Self-Consistent Regge Trajectory in a Multiperipheral Model*

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(Received 6 December 1968)

We examine the possibility of using multiparticle unitarity to write self-consistency equations for Regge trajectories. On the assumption that the dominant mechanism for inelastic processes at high energy is provided by multi-Reggeized meson exchange, and using the integral-equation approach suggested by Chew, Goldberger, and Low, the self-consistency conditions on the trajectory intercept and slope leave only one free parameter. The numerical values obtained are physically quite reasonable. It is also possible to generate the Pomeranchuk trajectory via the same dynamical mechanism. The slope of the Pomeranchuk trajectory thus obtained is similar to the slope of the meson Regge trajectory.

I. INTRODUCTION

SINCE the beginnings of Regge theory, it has been recognized that the expression of the imaginary part of the elastic amplitude as a sum of contributions from multiparticle intermediate states provides a scheme to study the high-energy behavior of two-body processes.[1] In other words, given a model for the many-particle production, it is possible to deduce high-energy properties of the elastic amplitude.

Amati, Bertocchi, Fubini, Stanghellini, and Tonin[2] used for the production amplitude a multiperipheral model with elementary-pion exchange, and were able to derive a Regge behavior for the absorptive part of the elastic scattering. More recently, a multiperipheral model in which the exchanged particles are Reggeized[3] has been considerably successful both in fitting experimental data on particle production[4] and in calculating shadow elastic scattering.[5] It has also been suggested by Chew and Pignotti (CP)[6] that the multi-Regge model, in conjunction with the unitarity relation, can be used to obtain bootstrap-like conditions involving Regge parameters. This is achieved by imposing the condition that the Regge behavior obtained for a suitably chosen two-body amplitude at high energies is controlled by the same trajectory that is exchanged along the multi-Regge line. Here we pursue this approach and succeed to impose self-consistency conditions on the value and slope of a Regge trajectory at zero momentum transfer. In this way we are able to construct a dynamical model in which there is only one free parameter.

To sum the multiperipheral graphs we use the integral-equation approach proposed by Chew, Goldberger, and Low,[7] which is a generalization of the method used in Ref. 2 to the more complicated case of multi-Regge exchanges. Because we are more interested in exploiting the restrictions imposed by the self-consistency requirements than in detailed data fitting, we try to minimize the number of parameters; therefore, we study the oversimplified model in which an average meson trajectory represents the combined effect of the various meson exchanges that we believe to be present. As a first approximation we neglect Pomeranchuk exchange, because we know that this mechanism is not important in production processes.[6] As in CP, we only consider ladder-type diagrams in which pions are produced in the intermediate state[8] and baryon exchange is neglected.

In Sec. II, we derive for completeness the integral equation for the absorptive part of the elastic scattering amplitude, with some simplifications. In Sec. III, we solve this equation, and in Sec. IV, we impose the self-consistency requirements on the meson trajectory. The generation of the Pomeranchuk trajectory is described in Sec. V. Section VI discusses the results and suggests some lines for possible future work.

II. DERIVATION OF INTEGRAL EQUATION

We consider the high-energy and small-momentum-transfer elastic scattering of two particles, which we assume for simplicity to be spinless and identical. We introduce for convenience three independent four-vectors, Q, k_0, and k_0', in terms of which the momenta of the initial particles are $Q-k_0$ and $-Q+k_0'$, and those of the final particles $-Q-k_0$ and $Q+k_0'$ (see Fig. 1). The total energy squared is $s=(k_0'-k_0)^2$ and the invariant momentum transfer is $4Q^2$. We expect the asymptotic behavior of the amplitude at fixed Q^2 to be characterized by the "t-channel" internal quantum numbers, and

* Work supported in part by the National Science Foundation.
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[3] The multi-Regge model was first proposed by T. W. B. Kibble, Phys. Rev. **131**, 2282 (1963), and K. A. Ter Martirosyan, Zh. Eksperim i Teor. Fiz. **44**, 341 (1963) [English transl.: Soviet Phys.—JETP **17**, 233 (1963)]. A more complete list of contributions to this model can be found in Ref. 6.
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[5] L. Michejda, J. Turnau, and A. Białas, Nuovo Cimento **56A**, 241 (1968).
[6] G. F. Chew and A. Pignotti, Phys. Rev. **176**, 2112 (1968); hereafter referred to as CP.
[7] W. R. Frazer, Rapporteur talk, in *Proceedings of the Fourteenth International Conference on High Energy Physics, Vienna, 1968* (CERN, Geneva, 1968), p. 415. A similar approach was independently proposed by I. G. Halliday, Imperial College, London, report ICTP/67/36.
[8] G. F. Chew and A. Pignotti, Phys. Rev. Letters **20**, 1078 (1968). In this paper it is pointed out that addition of resonance production can lead to double counting.

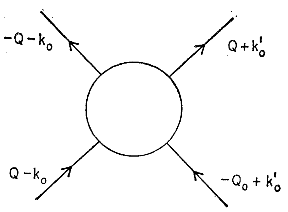

FIG. 1. Definition of momenta for the two-body amplitude.

therefore we study amplitudes in which these quantum numbers are well defined.

As indicated in the Introduction, we express the imaginary part of the two-body amplitude using the unitarity relation and summing over the contribution of intermediate states in which n pions are produced:

$$\mathrm{Im}A(s,Q^2)=\sum_n \mathrm{Im}A^n(s,Q^2)\,, \qquad (2.1)$$

with

$$\mathrm{Im}A^n(s,Q^2)=\int d\Phi^n A_{2n}A_{2'n}{}^*. \qquad (2.2)$$

Here $d\Phi^n$ is the usual phase space for the intermediate state, and A_{2n} is the n-pion production amplitude. We are going to adopt for this amplitude a Reggeized multiperipheral model, namely to consider contributions of Reggeized ladder diagrams, with the kinematics defined in Fig. 2. In this figure, wavy lines represent the exchanged Regge pole, dotted lines represent pions, and solid lines represent the external particles. With these variables the phase space can be written

$$\begin{aligned}d\Phi^n=\delta[(k_1-k_0)^2-m^2]d^4k_1\delta[(k_2-k_1)^2-\mu^2]d^4k_2\cdots\\ d^4k_n\delta[(k_{n+1}-k_n)^2-\mu^2]d^4k_{n+1}\\ \times\delta[(k_0'-k_{n+1})^2-m^2]\,,\end{aligned} \qquad (2.3)$$

where μ and m are the masses of the pion and of the external particles. In this model the amplitude A_{2n} can be written as a product of Regge factors and vertex functions (we identify k_{n+2} with k_0'),

$$\begin{aligned}A_{2n}(Q,k_0,k_1,\cdots,k_{n+1},k_0')=Z(Q,k_1)Z(Q,k_{n+1})\\ \times\prod_{i=1}^{n} z(Q,k_{i-1},k_i,k_{i+1},k_{i+2})\prod_{j=1}^{n+1} F(Q,k_{j-1},k_j,k_{j+1})\,.\end{aligned} \qquad (2.4)$$

Here $F(Q,k_{j-1},k_j,k_{j+1})$ is a function having the asymptotic Regge behavior and Regge signature factor, i.e.,

$$F(Q,k_{j-1},k_j,k_{j+1})=-\left(\frac{k_{j+1}-k_{j-1}}{\sqrt{s_0}}\right)^{2\alpha((Q-k_j)^2)} \times\left(\frac{\exp\{-i\pi\alpha((Q-k_j)^2)\}\pm1}{\sin\pi\alpha((Q-k_j)^2)}\right), \qquad (2.5)$$

where α is the Regge trajectory of the pole considered, and s_0 is a parameter introduced as usual for dimensional reasons. The functions Z are the residues for the coupling of the Regge poles to two on-mass-shell particles and the functions z describe the coupling of two Regge poles to a pion. Therefore in our model we have

$$\begin{aligned}\mathrm{Im}A^n(Q,k_0,k_0')\\ =\int Z(Q,k_1)Z(Q,\,-k_1)Z(Q,k_{n+1})Z(Q,\,-k_{n+1})\\ \times\delta[(k_1-k_0)^2-m^2]\delta[(k_0'-k_{n+1})^2-m^2]\\ \times\prod_{i=1}^{n} z(Q,k_{i-1},k_i,k_{i+1},k_{i+2})\\ \times z(Q,\,-k_{i-1},\,-k_i,\,-k_{i+1},\,-k_{i+2})\\ \times\delta[(k_{i+1}-k_i)^2-\mu^2]\prod_{j=1}^{n+1} F(Q,k_{j-1},k_j,k_{j+1})\\ \times F^*(Q,\,-k_{j-1},\,-k_j,\,-k_{j+1})d^4k_j\,.\end{aligned} \qquad (2.6)$$

The zeros of the factor $\sin\pi\alpha((Q\pm k_j)^2)$ in the denominator of Eq. (2.5) cannot produce singularities in the region we are interested in, and the vertex functions have to provide a cancellation mechanism at the right-signature points. Therefore we extract from the product

$$F(Q,k_{j-1},k_j,k_{j+1})F^*(Q,\,-k_{j-1},\,-k_j,\,-k_{j+1})$$

a factor

$$R(Q,k_{j-1},k_j,k_{j+1})=\left(\frac{k_{j+1}-k_{j-1}}{\sqrt{s_0}}\right)^{2[\alpha((Q-k_j)^2)+\alpha((Q+k_j)^2)]} \times S((Q-k_j)^2,\,(Q+k_j)^2)\,, \qquad (2.7)$$

with

$$S(x,y)=\exp\{\tfrac{1}{2}i\pi[\alpha(y)-\alpha(x)]\}\,, \qquad (2.8)$$

and absorb the remaining factors in the definition of new vertex functions G and g, which are no longer required to provide cancellation of unwanted singularities. Furthermore, we introduce some simplifying assumptions on the structure of the internal vertex functions. These depend on three variables which we can choose to be the two invariant momentum transfers of the Regge lines, and an angular variable ω.[9] We neglect this last dependence and assume that the dependence on the first two factorizes in the following way at the vertex at

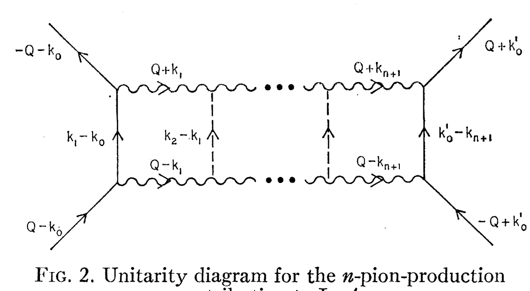

FIG. 2. Unitarity diagram for the n-pion-production contribution to ImA.

[9] N. F. Bali, G. F. Chew, and A. Pignotti, Phys. Rev. **163**, 1572 (1967).

which the ith pion is emitted:

$$g(Q,k_i,k_{i+1})\approx 2\sqrt{2}g\psi((Q-k_i)^2)\psi((Q-k_{i+1})^2). \quad (2.9)$$

The function ψ is normalized by requiring that

$$\int_{-\infty}^{0} dx\,\psi^4(x)=1. \quad (2.10)$$

With these assumptions we can write

$$\begin{aligned}\operatorname{Im}A^n(Q,k_0,k_0')&\\ =\int d^4k_{n+1}B^n(Q,k_0,k_{n+1},k_0')\delta[(k_0'-k_{n+1})^2-m^2]&\\ \times G((Q+k_{n+1})^2)G((Q-k_{n+1})^2).&\quad (2.11)\end{aligned}$$

Comparing (2.11) with Eq. (2.6), we find

$$\begin{aligned}B^0(Q,k_0,k_1,k_0')=G((Q-k_1)^2)G((Q+k_1)^2)&\\ \times\delta[(k_1-k_0)^2-m^2]R(Q,k_0,k_1,k_0'),&\quad (2.12)\end{aligned}$$

and the recursion relation

$$\begin{aligned}B^n(Q,k_0,k_{n+1},k_0')=\int d^4k_nB^{n-1}(Q,k_0,k_n,k_{n+1})&\\ \times K(Q,k_n,k_{n+1},k_0'),&\quad (2.13)\end{aligned}$$

with

$$\begin{aligned}K(Q,k,k',k'')&\\ =8g^2\psi((Q-k)^2)\psi((Q+k)^2)\psi((Q-k')^2)\psi((Q+k')^2)&\\ \times R(Q,k,k',k'')\delta[(k'-k)^2-\mu^2].&\quad (2.14)\end{aligned}$$

Of course, for a given energy of the two-body scattering process, there is a maximum number $\bar{n}$ of pions that can be produced in the intermediate state. The mechanism for the vanishing of B^n when n is bigger than $\bar{n}$ is provided by the structure of the function K. We can therefore define a function

$$B(Q,k_0,k',k'')=\sum_{n=0}^{\infty}B^n(Q,k_0,k',k''),$$

the actual contributions coming from the first $\bar{n}$ terms.

The function B satisfies the integral equation

$$\begin{aligned}B(Q,k_0,k',k'')=B_0(Q,k_0,k',k'')&\\ +\int d^4k\,K(Q,k,k',k'')B(Q,k_0,k,k'),&\quad (2.15)\end{aligned}$$

and the imaginary part of the scattering amplitude is given by

$$\begin{aligned}\operatorname{Im}A(Q,k_0,k_0')=\int d^4k\,B(Q,k_0,k,k_0')&\\ \times G((Q-k)^2)G((Q+k)^2)\delta[(k_0'-k)^2-m^2].&\quad (2.16)\end{aligned}$$

Following Chew, Goldberger, and Low,[7] we propose to use Eqs. (2.15) and (2.16) to study the high-energy behavior of $\operatorname{Im}A$.

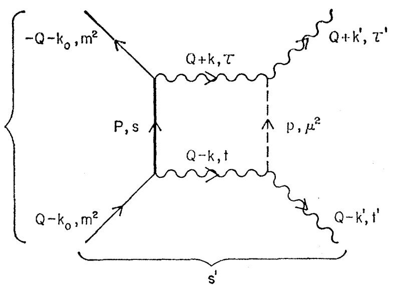

FIG. 3. Diagram showing the variables on which B depends. Next to each line the corresponding four-momentum and invariant mass squared have been indicated.

The function $B(Q,k_0,k,k')$ depends effectively on seven invariants, which we can choose to be (see Fig. 3)

$$\begin{aligned}&s=(k-k_0)^2,\quad t=(Q-k)^2,\\ &\tau=(Q+k)^2,\quad s'=(k'-k_0)^2,\\ &t'=(Q-k')^2,\quad \tau'=(Q+k')^2,\quad\text{and}\quad Q^2.\end{aligned}\quad (2.17)$$

Defining

$$P=k-k_0,\quad p=k'-k, \quad (2.18)$$

we can, in Eq. (2.15), replace

$$\int d^4k\delta[(k'-k)^2-\mu^2]\cdots,$$

with

$$\int_{m^2}^{(\sqrt{s'}-\mu)^2}ds\int d^4Pd^4p\delta(P^2-s)\delta(p^2-\mu^2)\times\delta^4(P+p-k'+k_0)\cdots, \quad (2.19)$$

which has the form of an integration over the phase space for an intermediate state of two particles of masses μ and $\sqrt{s}$, integrated over s (see Fig. 3). Because we are interested in the high-energy behavior of the solution, we neglect the pion mass with respect to the total energy and also terms of order m^2/s', t'/s', and τ'/s'. We can now go to a "center-of-mass" frame in which we have

$$P+p=(\sqrt{s'},0,0,0) \quad (2.20)$$

and define three angles Θ, θ, and φ such that

$$\begin{aligned}Q-k_0&=[\tfrac{1}{2}\sqrt{s'},\ \tfrac{1}{2}(\sqrt{s'})\sin\tfrac{1}{2}\Theta,\ 0,\ \tfrac{1}{2}(\sqrt{s'})\cos\tfrac{1}{2}\Theta],\\ -Q-k_0&=[\tfrac{1}{2}\sqrt{s'},\ -\tfrac{1}{2}(\sqrt{s'})\sin\tfrac{1}{2}\Theta,\ 0,\ \tfrac{1}{2}(\sqrt{s'})\cos\tfrac{1}{2}\Theta],\\ P&=\left[\frac{s'+s}{2\sqrt{s'}},\ -\frac{s'-s}{2\sqrt{s'}}\sin\theta\sin\varphi,\ \frac{s'-s}{2\sqrt{s'}}\sin\theta\cos\varphi,\ \frac{s'-s}{2\sqrt{s'}}\cos\theta\right].\end{aligned}\quad (2.21)$$

In the forward direction ($Q=0$) we have

$$t=\tau=\bar{t},$$

where

$$\bar{t}=-\tfrac{1}{2}(s'-s)(1-\cos\theta). \tag{2.22}$$

At $Q\neq 0$ we define two quantities ϵ and δ such that

$$\begin{aligned} t+\tau&=2\bar{t}+\epsilon,\\ t-\tau&=\delta. \end{aligned} \tag{2.23}$$

We find, for small Q^2,

$$\begin{aligned} \delta&\approx -4(1-s/s')^{1/2}(\bar{t}Q^2)^{1/2}\sin\varphi,\\ \epsilon&\approx 2(1-s/s')Q^2. \end{aligned} \tag{2.24}$$

We can now write the integration in (2.19) as

$$\int_{m^2}^{s'}\frac{ds}{8s'}\int_{s-s'}^{0}d\bar{t}\int_0^{2\pi}\frac{d\varphi}{2\pi}\cdots. \tag{2.25}$$

At this stage we achieve an important simplification by the replacement of the factor $(k_{j+1}-k_{j-1})^2/s_0$ in the Regge propagator of Eq. (2.5) with[10]

$$(k_{j+1}-k_0)^2/(k_j-k_0)^2. \tag{2.26}$$

This substitution is valid in the limit in which all the subenergies of pairs of adjacent particles, $s_i=(k_{i+1}-k_{i-1})^2$, are large compared with the masses squared and momentum transfers involved, and is equivalent to the assumptions leading to Eq. (B10) of CP.

We can now rewrite Eq. (2.15) as

$$\begin{aligned} &B(Q^2,t',\tau',s',s'')\\ &=G(t')G(\tau')\delta(s'-m^2)(s''/s')^{\alpha(t')+\alpha(\tau')}S(t',\tau')\\ &\quad+g^2\int_{m^2}^{s'}\frac{ds}{s'}\int_{-\infty}^{0}d\bar{t}\int_0^{2\pi}\frac{d\varphi}{2\pi}\psi(t)\psi(\tau)\psi(t')\psi(\tau')\\ &\quad\times\left(\frac{s''}{s'}\right)^{\alpha(t')+\alpha(\tau')}S(t',\tau')B(Q^2,t,\tau,s,s'). \end{aligned} \tag{2.27}$$

Because neither the kernel nor the inhomogeneous term depend on $t''=(Q-k'')^2$ and $\tau''=(Q+k'')^2$ in the approximation used, we have dropped these arguments from the function B. We have also extended the range of integration in $\bar{t}$ to $-\infty$; if the vertex function $\psi(t)$ decreases rapidly (e.g., exponentially) with negative t, this amounts to corrections of order $1/s''$.

III. SOLUTION OF THE INTEGRAL EQUATION FOR SMALL Q^2

We can substitute in Eq. (2.27)

$$\begin{aligned} &B(Q^2,t',\tau',s',s'')\\ &=(s''/s')^{\alpha(t')+\alpha(\tau')}S(t',\tau')[g^2b(Q^2,(s'/m^2))\psi(t')\psi(\tau')\\ &\qquad+G(t')G(\tau')\delta(s'-m^2)], \end{aligned} \tag{3.1}$$

and obtain an integral equation for the function $b(Q^2,x')$:

$$\begin{aligned} b(Q^2,x')&=\frac{1}{m^2}\int_{-\infty}^{0}d\bar{t}\int_0^{2\pi}\frac{d\varphi}{2\pi}G(t)G(\tau)\psi(t)\psi(\tau)\\ &\quad\times(x')^{\alpha(t)+\alpha(\tau)-1}S(t,\tau)+g^2\int_1^{x'}\frac{dx}{x'}\int_{-\infty}^{0}d\bar{t}\int_0^{2\pi}\frac{d\varphi}{2\pi}\\ &\quad\times\psi^2(t)\psi^2(\tau)\left(\frac{x'}{x}\right)^{\alpha(t)+\alpha(\tau)}S(t,\tau)b(Q^2,x). \end{aligned} \tag{3.2}$$

Expanding this integral equation to first order in Q^2 and performing the $d\varphi$ integration, we obtain

$$\begin{aligned} b(Q^2,x')&=\frac{1}{m^2}\int_{-\infty}^{0}d\bar{t}\,G^2(\bar{t})\psi^2(\bar{t})\\ &\quad\times(x')^{2[\alpha_M+\alpha_M'\bar{t}+\alpha_M'Q^2(x'-1)/x']-1}\\ &\quad\times[1+2((x'-1)/x')Q^2D(\bar{t},G,\psi)]\\ &\quad+g^2\int_1^{x'}\frac{dx}{x'}\int_{-\infty}^{0}d\bar{t}\,\psi^4(\bar{t})\left[1+2\left(\frac{x'-x}{x'}\right)Q^2D(\bar{t},\psi,\psi)\right]\\ &\quad\times\left(\frac{x'}{x}\right)^{2[\alpha_M+\alpha_M'\bar{t}+\alpha_M'Q^2(x'-x)/x']}b(Q^2,x). \end{aligned} \tag{3.3}$$

We have assumed here a linear trajectory $\alpha(t)=\alpha_M+\alpha_M't$ and we have used the notation

$$D(\bar{t},\varphi_1,\varphi_2)=D^1[\varphi_1(\bar{t})\varphi_2(\bar{t})]+\bar{t}(D^2[\varphi_1(\bar{t})\varphi_2(\bar{t})]-\tfrac{1}{2}\pi^2\alpha_M'^2)$$

with

$$D^n[F(\bar{t})]=(d^n/dt^n)\ln F(t)|_{t=\bar{t}}. \tag{3.4}$$

We look for a solution of Eq. (3.3) of the form of a Mellin transform:

$$b(Q^2,x)=\frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty}\tilde{b}(Q^2,\beta)x^\beta d\beta, \tag{3.5}$$

where C is chosen to leave all singularities of $\tilde{b}(Q^2,\beta)$ to the left of the path of integration, and we assume (and verify *a posteriori*) that $\tilde{b}(Q^2,\beta)$ vanishes at least as fast as $[\ln(\beta)]^{-2}$ for $|\beta|\to\infty$. It follows that the singularities of $\tilde{b}(Q^2,\beta)$ control the behavior of $b(Q^2,x)$ for large x.

Introducing the notation

$$H(x,F(x'))=\int_{-\infty}^{0}\frac{F(x')}{x-x'}dx', \tag{3.6}$$

we define the following functions:

$$\begin{aligned} U(Q^2,\beta)&=(1/2\alpha_M')H(y(Q^2,\beta),\psi^4(x')),\\ V(Q^2,\beta)&=(1/2\alpha_M')H(y(Q^2,\beta),G^2(x')\psi^2(x')),\\ W(Q^2,\beta)&=(1/2\alpha_M')H(y(Q^2,\beta),G^4(x')),\\ U_1(Q^2,\beta)&=(1/2\alpha_M')H(y(Q^2,\beta),\psi^4(x')D(x',\psi,\psi)),\\ V_1(Q^2,\beta)&=(1/2\alpha_M')H(y(Q^2,\beta),G^2(x')\psi^2(x')D(x',G,\psi)),\\ W_1(Q^2,\beta)&=(1/2\alpha_M')H(y(Q^2,\beta),G^4(x')D(x',G,G)), \end{aligned} \tag{3.7}$$

[10] The choice of the form (2.26) implies a choice for s_0, which in turn defines unambiguously the vertex functions.

where

$$y(Q^2,\beta)=(1/2\alpha_M')(\beta-2\alpha_M-2\alpha_M'Q^2+1). \quad (3.8)$$

We also define

$$\bar{U}(Q^2,\beta)=U(Q^2,\beta)-U(Q^2,\beta+1)+U(0,\beta+1) +2Q^2[U_1(Q^2,\beta)-U_1(Q^2,\beta+1)] \quad (3.9)$$

with analogous definitions for $\bar{V}(Q^2,\beta)$ and $\bar{W}(Q^2,\beta)$. By construction, these functions are analytic in β with cuts running from $-\infty$ to $2\alpha_M+2\alpha_M'Q^2-1$.

Substituting (3.5) in (3.3) we obtain, after some manipulation, the following solution for $\tilde{b}(Q^2,\beta)$[11]:

$$\tilde{b}(Q^2,\beta)=\frac{1}{m^2}\frac{\bar{V}(Q^2,\beta)}{1-g^2\bar{U}(Q^2,\beta)}. \quad (3.10)$$

Using now (3.10), (3.1), and (3.16), we can write the Mellin representation for the imaginary part of the amplitude:

$$\mathrm{Im}A(s,Q^2)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\int\frac{d\beta}{m^2}\left(\frac{s}{m^2}\right)^\beta \times\left(\frac{\bar{W}(Q^2,\beta)-g^2\{\bar{W}(Q^2,\beta)\bar{U}(Q^2,\beta)-[\bar{V}(Q^2,\beta)]^2\}}{1-g^2\bar{U}(Q^2,\beta)}\right). \quad (3.11)$$

Translating Eq. (3.11) into the usual Regge language, we can say that the asymptotic behavior of $\mathrm{Im}A(Q^2,s)$ is given by a Regge pole, the position of which, $\alpha_{\mathrm{out}}(Q^2)$, is defined by

$$\bar{U}(Q^2,\alpha_{\mathrm{out}}(Q^2))=1/g^2, \quad (3.12)$$

and a Regge cut starting at $2(\alpha_M+\alpha_M'Q^2)-1$. The slope of the trajectory at $Q^2=0$ is

$$\alpha_{\mathrm{out}}'=-\frac{1}{4}\frac{\partial\bar{U}(Q^2,\beta)/\partial Q^2}{\partial\bar{U}(Q^2,\beta)/\partial\beta}\bigg|_{\beta=\alpha_{\mathrm{out}},\,Q^2=0}, \quad (3.13)$$

where we have put

$$\alpha_{\mathrm{out}}(Q^2)=\alpha_{\mathrm{out}}+4\alpha_{\mathrm{out}}'Q^2.$$

IV. SELF-CONSISTENCY CONDITIONS ON THE MESON TRAJECTORY

In this section we consider the amplitude corresponding to the exchange of the quantum numbers of our meson. Therefore we require for the absorptive part of the amplitude a behavior $s^{\alpha_M+4\alpha_M'Q^2}$ at large s.

The equations in Sec. III are considerably simpler if we assume an exponential parametrization for the vertex function ψ, i.e.,

$$\psi(t)=(\sqrt[4]{a})e^{at/4}. \quad (4.1a)$$

We also use the same parametrization for $G(t)$:

$$G(t)=G(\sqrt[4]{a})e^{at/4}, \quad (4.1b)$$

although this choice of the coupling to the external particles is of no consequence on the output Regge trajectory [see Eq. (3.10)]. With these assumptions, we have

$$\bar{W}(Q^2,\beta)=G^4\bar{U}(Q^2,\beta),\quad \bar{V}(Q^2,\beta)=G^2\bar{U}(Q^2,\beta), \quad (4.2)$$

and

$$\bar{U}(Q^2,\beta)=u(Q^2,\beta)-u(Q^2,\beta+1)+u(0,\beta+1),$$

with

$$u(Q^2,\beta)=v(Q^2,\beta) \times[1+aQ^2-\tfrac{1}{2}Q^2\pi^2\alpha_M'(\beta-2\alpha_M+1-2\alpha_M'Q^2)]. \quad (4.3)$$

Here v is simply given in terms of the confluent hypergeometric function Ψ by[12]:

$$v(Q^2,\beta)=\frac{a}{2\alpha_M'}\Psi\left(1,1,\frac{a}{2\alpha_M'}(\beta-2\alpha_M+1)-aQ^2\right). \quad (4.4)$$

The slope of the output Regge trajectory at $Q^2=0$ is

$$\alpha_{\mathrm{out}}'=\tfrac{1}{4}a\left[\frac{1}{(\alpha_{\mathrm{out}}-2\alpha_M+1)(\alpha_{\mathrm{out}}-2\alpha_M+2)}+\frac{\pi^2\alpha_M'}{2a}\left((\alpha_{\mathrm{out}}-2\alpha_M+2)v(\alpha_{\mathrm{out}}+1,0)-\frac{\alpha_{\mathrm{out}}-2\alpha_M+1}{g^2}\right)\right]\Bigg/\left[\frac{a}{2\alpha_M'}\left(\frac{1}{\alpha_{\mathrm{out}}-2\alpha_M+1}-\frac{1}{g^2}\right)\right]. \quad (4.5)$$

Defining $\rho_M=2\alpha_M'/a$ and $\rho_{\mathrm{out}}=2\alpha_{\mathrm{out}}'/a$, we see that α_{out} and ρ_{out} depend on ρ_M, but not on α_M' and a separately.

In the limit $\rho_M\to 0$, the function $v(0,\beta)$ becomes

$$v(0,\beta)\xrightarrow[\rho_M\to 0]{}\frac{1}{\beta-2\alpha_M+1}-\frac{\rho_M}{(\beta-2\alpha_M+1)^2}+O(\rho_M^2), \quad (4.6)$$

and we find again the results of CP in which no Regge cut is present. By an opportune choice of g^2 we can satisfy the self-consistency condition on the intercept of the meson trajectory, namely $\alpha_{\mathrm{out}}=\alpha_M$. We observe, however, from Eqs. (4.5) and (4.6) that

$$\lim_{\rho_M\to 0}\rho_{\mathrm{out}}=[g^2/(1+g^2)],$$

[11] A similar result at $Q^2=0$ was obtained by G. F. Chew, C. de Tar, P. Ting, and A. Pignotti (unpublished).

[12] *Bateman Manuscript Project, Higher Transcendental Functions* (McGraw-Hill Book Co., Inc., New York, 1953), Vol. 1, p. 255.

TABLE I. Some numerical solutions of the self-consistency equations. The various quantities are defined in the text.

α_M	g^2	ρ_M	α_P'/α_M'	g_M^2/g^2	A	B
0.4	0.96	0.52	1.19	1.67	1.58	0.16
0.45	0.88	0.48	1.18	1.67	1.44	0.09
0.5	0.81	0.45	1.18	1.66	1.27	0.05
0.55	0.75	0.42	1.18	1.66	1.11	−0.07
0.6	0.66	0.38	1.18	1.65	1.00	−0.16

and therefore this limit does not correspond to a self-consistent meson generation. This happens because the t dependence of the vertex functions produces an output Regge slope, even in the absence of an input one. We therefore look for self-consistent solutions satisfying $\alpha_{\rm out}=\alpha_M$ and $\rho_{\rm out}=\rho_M$. These conditions force our three parameters α_M, ρ_M, and g^2 to satisfy two nonlinear constraints. We find that upon choosing, for instance, reasonable values for α_M, a solution exists, and the other two parameters determined by the self-consistency equations are also quite reasonable. (See Table I, columns 1–3.)

Indeed, if we take $\alpha_M'=1/\text{GeV}^2$, the range of values obtained for ρ corresponds to a width $1/a$ in the internal momentum transfer distribution between 0.2 and 0.25 GeV^2. The parameter g^2 gives good results when related to the average number of particles produced. (See the discussion at the end of Sec. V).

In the solutions obtained, ρ_M being different from zero, we have both a Regge pole at $\beta=\alpha_M$ and a cut starting at $\beta_c=2\alpha_M-1$. The function $v(0,\beta)$ is positive for $\beta>\beta_c$, has a logarithmic singularity at $\beta=\beta_c$, and goes to 0 as $\beta\to\infty$. Therefore it is guaranteed that a pole exists and that it is located to the right of the cut, and this implies that the intercept of the self-consistent meson must be smaller than 1.[13] The high-energy behavior of ImA in the forward direction can be written[11]

$$\frac{m^2}{G^4}\,\mathrm{Im}A(0,s)=\frac{v(0,\alpha_M)}{-g^2[\partial v(0,\beta)/\partial\beta]\big|_{\beta=\alpha_M}}\left(\frac{s}{m^2}\right)^{\alpha_M}+\int_{-\infty}^{2\alpha_M-1}\frac{d\beta}{\rho}\left(\frac{s}{m^2}\right)^{\beta}\times\frac{e^{(\beta-2\alpha_M+1)/\rho}}{[1-g^2\,\mathrm{Re}\,v(0,\beta)]^2+(g^2\pi/\rho)^2e^{2(\beta-2\alpha_M+1)/\rho}}. \quad (4.7)$$

The residue of the pole, i.e., the coefficient of the term $(s/m^2)^{\alpha_M}$, is approximately 0.89. The discontinuity across the cut has a logarithmic zero at the upper branch point and is a rapidly decreasing function of β; its width is the order of $2\rho\approx0.9$. The integrated discontinuity of the cut is 0.11, and we see that the summation has depressed it with respect to the "Born term" [in which the denominator in the integrand of Eq. (4.7) is 1] by a factor of 9.

[13] A similar argument in models with repeated Pomeranchuk exchange leads to violation of unitarity if the Pomeranchuk trajectory has intercept strictly equal to one, unless a zero in the vertex functions avoids the logarithmic singularity. See I. A. Verdiev, O. V. Kancheli, S. G. Matinyan, A. M. Popova, and K. A. Ter-Martirosyan, Zh. Eksperim i Teor. Fiz **46**, 1700 (1964) [English transl.: Soviet Phys.—JETP **19**, 1148 (1964)]; J. Finkelstein and K. Kajantie, Phys. Letters **26B**, 305 (1968).

V. GENERATION OF THE POMERANCHUK TRAJECTORY

Having achieved self-consistency for the meson trajectory, we want to examine in our formalism the possibility of generating the Pomeranchuk pole via the same dynamical mechanism, i.e., multi-Regge meson exchange. In Sec. V, we have evaluated the contribution of ladders of the type of Fig. 2 to the amplitude with the t-channel quantum numbers of our self-consistent meson. The same kind of ladders also contribute to the amplitude for the exchange of the quantum numbers of the vacuum. In specific models, the relation between these two amplitudes can be easily expressed in terms of Clebsch-Gordan coefficients, which effectively only redefine the coupling constants. For instance, in a model in which our building meson has isospin 1 (e.g., alternate exchange of degenerate ρ and A_2 poles), we have to replace g^2 by $g_M^2=2g^2$ in the vacuum exchange amplitude. On the other hand, if the dominant mechanism is alternate $I=0$ and $I=1$ exchanges (such as degenerate ω and ρ, or P' and A_2 trajectories), the substitution leading from the $I=1$ exchange to the vacuum exchange amplitude is $g^2\to g_M^2=\sqrt{3}g^2$. Of course, any one of these models is oversimplified, and we adopt the attitude of determining this model-dependent ratio $\gamma=g_M^2/g^2$ by requiring an intercept for the Pomeranchuk trajectory equal or very close to 1. Thus, γ is given by the equation

$$\gamma=v(0,\alpha_M)/v(0,\alpha_P). \quad (5.1)$$

The values of γ for the set of solutions obtained in Sec. IV are listed in Table I, and turn out to be very stable around the value 1.66. This value does not change appreciably if we impose an intercept for the Pomeranchuk trajectory slightly below one. From Eq. (4.5) we can now compute the ratio α_P'/α_M', where α_P' is the slope of the Pomeranchuk trajectory. This ratio is also stable, and its value is slightly larger than 1 (see Table I).

We can compute from this model the average number of particles produced, using the optical theorem and the relation

$$\bar{n}(s)=\frac{\sum n\,\sigma^n(s)}{\sigma^{\rm tot}(s)}=g_M^2\frac{\partial}{\partial g_M^2}\,ln\sigma^{\rm tot}(s). \quad (5.2)$$

We obtain $\bar{n}(s)=A\ln(s/m^2)+B$, with

$$A=\rho/(R-1),\qquad B=[\rho R^2-g_M^2(R-1)]/[g_M^2(R-1)^2]-2, \quad (5.3)$$

where

$$R=\tfrac{1}{2}g_M^2/(1-\alpha_M). \quad (5.4)$$

The values of A and B in our solutions are also in Table I. The logarithmic growth of multiplicity with energy is a common feature of multiperipheral models and is compatible with cosmic-ray data.[14] The predicted values of A and B listed in Table I compare well with experiment, the data from cosmic rays slightly preferring solutions of lower α_M than the accelerator experiments, in both cases within the range considered in Table I.

VI. CONCLUSION

We have examined in this paper the possibilities of the bootstrapping of Regge poles suggested in CP. The essential ingredients in this approach are multiparticle unitarity and the assumption that the multi-Regge model gives a satisfactory representation for the production amplitude, not only in the multi-Regge asymptotic region but throughout phase space.

With these assumptions we have found some reasonable solutions in which the meson trajectory satisfies self-consistency conditions. We have indicated how the Pomeranchuk pole can also be generated by multiple meson exchange. Not surprisingly, the slope obtained in this way for the Pomeranchuk trajectory is similar to the meson slope. Present experimental data seem to favor a flatter Pomeranchuk trajectory.[15] We know, however, that secondary trajectories are still important at present energies, and that therefore it is not easy to isolate the Pomeranchuk contribution.

Our bootstrap is not complete in many senses. In particular, in the approximation in which the Pomeranchuk trajectory is not exchanged, we cannot force it to obey self-consistency conditions. To consider also Pomeranchuk exchange, it is necessary to solve a coupled-channel problem, and this is possible, but is more complicated and involves more parameters.

Whereas here we have kept only the channel in which mesons are exchanged, Goldberger and Low[16] have suggested the alternative point of view of considering that multiple Pomeranchuk exchange, even though responsible for a very small fraction of the observed multiplicity, is still the dominant mechanism for the generation of the Pomeranchuk trajectory itself. As a result, they obtain a solution in which there are three small numbers of the same order of magnitude; the Pomeranchuk-Pomeranchuk-meson coupling constant squared, the slope of the Pomeranchuk trajectory, and the distance from 1 of the intercept of the Pomeranchuk trajectory.[17] If one takes this attitude, one may also have to keep Pomeranchuk exchange in the diagrams generating the meson trajectory, instead of the mechanism considered here.

The solution of the more complicated problem involving coupled channels, and a good knowledge of some quantities like the intercept of the Pomeranchuk trajectory and its coupling constant in inelastic processes, may clarify the relative importance of the various channels, and therefore the validity of these two complementary points of view. For the time being, we find our approach more appealing, because in it the high-energy behavior of the total cross section is intimately related to the dominant mechanism for particle production.

ACKNOWLEDGMENTS

The underlying ideas in this work have been repeatedly discussed in the past with Professor G. Chew, to whom we are deeply indebted. We are also very grateful to him for communication of details of his work with Goldberger and Low prior to publication. C. de Tar and P. Ting also participated in early discussions of this approach, and we want to acknowledge their extremely helpful contribution. We also thank R. Haymaker for useful discussions.

[14] M. Koshiba, Rapporteur's talk at the Tenth International Conference on Cosmic Rays, Denver, 1967 (unpublished).

[15] See, for example, W. Rarita, R. J. Riddell, Jr., C. B. Chiu, and R. J. N. Phillips, Phys. Rev. **165**, 1615 (1968).

[16] F. Low (private communication).

[17] A result of this type can formally be reproduced in our solutions for a self-consistent trajectory (Sec. IV), if we explore the neighborhood of $\alpha_M=1$. We find $g^2\approx2(1-\alpha_M)$ and $\rho_M\approx1.64\times(1-\alpha_M)$.