{}^2=m^2/A. \tag{18}$$

[5] More precisely, J_0 is the number of particles with positive energy, minus the number of particles with negative energy.

[6] In this simple case we may simply use (14) to calculate

$$pCp=\begin{pmatrix}p_0{}^2 & -p_0\mathbf{p}\\ p_0\mathbf{p} & -\mathbf{pp}\end{pmatrix}$$

from which it is trivial to find the eigenvectors (19).

The four solutions (there are eight, counting the two signs of the energy) are

$$\psi_0=\begin{bmatrix}E_0\\p_1\\p_2\\p_3\end{bmatrix},\quad \psi_{11}=\begin{bmatrix}p_1\\E_1\\0\\0\end{bmatrix},\quad \psi_{12}=\begin{bmatrix}p_2\\0\\E_1\\0\end{bmatrix},\quad \psi_{13}=\begin{bmatrix}p_3\\0\\0\\E_1\end{bmatrix} \tag{19}$$

with

$$E_0{}^2=m_0{}^2+\mathbf{p}^2,\quad E_1{}^2=m_1{}^2+\mathbf{p}^2. \tag{20}$$

The conserved canonical current density is

$$J_\mu(p,q)=\psi^\dagger(p)I_\mu(p,q)\psi(q), \tag{21}$$

where

$$I_\mu(p,q)=(p+q)^\nu\Gamma_{\mu\nu},\quad \Gamma_{\mu\nu}=\bar{C}_{\mu\nu}+Ag_{\mu\nu} \tag{22}$$

and $\bar{C}_{\mu\nu}$ is the symmetric part of $C_{\mu\nu}$. In the space of fixed three-momentum $J_0(p_0,\mathbf{p};q_0,\mathbf{p})$ defines orthogonality and physical normalization of the wave functions:

$$\begin{aligned}\psi_j{}^\dagger(p)I_0(p,q)\psi_{j'}(q)\,|_{\mathbf{p}=\mathbf{q}}&=\delta_{jj'}\eta_j(p),\quad p_0q_0>0;\\&=0,\quad p_0q_0<0;\end{aligned} \tag{23}$$

where the single index j takes the values 0, 11, 12, 13, and

$$\begin{aligned}\eta_0(p)&=2E_0m^2,\\ \eta_{1a}(p)&=-2E_1m^2.\end{aligned} \tag{24}$$

Obviously, if $\mathbf{p}=\mathbf{q}$,

$$\sum_{j\pm}\frac{\psi_j(q)\psi_j{}^\dagger(q)}{\eta_j(q)}I_0(q,p)\psi(p)=\psi(p). \tag{25}$$

As in the Klein-Gordon theory, this remains valid even if $p\neq q$; this is equivalent to a pair of sum rules, analogous to Eqs. (10) and (11)

$$\sum_{j\pm}\frac{\psi_i(q)\psi_j{}^\dagger(q)}{\eta_j(q)}=0,\qquad \text{(first sum rule)}; \tag{26}$$

$$\sum_{j\pm}\frac{\psi_j(q)\psi_j{}^\dagger(q)}{\eta_j(q)}q_0\Gamma_{00}=1,\qquad \text{(second sum rule)}. \tag{27}$$

The matrix $\Gamma_{00}\equiv C_{00}+A$ is the coefficient of q_0 in $I_0(q,p)$. These results are obtained by direct calculation from (19) for the case at hand; the more general case of higher-dimensional representation of $U(3,1)$ is treated in the Appendix.

Fourth-Order Lagrangian

If the Lagrangian is of fourth order in p_μ, then the number of c-number solutions needed for completeness is four times the number of field components. We study a theory with the same field $\psi(x)$ as in the preceding example, but with the second-order Lagrangian operator (13) replaced by

$$L(p)=p^4+pCp+Ap^2+B. \tag{28}$$

Diagonalizing the operator $n(p)$ defined by (15) we have

$$L_n(p^2)=p^4+p^2(1-n+A)+B. \tag{29}$$

The zeros of this polynomial give the masses

$$p^2=\tfrac{1}{2}(-A+n-1)\pm[\tfrac{1}{4}(-A+n-1)^2-B]^{1/2}=\begin{Bmatrix}m_n{}^2\\ \mu_n{}^2\end{Bmatrix}. \tag{30}$$

The solutions are given by (19), and by another set of four wave functions obtained by replacing m_n by μ_n. These additional solutions will be labeled ψ_0', ψ_{1a}'; $a=1, 2, 3$.

The theory may be bilinearized by introducing the eight-component field

$$\chi=\begin{pmatrix}\varphi\\ \psi\end{pmatrix},\quad \varphi=p^2\psi. \tag{31}$$

Then the Lagrangian takes the form

$$\mathcal{L}=\int d^4x\,\chi^\dagger(x)L'\Big(i\frac{\partial}{\partial x}\Big)\chi(x) \tag{32}$$

with

$$L'(p)=\begin{pmatrix}-1 & p^2\\ p^2 & pCp+Ap^2+B\end{pmatrix}. \tag{33}$$

The solutions are of the form

$$\chi_1=\begin{pmatrix}m_0{}^2\psi_0\\ \psi_0\end{pmatrix},\ \cdots,\ \chi_8=\begin{pmatrix}\mu_1{}^2\psi_{13}'\\ \psi_{13}'\end{pmatrix}. \tag{34}$$

The canonical conserved current is

$$J_\mu(p,q)=\chi^\dagger(p)I_\mu'(p,q)\chi(q), \tag{35}$$

$$I_\mu'(p,q)=\begin{pmatrix}0 & (p+q)_\mu\\ (p+q)_\mu & (p+q)^\nu(\bar{C}_{\mu\nu}+Ag_{\mu\nu})\end{pmatrix}. \tag{36}$$

For fixed three-momentum, J_0 defines orthogonality

$$\begin{aligned}\chi_j{}^\dagger(p)I_0'(p,q)\chi_{j'}(q)&=\delta_{jj'}\eta_j(q),\quad p_0q_0>0;\\&=0,\quad p_0q_0<0;\end{aligned} \tag{37}$$

where j and j' run through the eight values 1, $\cdots$, 8,

$$\begin{aligned}\eta_1(q)&=2E_0(m_0{}^2-\mu_0{}^2)m_0{}^2,\\ \eta_2=\eta_3=\eta_4&=-2E_1(m_1{}^2-\mu_1{}^2)m_1{}^2,\end{aligned} \tag{38}$$

and η_5, $\cdots$, η_8 are found by interchanging m_n and μ_n in $\eta_1\cdots\eta_4$. Obviously, if $\mathbf{p}=\mathbf{q}$,

$$\sum_{j\pm}\frac{\chi_j(q)\chi_j{}^\dagger(q)}{\eta_j(q)}I_0'(q,p)\chi(p)=\chi(p). \tag{39}$$

As in the previous examples, this remains valid if $\mathbf{p}\neq\mathbf{q}$, because the following sum rules are satisfied:

$$\sum_{j\pm}\frac{\chi_j(q)\chi_j^\dagger(q)}{\eta_j(q)}=0\,, \quad \text{(first sum rule)}; \tag{40}$$

$$\sum_{j\pm}\frac{\chi_j(q)\chi_j^\dagger(q)}{\eta_j(q)}q_0\Gamma_{00}'=1\,, \quad \text{(second sum rule)}; \tag{41}$$

where

$$\Gamma_{00}'=\begin{pmatrix}0 & 1\\ 1 & C_{00}+A\end{pmatrix} \tag{42}$$

is the coefficient of q_0 in I_0'.

In terms of the original field ψ, these sum rules are rather astonishing

$$\sum_{j\pm}\frac{\psi_j(q)\psi_j^\dagger(q)}{\eta_j(q)}\times\begin{cases}1 & =0\,,\\ q^2 & =0\,,\\ q^4 & =0\,,\\ q_0 & =0\,,\\ q_0q^2 & =1\,,\\ q_0q^4 & =-A-C_{00}.\end{cases} \tag{43}$$

Here the sum as always, includes all the solutions, in in this case 16 in number. From (43) it follows that the usual completeness relation is satisfied in the fourth-order theory as well,

$$\sum_{j\pm}\frac{\psi_j(q)\psi_j^\dagger(q)}{\eta_j(q)}I_0(q,p)=1 \tag{44}$$

for arbitrary $\mathbf{q}$ and $\mathbf{p}$. Here

$$I_\mu(q,p)=(p+q)_\mu(p^2+q^2)+(p+q)^\nu(\bar{C}_{\mu\nu}+Ag_{\mu\nu}) \tag{45}$$

defines the canonical conserved current of the fourth-order Lagrangian,

$$J_\mu(q,p)=\psi^\dagger(q)I_\mu(q,p)\psi(p)\,. \tag{46}$$

This current has the same matrix elements as (35); in particular $\eta_j(q)$ is always given by (38).

All these results are generalized in the Appendix.

General Second-Order Lagrangian

Let $L(p)$ be a second-order polynomial in p_μ, with Hermitian matrix coefficients. Let d be the dimension of the matrices, and let $\psi_j(q)$, $j=1, \cdots, d$, be a set of d linearly independent positive energy solutions of the Lagrangian field equation. Let m_j be the corresponding masses. Let us further postulate invariance of $L(p)$ under Poincaré transformations; then there exists an equal number of linearly independent negative energy solutions, with the same masses,[7] all the poles of $1/L(p)$ are simple, and no solutions have zero norm.

[7] W. Pauli, Phys. Rev. 58, 716 (1940)

The general second-order Lagrangian may be written

$$L(p)=\Gamma_{\mu\nu}p^\mu p^\nu+\Gamma_\mu p^\mu-\mathfrak{M}^2. \tag{47}$$

The conserved canonical current is then defined in terms of the operator

$$I_\mu(p,q)=(p+q)^\nu\Gamma_{\nu\mu}+\Gamma_\mu. \tag{48}$$

Let ψ_{in} and ψ_{fi} be two of the solutions, and consider

$$F(q)=\psi_{\text{in}}^\dagger(p')[L(q)]^{-1}I_0(q,p)\psi_{\text{fi}}(p)\,. \tag{49}$$

This is a meromorphic function of q_0, $\mathbf{q}$ fixed, with simple poles at $q_0=\pm(\mathbf{q}+m_j^2)^{1/2}$. Furthermore

$$\lim_{q_0\to\infty}q_0F(q)=\psi_{\text{in}}^\dagger(p')\psi_{\text{fi}}(p) \tag{50}$$

from which it follows that

$$\frac{1}{2\pi i}\int F(q)dq_0=\psi_{\text{in}}^\dagger(p')\psi_{\text{fi}}(p)\,. \tag{51}$$

The contour surrounds all the poles, and the left-hand side has the value

$$\sum\{\text{Residues of } F(q) \text{ at } q_0=\pm(\mathbf{q}^2+m_j^2)^{1/2}\}$$
$$=\sum_{j\pm}\psi_{\text{in}}^\dagger(p')\frac{\psi_j(q)\psi_j^\dagger(q)}{\eta_j(q)}I_0(q,p)\psi_{\text{fi}}(p)\,. \tag{52}$$

Thus,

$$\sum_{j\pm}\frac{\psi_j(q)\psi_j^\dagger(q)}{\eta_j(q)}I_0(q,p)=1 \tag{53}$$

for all $\mathbf{q}$ and all p_μ. This is equivalent (provided $\Gamma_{\mu\nu}\neq0$) to the two sum rules

$$\sum_{j\pm}\frac{\psi_j(q)\psi_j^\dagger(q)}{\eta_j(q)}=0\,, \quad \text{(first sum rule)}; \tag{54}$$

$$\sum_{j\pm}\frac{\psi_j(q)\psi_j^\dagger(q)}{\eta_j(q)}q_0\Gamma_{00}=1\,, \quad \text{(second sum rule)}. \tag{55}$$

We see that this pair of sum rules, already obtained in special cases, is a general feature of Lagrangian field theories. Their validity depends only on the very general postulates enumerated in the first paragraph of this subsection. Lagrangian field theories that satisfy these postulates will be referred to as canonical field theories. An example of a noncanonical Lagrangian field theory is obtained by setting $B=0$ in (28); in this limiting case the number of linearly independent solutions with positive energy is 3 instead of 4; $F(q)$ has a double pole, and (44) is not satisfied.

Finally, it is clear that the restriction to second-order Lagrangians is not essential. The main result (53) is valid for a Lagrangian of any finite order, provided the field equations have a sufficient number of independent solutions.

For example, let $\Gamma_{\mu\nu}=0$, to treat the most general first-order Lagrangian. The main result (53) is always valid, but instead of (54), (55) we now have the single sum rule

$$\sum_{j\pm}\frac{\psi_j(q)\psi_j^\dagger(q)}{\eta_j(q)}\Gamma_0=1. \tag{56}$$

For the case of a fourth-order Lagrangian an illustrative example was given above.

In addition to (54) and (55) an equally important third sum rule is valid under the same conditions; see Eq. (69) below.

III. CURRENT ALGEBRAS

The current algebras considered here are all related to the canonical conserved current. We construct scalar and vector densities in terms of sets of fields that form representations of some "charge group" like isospin or unitary symmetry. The Lagrangian is a sum of the Lagrangians of the individual charge components, and is invariant with respect to the transformations of the charge group. The charge index will always be suppressed. A set of constant matrices λ^σ, satisfying the commutation relations of the charge group

$$[\lambda^\sigma,\lambda^\tau]=f^{\sigma\tau\rho}\lambda^\rho,$$

act on the new degree of freedom, i.e., on the charge index. They are used to define

$$\begin{aligned}J^\sigma(p,q)&=\psi^\dagger(p)\lambda^\sigma\psi(q),\\ J^{\sigma\tau}(p,q)&=\psi^\dagger(p)\lambda^\sigma\lambda^\tau\psi(q),\\ J_\mu{}^\sigma(p,q)&=\psi^\dagger(p)I_\mu(p,q)\lambda^\sigma\psi(q),\\ J_\mu{}^{\sigma\tau}(p,q)&=\psi^\dagger(p)I_\mu(p,q)\lambda^\sigma\lambda^\tau\psi(q).\end{aligned} \tag{57}$$

These quantities are matrix elements of the Fourier transforms of local operators, e.g.,

$$\begin{aligned}J^\sigma(k)&=\int e^{ikx}J^\sigma(x)d^4x,\\ \langle p|J^\sigma(k)|q\rangle&=\delta^{(4)}(q-p-k)J^\sigma(p,q).\end{aligned} \tag{58}$$

The physical norm is defined by

$$\psi_j^\dagger(p)I_0(p,p)\psi_j(p)=\eta_j(p); \tag{59}$$

hence the operator product $J^\sigma(x')J_\mu{}^\tau(x)$ is given by

$$\begin{aligned}&\langle p'|J^\sigma(k')J_\mu{}^\tau(-k)|p\rangle\\ &=\sum_{j\pm}\int\frac{d^3q}{\eta_j(q)}\langle p'|J^\sigma(k')|j,\mathbf{q}\rangle\langle j,\mathbf{q}|J_\mu{}^\tau(-k)|p\rangle\\ &=\delta^{(4)}(p'-p+k'-k)\sum_{j\pm}(p_0'+k_0'-E_j)\psi^\dagger(p')\lambda^\sigma\\ &\quad\times\frac{\psi_j(p'+k')\psi_j^\dagger(p'+k')}{\eta_j(p'+k')}I_\mu(p'+k',p)\lambda^\tau\psi(p).\end{aligned} \tag{60}$$

If we integrate over the time component of k with p, p' and $k'-k$ fixed, and use the sum rule (53), then we find that

$$\begin{aligned}&\int dk_0\,\langle p'|J^\sigma(k')J_0{}^\tau(-k)|p\rangle\\ &\qquad=\delta^{(4)}(p'-p+k'-k)\psi^\dagger(p')\lambda^\sigma\lambda^\tau\psi(p)\end{aligned} \tag{61}$$

or

$$\int dk_0\,J^\sigma(k')J_0{}^\tau(-k)=J^{\sigma\tau}(k'-k). \tag{62}$$

Similarly

$$\int dk_0\,J_0{}^\tau(-k)J^\sigma(k')=J^{\tau\sigma}(k'-k), \tag{63}$$

and therefore,

$$\int dk_0[J^\sigma(k'),\,J_0{}^\tau(-k)]=f^{\sigma\tau\rho}J^\rho(k'-k). \tag{64}$$

Transformed to configuration space, this is an equal-time commutator:

$$[J^\sigma(x'),J_0{}^\tau(x)]\delta(x_0'-x_0)=f^{\sigma\tau\rho}J^\rho(x)\delta^{(4)}(x'-x). \tag{65}$$

Similar results are easily obtained for the commutator of $J_0{}^\tau(x)$ with any local density that does not involve time derivatives of the fields. The final result (65) also remains valid if $\Gamma_{\mu\nu}=0$.

Next, to calculate the "time-time commutator" $[J_0{}^\tau(x'),J_0{}^\sigma(x)]$, consider the meromorphic function

$$F(q)=\psi_{\mathrm{in}}^\dagger(p')I_0(p',q)[1/L(q)]I_0(q,p)\psi_{\mathrm{fi}}(p). \tag{66}$$

As q_0 tends to infinity,

$$F(q)\to\psi_{\mathrm{in}}^\dagger(p')[\Gamma_{00}+(1/q_0)I_0(p',p)]\psi_{\mathrm{fi}}(p). \tag{67}$$

Applying the residue theorem as before we obtain

$$\sum_{j\pm}I_0(p',q)\frac{\psi_j(q)\psi_j^\dagger(q)}{\eta_j(q)}I_0(q,p)=I_0(p',p) \tag{68}$$

for arbitrary p_μ', p_μ, and $\mathbf{q}$. This again implies (provided $\Gamma_{\mu\nu}\neq0$) the validity of the sum rules (54) and (55), as well as the further result

$$\sum_{j\pm}(q^\nu\Gamma_{\nu0}+\tfrac{1}{2}\Gamma_0)\frac{\psi_j(q)\psi_j^\dagger(q)}{\eta_j(q)}(q^\mu\Gamma_{\mu0}+\tfrac{1}{2}\Gamma_0)=0 \quad\text{(third sum rule)}. \tag{69}$$

In exactly the same way that (53) led to the equal-time commutator (65), Eq. (68) gives

$$[J_0{}^\sigma(x'),J_0{}^\tau(x)]\delta(x_0'-x_0)=f^{\sigma\tau\rho}J_0{}^\rho(x)\delta^{(4)}(x'-x). \tag{70}$$

This result remains valid when $\Gamma_{\mu\nu}=0$; i.e., in the case of a general linear Lagrangian.

The "space-time commutator" is more complicated and involves Schwinger terms. The function

$$F(q)=\psi_{\mathrm{in}}^\dagger(p')I_1(p',q)[L(q)]^{-1}I_0(q,p)\psi_{\mathrm{fi}}(p) \tag{71}$$

has the limit

$$\psi_{\rm in}^\dagger(p')[\Gamma_{01}+(1/q_0)\{I_1(p,p')+(q-p)^\mu \times(\Gamma_{\mu 1}-\Gamma_{01}\Gamma_{00}^{-1}\Gamma_{\mu 0})\}]\psi_{\rm fi}(p) \quad (72)$$

as q_0 tends to infinity; whence the sum rule

$$\sum_{j\pm} I_1(p',q)\frac{\psi_j(q)\psi_j^\dagger(q)}{\eta_j(q)}I_0(q,p) =I_1(p,p')+(q-p)^\mu(\Gamma_{\mu 1}-\Gamma_{01}\Gamma_{00}^{-1}\Gamma_{\mu 0}). \quad (73)$$

This is not independent of the previous results; it is equivalent to (68) and can be obtained directly from the three basic sum rules (54), (55) and (69). By the usual arguments we obtain

$$\int dk_0[J_1{}^\sigma(k'), J_0{}^\tau(-k)]=f^{\sigma\tau\rho}J_1{}^\rho(k'-k) +k^\mu\psi_{\rm in}^\dagger(p')(\Gamma_{\mu 1}-\Gamma_{01}\Gamma_{00}^{-1}\Gamma_{\mu 0})[\lambda^\sigma,\lambda^\tau]_+\psi_{\rm fi}(p) \quad (74)$$

or, equivalently in configuration space,

$$[J_1{}^\sigma(x'),J_0{}^\tau(x)]\delta(x_0-x_0') =\left[f^{\sigma\tau\rho}J_1{}^\rho(x)+i\Sigma_{\mu 1}{}^{\sigma\tau}(x')\frac{\partial}{\partial x_\mu}\right]\delta^{(4)}(x-x'), \quad (75)$$

where

$$\Sigma_{\mu\nu}{}^{\sigma\tau}(x)=\psi_{\rm in}^\dagger(x)(\Gamma_{\mu\nu}-\Gamma_{0\nu}\Gamma_{00}^{-1}\Gamma_{\mu 0})[\lambda^\sigma,\lambda^\tau]_+\psi_{\rm fi}(x). \quad (76)$$

In the case of a linear Lagrangian, Eq. (75) remains valid; in this case the Schwinger terms vanish. (After field quantization, new Schwinger terms appear in both first- and second-order theories.)

IV. FUBINI SUM RULES

To show that the preceding results contain sum rules of the type considered by Fubini *et al.*[8] it is sufficient to show that the meromorphic functions $F(q)$ are related to scattering amplitudes. Let us introduce interactions into the Lagrangian by adding the free Lagrangians of scalar fields $A(x)$ and vector fields $A_\mu(x)$, and local interaction terms

$$\mathcal{L}_I=\int d^4x\{A_\sigma(x)J^\sigma(x)+A_\sigma{}^\mu(x)J_\mu{}^\sigma(x)\}. \quad (77)$$

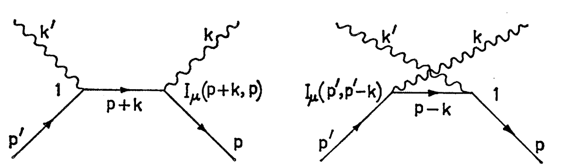

FIG. 1. Feynman diagrams illustrating the scattering amplitude $T_\mu{}^{\sigma\tau}$ of Eq. (78). The incoming quantum (marked k') is a scalar with internal quantum number σ; the outgoing one is a vector.

[8] V. de Alfaro, S. Fubini, C. Rossetti, and G. Furlan, Phys. Letters **21**, 576 (1966).

Using straightforward Feynman rules, we obtain the following amplitude for "Compton" scattering (see Fig. 1):

$$T_\mu{}^{\sigma\tau}(p,p',k)=\psi_{\rm in}^\dagger(p')\{\lambda^\sigma[1/L(p+k)]I_\mu(p+k, p)\lambda^\tau +\lambda^\tau I_\mu(p', p'-k)[1/L(p'-k)]\lambda^\sigma\}\psi_{\rm fi}(p). \quad (78)$$

The time component $T_0{}^{\sigma\tau}$ is a function of the type used in the proof of the sum rules. It is a meromorphic function of k_0 that behaves like

$$(1/k_0)\psi_{\rm in}^\dagger(p')[\lambda^\sigma,\lambda^\tau]I_0(p',p)\psi_{\rm fi}(p) \quad (79)$$

as k_0 tends to infinity. Thus, if $t_0{}^{\sigma\tau}$ is the absorptive part of $T_0{}^{\sigma\tau}$, then by the residue theorem,

$$\int dk_0 t_0{}^{\sigma\tau}=f^{\sigma\tau\rho}J^\rho(p',p). \quad (80)$$

Similarly, when both vertices are of the vector type,

$$\int dk_0 t_{0\mu}{}^{\sigma\tau}=f^{\sigma\tau\rho}J_\mu{}^\rho(p',p)+k^\nu\Sigma_{\mu\nu}{}^{\sigma\tau}(p',p), \quad (81)$$

where $\Sigma_{\mu\nu}{}^{\sigma\tau}$ was defined by (76).

V. INFINITE-MOMENTUM FRAME

The time direction has played a distinguished role in the whole development. The time component of the canonical current plays a major role in defining the physical norm, and all the sums and integrals are carried out with the three-momenta fixed. The Lorentz invariance of the Lagrangian guarantees that any positive timelike direction λ^μ can be used to define the time axis: in general, sums and integrals are then to be evaluated with the transverse components of momenta kept fixed, for example[9]

$$\int dk_0 T(k_\mu,\cdots)\rightarrow\int d\nu\ T(k_\mu+\nu\lambda_\mu,\cdots).$$

Although the sum rules are Lorentz-invariant, the individual terms are not, and it is possible to suppress some contributions by a judicious choice of λ_μ and a restriction on the momentum transfer. In particular it is possible to suppress the contribution of negative-energy states by taking the limit of lightlike λ_μ.[10]

For definiteness consider the sum rule

$$\sum_{j\pm}\frac{\psi_j(q)\psi_j^\dagger(q)}{\eta_j(q)}\lambda^\mu I_\mu(q,p)\psi_{\rm fi}(p)=\psi_{\rm fi}(p), \quad (82)$$

[9] C. G. Bollini, J. J. Giambiagi, and J. Tiomno, Nuovo Cimento **51A**, 717 (1967).

[10] S. Fubini and G. Furlan, Physics 1, 229 (1964); M. Gell-Mann, Lecture Notes, International School of Physics, "Ettore Majorana," Erice, Sicily, July (1966) (unpublished). The infinite-momentum limit considered as a transformation to a lightlike coordinate system has been discussed by H. Leutwyler, presented at the VII Internationale Universitätswochen für Kernphysik, Schladming, 1968 (unpublished).

where $\psi_{\mathrm{fi}}(p)$ is a positive-energy solution of the field equation.

Let ϵ be a small positive number and put

$$\begin{aligned}\lambda^\mu&=\{1+\tfrac{1}{4}\epsilon,\quad 1-\tfrac{1}{4}\epsilon,\quad 0,\quad 0\},\\ \lambda'^\mu&=\{1-\tfrac{1}{4}\epsilon,\quad 1+\tfrac{1}{4}\epsilon,\quad 0,\quad 0\}.\end{aligned}\tag{83}$$

Then $\lambda^2=-\lambda'^2=\epsilon$, and

$$k^\mu I_\mu(q,p)=(1/\epsilon)[(\lambda\cdot k)(\lambda\cdot I)-(\lambda'\cdot k)(\lambda'\cdot I)]-(\mathbf{k}_\perp\cdot\mathbf{I}).\tag{84}$$

Putting $k=q-p$ and using current conservation, we get

$$(\lambda\cdot k)(\lambda\cdot J)-(\lambda'\cdot k)(\lambda'\cdot J)=\epsilon\mathbf{k}_\perp\cdot\mathbf{J}.\tag{85}$$

The three fixed components of momenta are $(\lambda'\cdot k)$ and $k_\perp$. We now specialize to the case $(\lambda' k)=0$, or

$$(\lambda'\cdot q)=(\lambda'\cdot p)\equiv\kappa>0.\tag{86}$$

Then (85) becomes

$$\psi_j^\dagger(q)\lambda^\mu I_\mu(q,p)\psi_{\mathrm{fi}}(p)=(\epsilon/\lambda k)\psi_j^\dagger(q)\mathbf{k}_\perp\cdot\mathbf{I}(q,p)\psi_{\mathrm{fi}}(p).\tag{87}$$

We have to determine the limits of p_μ and q_μ as ϵ tends to zero for fixed κ. If $p^2=m_{\mathrm{fi}}^2$ and $q^2=m_j^2$, then

$$\begin{aligned}(p\cdot\lambda)&=+[(p\cdot\lambda')^2+\epsilon(m_{\mathrm{fi}}^2+\mathbf{p}_\perp^2)]^{1/2}\to\kappa,\\ (q\cdot\lambda)&=\pm[(q\cdot\lambda')^2+\epsilon(m_j^2+\mathbf{q}_\perp^2)]^{1/2}\to\pm\kappa.\end{aligned}\tag{88}$$

Both p_0 and p_1 remain finite and tend to $\frac{1}{2}\kappa$. If $(q\cdot\lambda)$ tends to $+\kappa$, then q_0 and q_1 tend to the same finite limit, while $(k\cdot\lambda)$ tends to zero like ϵ. If $(q\cdot\lambda)$ tends to $-\kappa$, then q_0 and q_1 tend to infinity like ϵ^{-1}, while $(k\lambda)$ tends to 2κ. Thus

$$\begin{aligned}(\lambda J)&\approx\psi_j^\dagger(q)\mathbf{k}_\perp\cdot\mathbf{I}\psi_{\mathrm{fi}}(p),\quad (q\cdot\lambda)\approx\kappa;\\ (\lambda J)&\approx\epsilon\psi_j^\dagger(q)\mathbf{k}_\perp\cdot\mathbf{I}\psi_{\mathrm{fi}}(p),\quad (q\cdot\lambda)\approx-\mu.\end{aligned}\tag{89}$$

If $(q\cdot\lambda)\approx\kappa$ then all matrix elements remain finite, and the positive-energy solutions contribute a finite amount to the sum (82). To show that the negative-energy solutions contribute nothing in the limit is not so simple, because the components of the wave function may tend to zero or to infinity as ϵ tends to zero and q_0 and q_1 tend to infinity.

Let us normalize $\psi_j(q)$ so that the individual components remain finite; then the matrix elements in (89) are finite, and

$$\psi_j^\dagger(q)(\lambda I)\psi_{\mathrm{fi}}(p)\approx\epsilon|\mathbf{k}_\perp|,\quad (q\cdot\lambda)\approx-\kappa\tag{90}$$

for a negative-energy solution $\psi_j(q)$. It remains to consider the phase-space factor

$$\eta_j(q)=\psi_j^\dagger(q)(\lambda\cdot I)\psi_j(q),\quad (q\cdot\lambda)\approx-\kappa.\tag{91}$$

From the field equation it follows easily that:

$$\begin{aligned}\eta_j(q)&\approx 1,\quad \Gamma_{\mu\nu}\neq 0;\\ \eta_j(q)&\approx\epsilon,\quad \Gamma_{\mu\nu}=0.\end{aligned}\tag{92}$$

We have not been able to exclude the possibility that special cancellations could produce higher powers of ϵ in (90) or in (92), but no examples of such cancellations have been found.

The meaning of this is that, in the limit of lightlike λ^μ, with momentum transfers restricted to $(\lambda' k)=k_0+k_1=0$, and "external" momenta restricted to $p_0>0$, $p_1>0$, negative-energy intermediate states do not contribute to the operator products $(\lambda^\mu J_\mu{}^\sigma)(\lambda^\nu J_\nu{}^\tau)$. In second-order canonical theories, with $\Gamma_{\mu\nu}\neq 0$, the same is true of the products $J^\sigma(\lambda^\mu J_\mu{}^\tau)$ and $J_\mu{}^\sigma(\lambda^\nu J_\nu{}^\tau)$.

VI. CANONICAL FIELD QUANTIZATION

We have defined a "canonical" Lagrangian c-number field theory in Sec. II as one that has a sufficient number of solutions, and we showed that the sum rules (54), (55), and (69) result. We shall now show that these sum rules are intimately connected with the locality of the corresponding q-number field theory.

Let us distinguish positive- and negative-energy wave functions by a sign following the subscript, viz., $\psi_{j+}(q)$ and $\psi_{j-}(q)$. For each positive-energy solution we introduce an annihilation operator $a_j(q)$, and for every negative-energy solution an operator $b_j(q)$, and postulate canonical commutation or anticommutation relations;

$$[a_j(q),a_k{}^*(q')]=\delta_{jk}\eta_{j+}(q)\delta^{(3)}(\mathbf{q}-\mathbf{q}'),\tag{93}$$

$$[b_j{}^*(q),b_k(q')]=\delta_{jk}\eta_{j-}(q)\delta^{(3)}(\mathbf{q}-\mathbf{q}').\tag{94}$$

Now define a quantized field by

$$\psi(x)=\sum_j\int d^3q\, e^{iq\cdot x}\left\{\frac{1}{\eta_{j+}(q)}\psi_{j+}(q)a_j(q)+\frac{1}{\eta_{j-}(q)}\psi_{j-}(q)b_j{}^*(q)\right\}.\tag{95}$$

Then,

$$[\psi(x),\psi^\dagger(x')]=\int d^3q\, e^{i(x-x')\cdot q}\times\sum_j\left[\frac{\psi_{j+}(q)\psi_{j+}{}^\dagger(q)}{\eta_{j+}(q)}+\frac{\psi_{j-}(q)\psi_{j-}{}^\dagger(q)}{\eta_{j-}(q)}\right].\tag{96}$$

This can be evaluated when $x_0=x_0'$ by means of the sum rules (54)–(56). The physical interpretation requires that $\eta_{j+}(q)$ be positive-definite, and that $\eta_{j-}(q)$ be positive-definite (negative-definite) if the quantization is made with anticommutators (commutators). Here we compromise on positivity, as explained in the introduction. (According to a well-known theorem due to Pauli,[7] $\eta_{j-}=-\eta_{j+}$ in a finite-component field theory of integer spin fields, and $\eta_{j-}=\eta_{j+}$ in the case of half-integral spins.)

Let us consider the linear theories first. Then the sum rule (56), together with (96), gives the equal-time canonical commutator

$$[\psi(x),\psi^\dagger(x')\Gamma_0]|_{x_0=x_0'}=\delta^{(3)}(\mathbf{x}-\mathbf{x}')\tag{97}$$

and this need only be supplemented with the observation that $\psi(x)$ commutes with $\psi(x')$.

For a general second-order Lagrangian the complete set of canonical equal-time commutators is given by (96) and the three sum rules (54), (55), and (69) are given by

$$[\psi(x),\psi^\dagger(x')]|_{x_0=x_0'}=0, \tag{98}$$

$$[\psi(x),\Pi(x')]|_{x_0'=x_0}=i\delta^{(3)}(\mathbf{x}-\mathbf{x}'), \tag{99}$$

$$[\Pi(x),\Pi^\dagger(x')]|_{x_0'=x_0}=0. \tag{100}$$

Here we have defined

$$\Pi(q)=i\psi^\dagger(q)(q^\mu\Gamma_{\mu 0}+\tfrac{1}{2}\Gamma_0) \tag{101}$$

so that

$$J_0(q,p)=\Pi(q)\psi(p)+\psi^\dagger(q)\Pi^\dagger(p). \tag{102}$$

Our three sum rules may be written

$$\sum_{j\pm}\frac{\psi_j(q)\psi_j^\dagger(q)}{\eta_j(q)}=0, \quad \text{(first sum rule)}; \tag{103}$$

$$\sum_{j\pm}\frac{\psi_j(q)\Pi_j(q)}{\eta_j(q)}=1, \quad \text{(second sum rule)}; \tag{104}$$

$$\sum_{j\pm}\frac{\Pi_j^\dagger(q)\Pi_j(q)}{\eta_j(q)}=0, \quad \text{(third sum rule)}. \tag{105}$$

which emphasizes the close relationship between c-number completeness relations and q-number commutation relations. It is now easy to see that

$$\int dq_0 J_0(p',q)J_0(q,p)=J_0(p',p). \tag{106}$$

VII. CANONICAL FIELD QUANTIZATION FOR FOURTH-ORDER LAGRANGIAN

Let us return to the example of Sec. II of a canonical theory with a fourth-order Lagrangian. The current is given by (45) and (46). The time component can be written without explicit appearance of derivatives as follows:

$$J_0(q,p)=\Pi(q)\psi(p)+\Pi'(q)\psi'(p)+\text{H.c.}, \tag{107}$$

where

$$\psi'(p)=p^2\psi(p), \tag{108}$$

$$\Pi'(p)=\psi^\dagger(p)p_0, \tag{109}$$

$$\Pi(p)=\psi^\dagger(p)(p^2p_0+p^\nu\bar{C}_{0\nu}+Ap_0). \tag{110}$$

The sum rules (40) and (41), together with the analog of (69), give 10 sum rules of the type (103)–(105), the only nonvanishing sums being

$$\sum_{j\pm}\frac{\psi_j(q)\Pi_j(q)}{\eta_j(q)}=\sum_{j\pm}\frac{\psi_j'(q)\Pi_j'(q)}{\eta_j(q)}=1. \tag{111}$$

We can introduce local fields $\psi(x)$ and $\psi'(x)$, with corresponding momenta $\Pi(x)$ and $\Pi'(x)$ and verify that a complete set of canonical equal-time commutation relations are satisfied. Thus we learn that a complete canonical formalism can be developed for some quantum field theories with Lagrangians of order higher than 2.

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APPENDIX

In the preceding sections we have discussed models based on one of the simplest representations of $U(3,1)$, i.e., the representations where the wave function is a four-vector. In this Appendix we want to generalize some of the results to the representations where the wave function is a symmetric four-tensor $\psi_{\mu_1\cdots\mu_N}$ with N indices, N being an arbitrary finite positive integer.

If $U(3,1)$ is reduced along the canonical chain $U(3,1)\supset U(3)\supset U(2)\supset U(1)$, the wave functions are labeled by the integers τ, λ, η, satisfying $N\geq\tau\geq\lambda\geq\eta$. Explicitly

$$\psi_{\mu_1\cdots\mu_N}{}^{(\tau,\rho,n)}=S\binom{N}{\tau}^{1/2}\tilde{\psi}_{\mu_1\cdots\mu_\tau}{}^{(\lambda,n)}(\eta)(\eta_\mu)^{N-\tau}$$
$$=S\binom{N}{\tau}^{1/2}\binom{\tau}{\lambda}^{1/2}\binom{\lambda}{n}^{1/2}(\tilde{\eta}_{1\mu})^n(\tilde{\eta}_{2\mu})^{\lambda-n}$$
$$\times(\tilde{\eta}_{3\mu})^{\tau-\lambda}(\eta_\mu)^{N-\tau}, \tag{A1}$$

where η_μ is the four-velocity and $\tilde{\eta}_{i\mu}$ is a set of spacelike unit vectors orthogonal to η_μ. The symbol S is a symmetrization operator.

The D functions are defined by

$$D_{\tau'\lambda'n',\tau\lambda n}(\eta',\eta)$$
$$=\bar{\psi}^{(\tau',\lambda',n')\mu_1\cdots\mu_N}(\eta')\psi_{\mu_1\cdots\mu_N}{}^{(\tau,\lambda,n)}(\eta)$$
$$=\bar{\psi}^{(\tau',\lambda',n')\mu_1\cdots\mu_N}(\mathbf{0})e^{iL(\eta\to\eta')}\psi_{\mu_1\cdots\mu_N}{}^{(\tau,\lambda,n)}(\mathbf{0}). \tag{A2}$$

We may pass from η to η' by means of a pure Lorentz transformation in the plane of η' and η. Therefore we choose the direction of the z axis to lie in this plane.

From (A1) and (A2) we obtain

$$D_{\tau'\lambda'n',\tau\lambda n}(\eta',\eta)$$
$$=\delta_{\lambda'\lambda}\delta_{n'n}\binom{N-\lambda}{\tau-\lambda}^{1/2}\binom{N-\lambda}{\tau'-\lambda}^{1/2}[\cosh(\theta)]^{N-\lambda}$$
$$\times(\tanh\theta)^{\tau+\tau'-2\lambda}\,{}_2F_1(\lambda-\tau,\,\lambda-\tau',\,\lambda-N;\,-1/\sinh^2\theta), \tag{A3}$$

where $\cosh\theta=\eta\cdot\eta'$.

In a similar manner it is possible to obtain the matrix elements of the $U(3,1)$ generators $C_\mu{}^\nu$ between states of different $U(3)$ quantum numbers and different fourvelocities. For instance,

$$\langle\bar{\psi}^{(\tau',\lambda',n')}(\eta')|C_0{}^0|\psi^{(\tau,\lambda,n)}(\eta)\rangle=\frac{1}{\eta'\cdot\eta}\left[(N-\lambda)\eta_0\eta_0'+\frac{(\tau-\lambda)\eta_3'\eta_0'-(\tau-\lambda)\eta_3p_0}{\eta_3\eta_0'-\eta_3'\eta_0}\right]D_{\tau'\lambda'n',\tau\lambda n}(\eta',\eta)\,,\tag{A4}$$

$$\langle\bar{\psi}^{(\tau',\lambda',n')}(\eta')|C_0{}^3-C_3{}^0|\psi^{(\tau,\lambda,n)}(\eta)\rangle=-\frac{1}{\eta'\cdot\eta}\left[(N-\lambda)(\eta_3\eta_0'+\eta_3'\eta_0)+\frac{(\tau-\lambda)(1+2\eta_3'^2)-(\tau'-\lambda)(1+2\eta_3^2)}{\eta_3\eta_0'-\eta_3'\eta_0}\right]\times D_{\tau'\lambda'n',\tau\lambda n}(\eta',\eta)\,.\tag{A5}$$

If the spacelike vectors are chosen in an arbitrary manner with respect to $\eta'-\eta$, the preceding expressions will involve $U(3)$ rotation matrices in addition to the expressions for pure Lorentz transformations.

Formulas (A3)–(A5) allow us to write the sum rules of Sec. II in terms of the D's and to obtain a series of rather complicated addition theorems for the hypergeometric functions. In particular, for the quadratic model the sum rules of Eqs. (26) and (27) may be written

$$\sum_{\sigma,q_0=\pm(\mathbf{q}_3{}^2+m_\sigma{}^2)^{1/2}}\binom{N}{\sigma}\left[\frac{(p'\cdot q)(p\cdot q)}{m_\tau m_{\tau'}m_\sigma{}^2}\right]^{N-\sigma}\left[\frac{(p'\cdot q)^2-m_{\tau'}{}^2m_\sigma{}^2}{(p'\cdot q)^2}\right]^{\frac{1}{2}(\tau'+\sigma)}\left[\frac{(p\cdot q)^2-m_\tau{}^2m_\sigma{}^2}{(p\cdot q)^2}\right]^{\frac{1}{2}(\tau+\sigma)}$$
$$\times(-)^\sigma\,{}_2F_1\left(-\tau',\,-\sigma;\,-N;\,\frac{m_{\tau'}{}^2m_\sigma{}^2}{m_\tau{}^2m_\sigma{}^2-(p'\cdot q)^2}\right){}_2F_1\left(-\tau,\,-\sigma,\,-N;\,\frac{m_\tau{}^2m_\sigma{}^2}{m_\tau{}^2m_\sigma{}^2-(p\cdot q)^2}\right)$$
$$\times\left[A+B\left(N\frac{p_0q_0}{p\cdot q}+\frac{m_\tau m_\sigma}{p\cdot q}\,\frac{\tau q_3q_0-\sigma p_3p_0}{p_3q_0-p_3p_0}\right)\right]m_\sigma{}^2q_0{}^n$$
$$=0\quad\text{if}\quad n=0\,,$$
$$=\left(\frac{p\cdot p'}{m_\tau m_{\tau'}}\right)^N\left[\frac{(p'\cdot p)^2-m_\tau{}^2m_{\tau'}{}^2}{(p\cdot p')^2}\right]^{\tau+\tau'}{}_2F_1\left(-\tau,\,-\tau',\,-N;\,\frac{m_\tau{}^2m_{\tau'}{}^2}{m_\tau{}^2m_{\tau'}{}^2-(p\cdot p')^2}\right)\quad\text{if}\quad n=1,\tag{A6}$$

where $m_\tau{}^2=1/(A+BN-B\tau)$.

In (A6) we have considered the simple case when $\mathbf{p}$, $\mathbf{p}'$, and $\mathbf{q}$ are collinear.

For the more general case of arbitrary orientation of the 3-vectors, one obtains more complicated addition theorems including extra hypergeometric functions coming from $U(3)$ rotations.

Similar expressions may be obtained from the fourth-order model.

In general the addition theorems are rather complicated and difficult to check by direct computation. We have done it in a few particular cases. We produce the "check" for the simple case when $\mathbf{p}'=\mathbf{p}=0$ and $\tau=\tau'=0$. Then (A6) becomes

$$\sum_\sigma\binom{N}{\sigma}\left(1+\frac{\mathbf{q}^2}{m_\sigma{}^2}\right)^{N-\sigma}\left(\frac{\mathbf{q}^2}{m_\sigma{}^2}\right)(-)^\sigma\frac{m_\sigma{}^2}{m_0{}^2}=1.\tag{A7}$$

The left-hand side of (A7) can be written

$$\sum_\sigma\sum_n\binom{N}{\sigma}\binom{N-\sigma}{n}(-)^\sigma\left(\frac{\mathbf{q}^2}{m_\sigma{}^2}\right)^{\sigma+n}\frac{m_\sigma{}^2}{m_0{}^2}.$$

As both sums are of finite range (N=positive integer), we may exchange the order of summation and write

$$\sum_u\frac{N!}{(N-u)!}\left(\frac{\mathbf{q}^2}{m_0}\right)^u\sum_\sigma\frac{[1-(B/A)\sigma]^{u-1}}{\sigma!(u-\sigma)!}(-)^\sigma=1+\frac{\mathbf{q}^2}{m_0{}^2}$$
$$\times\sum_{s=0}\frac{N!}{(N-s-1)!}\left(\frac{\mathbf{q}^2}{m_0{}^2}\right)^s\sum_\sigma\frac{(1-B/A)^s(-)^\sigma}{\sigma!(1+s-\sigma)!}.$$

But

$$\sum_\sigma\frac{[1-(B/A)\sigma]^s(-)^\sigma}{\sigma!(1+s-\sigma)!}=\sum_{r=0}^{s}a_r\frac{(-)^\sigma}{\sigma!(1+s-\sigma-r)!}=0$$

as was to be shown.