Feynman Rules and Quantum Electrodynamics at Infinite Momentum

SHAU-JIN CHANG* AND SHANG-KENG MA†
The Institute for Advanced Study, Princeton, New Jersey 08540
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We have studied the Feynman rules in terms of the new variables $s=p^0-p^3$, $\eta=p^0+p^3$, and $\mathbf{q}=(p^1,p^2)$ in the ϕ^3 model and in quantum electrodynamics. The connection between the new variables and the dynamics at infinite momentum is established. In the ϕ^3 model, one easily deduces Weinberg's rules at infinite momentum upon integrating over the s variables in the propagators without taking the $p^3\to\infty$ limit. The new Feynman rules lead to much simpler calculation of the second-order self-energies and the magnetic moment in quantum electrodynamics. It is still unclear if there is advantage in computing higher-order terms in quantum electrodynamics with the new rules.

I. INTRODUCTION

RECENTLY, there appears to be increasing interest in analyzing the structure of Feynman diagrams in the reference frame in which the momenta of the particles approach infinity. This interest is mainly motivated by the recent development in the current algebra.[1] It is suggested that the Fubini–Dashen–Gell-Mann sum rules obtained by sandwiching the local commutator of two current densities between hadron states with infinite momentum may be used as an alternative framework to handle the strong interactions. The advantages for introducing the infinite-momentum frames are[2]: (i) The disconnected diagrams and the pair creation and annihilation from the vacuum are suppressed; (ii) all intermediate states will have the same infinite momentum, and consequently, we may saturate the current commutator by states of the same infinite momentum; (iii) the operator current algebra gives rise to a simpler matrix algebra of form factors. Since the properties of the infinite-momentum frame is crucial in this analysis, it is important to study the Feynman rules in this special frame.

Weinberg made important advances by examining the infinite-momentum limit of the old-fashioned perturbation theory.[3] He showed that in the infinite-momentum frame, the contribution of the old-fashioned diagrams, at least in the ϕ^3 or ϕ^4 coupling theory, either vanishes or tends to a finite limit. He has also shown that those diagrams which drop out at the infinite-momentum limit are precisely those diagrams that cause most trouble. These diagrams include all processes which involve the creation and annihilation of particles from the vacuum.

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† On leave of absence from the University of California, San Diego, La Jolla, Calif.
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[2] See, for example, S. L. Adler and R. Dashen, *Current Algebras* (W. A. Benjamin, Inc., New York, 1968).
[3] S. Weinberg, Phys. Rev. **150**, 1313 (1966). See also L. Susskind and G. Frye, *ibid.* **165**, 1535 (1968); **165**, 1547 (1968); **165**, 1553 (1968); K. Bardakci and M. B. Halpern, *ibid.* **176**, 1686 (1968). A related formulation was also studied by P. A. M. Dirac, Rev. Mod. Phys. **21**, 392 (1949), in his "front form" of relativistic dynamics. The authors wish to thank Professor C. G. Itzykson for bringing their attention to this paper.

In this paper, we shall give an alternative derivation of Weinberg's result without actually going to the infinite-momentum limit. We introduce, for every momentum p^μ, a set of new variables $\eta=p^0+p^3$, $s=p^0-p^3$, and $\mathbf{q}=\mathbf{p}_\perp$. This is equivalent to a rotation in the 0-3 plane. We then obtain a new set of Feynman rules by expressing the usual covariant Feynman rules in terms of these new variables. As we shall see, Weinberg's infinite-momentum rules can be recovered after the s integrations. Since there is no limiting process involved in the derivation, the validity of the new rules can be justified step by step.

By introducing a new time variable τ conjugate to s, and after a Fourier transformation, we have a clear physical picture that all particles with $\eta>0$ move forward in τ, and those with $\eta<0$ move backward in τ. This variable τ, apart from an irrelevant infinite constant, can be identified as the usual time variable at the infinite-momentum frame. There is then a simple and intuitive reason why certain diagrams must drop out at infinite momentum. We have applied the new rules to quantum electrodynamics (QED). The calculations based on the new rules are simpler and lead to correct results for all the second-order self-energy diagrams and for the lowest-order anomalous magnetic moment. However, it seems to us that there are complications in higher-order diagrams to which the applicability of the new technique is still uncertain.

The order of this paper is as follows: In Sec. II, we give a simple derivation and discussion of the new Feynman rules. In Sec. III, the connection between our derivation and the infinite-momentum frame is established. In Secs. IV and V, we apply our technique to QED and carry out some second-order calculations explicitly. The difficulties in calculating higher-order diagrams are discussed in Sec. VI.

II. FEYNMAN RULES

In this section, we shall obtain a set of new rules from the usual Feynman rules by a simple change of variables. Some simple features of the new rules are studied. We shall restrict our discussion in this section to a ϕ^3 model.

A. New Rules

Let us introduce the new variables η, s, and $\mathbf{q}$ defined by

$$\eta = p^0 + p^3, \quad s = p^0 - p^3, \quad \mathbf{q} = (p^1, p^2), \tag{2.1}$$

where p^μ is the four-momentum of a single particle. These new variables play an essential role in our new rules.

Our new rules are simply the usual Feynman rules written in terms of the new variables $(s,\eta,\mathbf{q})$. The propagator has the form $G = i\Delta_F$,

$$G(p) = i(\eta s - \mathbf{q}^2 - m^2 + i\epsilon)^{-1}, \tag{2.2}$$

since $p^\mu p_\mu = \eta s - \mathbf{q}^2$. The momentum-space integral becomes

$$\int d^4p = \tfrac{1}{2} \int d^2q\, d\eta ds. \tag{2.3}$$

The energy-momentum conservation at each vertex becomes the conservation of s, η, and $\mathbf{q}$.

Although the new rules are identical to the old ones except for a 45° rotation of the p^0 and p^3 axes, they make the practical calculation very different. As a simple example, consider the self-energy diagram shown in Fig. 1(a). Taking $p = (s,\eta,0)$, we have

$$\Sigma(p) = i\tfrac{1}{2}(-ig)^2(2\pi)^{-4} \int d^2q' d\eta' ds' G(p') G(p'+p). \tag{2.4}$$

Using (2.2), the s' integral is

$$[\eta'(\eta+\eta')]^{-1} \int ds' \left[s' - \frac{1}{\eta'}(m^2 + \mathbf{q}'^2 - i\epsilon) \right]^{-1} \times \left[s + s' - \frac{1}{\eta+\eta'}(m^2 + \mathbf{q}'^2 - i\epsilon) \right]^{-1}. \tag{2.5}$$

Suppose $\eta > 0$. Clearly, if $\eta' > 0$, both of the poles of the integrand are below the real axis, and the integral therefore vanishes. For $\eta' < -\eta$, both poles are above the real axis, and the integral again vanishes. It is nonvanishing only when

$$0 < -\eta' < \eta, \tag{2.6}$$

which sets the limits for the η' integration. If $\eta < 0$, the same argument leads to $0 < \eta' < |\eta|$. The fact that η' has a *finite* range after the s' integral is an outstanding feature of the new rules. Performing the s' integration, one obtains

$$\Sigma(p) = -\tfrac{1}{2} g^2 (2\pi)^{-3} \int d^2q' \int d\eta' [\theta(\eta+\eta') - \theta(\eta')] \times [\eta'(\eta+\eta')s + \eta(\mathbf{q}'^2 + m^2 - i\epsilon)]^{-1}. \tag{2.7}$$

If we set $\eta = 1$, this expression becomes identical to the expression for $M(s)$ above Eq. (30) of Ref. 3, i.e.,

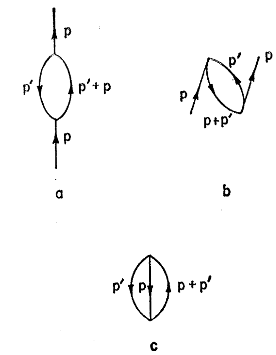

FIG. 1. (a) and (b) Second-order self-energy in two different τ orderings. (c) Lowest-order vacuum diagram. τ axis points upward.

$$\tfrac{1}{2} g^2 (2\pi)^{-3} \int d^2q \int_0^1 d\alpha [s\alpha(1-\alpha) - \mathbf{q}^2 - m^2 + i\epsilon]^{-1}, \tag{2.8}$$

where $\alpha = -\eta'$. Equation (2.8) is derived in Ref. 3 to illustrate the advantage of the diagram rules based on the old-fashioned perturbation theory with all particles having infinite momenta. The variable α, which is shown in Ref. 3 to be just the Feynman parameter of combining denominators, may now be related directly to $\eta = p^0 + p^3$. It seems a bit surprising that a simple change of variables has the advantage, which the infinite-momentum rules of Ref. 3 has, of bypassing some complicated steps of combining denominators. In the following few paragraphs, we shall show that our new rules are *already* the rules at infinite momentum in the sense that the rules derived in Ref. 3, plus some corrections, follow immediately from our rules without taking any limit of the form $p^3 \to \infty$.

B. Ordered Diagrams

We define the new time variable τ conjugate to s by

$$\tau = \tfrac{1}{2}(t+z),$$

and the propagator in the τ representation by

$$\begin{aligned} G(\tau,\eta,\mathbf{q}) &= \int \frac{ds}{2\pi} G(s,\eta,\mathbf{q}) e^{-i\tau s} = \theta(\eta\tau)|\eta|^{-1} e^{-i\tau(\mathbf{q}^2+m^2)/\eta}, && \text{for } \eta \neq 0 \\ &= -i(m^2+\mathbf{q}^2)^{-1}\delta(\tau), && \text{for } \eta = 0. \end{aligned} \tag{2.9}$$

Apart from the case $\eta = 0$, which will be shown later to be important only for vacuum diagrams, (2.9) shows that for $\eta > 0$, $G(\tau,\eta,\mathbf{q})$ is nonzero only when $\tau > 0$, and for $\eta < 0$, only when $\tau < 0$. If we call the quantity $(\mathbf{q}^2 + m^2)/\eta$ in (2.9) the "single-particle energy," then (2.9) says that positive-energy states propagate forward in τ and negative-energy states propagate backward. The latter may also be regarded as an antiparticle (which is the same as a particle in this case) with energy $(\mathbf{q}^2 + m^2)/|\eta|$ propagating forward in τ.

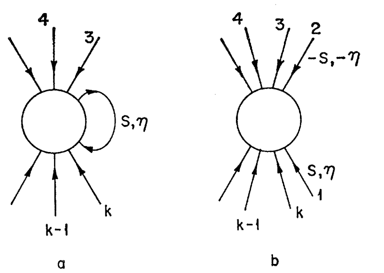

FIG. 2. (a) General diagram in the ϕ^3 model. (s,η) labels an arbitrary internal line. (b) Diagram (a) with (s,η) pulled out.

By a Fourier transformation, we may use $G(\tau,\eta,\mathbf{q})$ for the propagators in a Feynman diagram, which can now be interpreted as a set of $n!$ τ-ordered sequences of interactions. n is the order of the diagram. In a given ordering and between any two interactions, the intermediate state γ has a total energy

$$S_\gamma=\sum_k (\mathbf{q}_k^2+m^2)/|\eta_k|\,, \qquad (2.10)$$

where the sum is taken over all the particles in the intermediate state. Now the rules of the old-fashioned perturbation theory say that, for each intermediate state γ, one should write down an energy denominator $(S_\alpha-S_\gamma+i\epsilon)^{-1}$ and sum over γ, where α denotes the initial state. The η-conservation law can be written as

$$\sum_k |\eta_k| = \text{constant}, \qquad (2.11)$$

as one passes from one intermediate state to the next. The constant in (2.11) is determined by the initial state, of course.

What we have just obtained is the prescription of calculating the contribution of a given τ-ordered diagram analogous to that of calculating a time-ordered diagram in the old-fashioned perturbation theory. This prescription is exactly that derived by Weinberg in the infinite-momentum frame, which we shall discuss later in detail in Sec. III.

C. Bounds of the η Variables and Vacuum Diagrams

The fact that η is conserved at each vertex, and that a line with $\eta>0$ ($\eta<0$) must point forward (backward) in τ, enables one to find the range of the η variables by simply inspecting the diagram. For example, in Fig. 1(a) we must have $\eta'<0$ and $\eta'+\eta>0$ for $\eta>0$. If $\eta<0$, the two external lines must point downward, and the whole diagram must be turned upside down implying $\eta'>0$, $\eta'+\eta<0$. The diagram shown in Fig. 1(b) is forbidden, since it violates the η conservation at the vertices.

It also follows that all vacuum diagrams should vanish. This conclusion needs modification, however, due to the contribution from the point $\eta=0$ [see (2.9)] which we have ignored so far. We shall show that the point $\eta=0$ may be ignored except for vacuum diagrams.

Before we give a general proof, let us analyze the lowest-order vacuum diagram shown in Fig. 1(c), and see why the point $\eta=0$ may not be ignored.

Figure 1(c) may be obtained from Fig. 1(a) by closing the two external lines and integrating over η and s. The variable $\mathbf{q}$ is irrelvant for the present discussion and will be ignored for simplicity. This vacuum diagram is then expressed as

$$E=\int ds\, d\eta (s\eta-m^2+i\epsilon)^{-1}\Sigma(s,\eta)\,, \qquad (2.12)$$

where Σ is given by (2.4). By Lorentz invariance $\Sigma(s,\eta)=\Sigma(s\eta)$, i.e., Σ is a function of the product $s\eta$ only. According to our previous discussion, E is zero if the point $\eta=0$ is ignored. However, for $\eta=0$, the s integral of (2.12) diverges linearly suggesting that it is proportional to $\delta(\eta)$.

To exhibit $\delta(\eta)$ explicitly, we write $\Sigma(s\eta)$ in the form of a Fourier transform

$$\Sigma(s\eta)=\int d\lambda\, F(\lambda)e^{i\lambda s\eta}, \qquad (2.13)$$

where $F(\lambda)$ is found to be, after a little algebra,

$$F(\lambda)=\int_0^\infty d\xi_1 d\xi_2 \delta(\lambda(\xi_1+\xi_2)-\xi_1\xi_2)e^{-im^2\xi_1\xi_2/\lambda}. \qquad (2.14)$$

The ξ integrals come from the identity

$$(x+i\epsilon)^{-1}=-i\int_0^\infty d\xi\, e^{i\xi(x+i\epsilon)}\,, \qquad (2.15)$$

which we have used to replace the denominators in (2.4). Using (2.15) for the denominator in (2.12), and substituting (2.13) in (2.12), we have

$$E=\int d\eta\, ds\left[-i\int_0^\infty d\xi d\lambda\, F(\lambda)e^{is\eta(\lambda+\xi)-im^2\xi}\right]$$

$$=\int d\eta\left[-2\pi i\int_0^\infty d\xi d\lambda\, F(\lambda)(\lambda+\xi)^{-1}e^{-im^2\xi}\right]\delta(\eta)\,. \qquad (2.16)$$

It is thus clear that this vacuum diagram is contributed by the point $\eta=0$ only.

We proceed to generalize the above argument. Consider an arbitrary diagram $A(p_3, p_4, \cdots, p_k)$ with $k-2$ external lines shown in Fig. 2(a), which may be a part clipped out of a still bigger diagram. Now, we pull out an arbitrary internal line, labeled by (s,η), which is integrated over (i.e., by cutting this line, the diagram will not fall into two disconnected pieces). Let Γ denote the remainder of A with the (s,η) line removed. The removal of the (s,η) line generates two more vertices, 1 and 2. We write

$$A(p_3,\cdots,p_k)=\int d\eta\, ds(\eta s-m^2+i\epsilon)^{-1}\Gamma(p_1,\cdots,p_k)\,, \quad (2.17)$$

where $p_1=(\eta,s)$, and $p_2=(-\eta,\ -s)$. By Lorentz invariance, Γ must be a function of $\eta_i s_j+s_i\eta_j$, $i,\ j=1,\ \cdots,\ k$.[4] Analogous to (2.13), we write Γ, as well as the denominator in (2.17), in the form of a Fourier transform

$$\Gamma(p_1,\cdots,p_k)=\int(\prod_{j>i}d\lambda_{ij})F(\lambda_{ij})\times\exp i[\sum_{j>i}\lambda_{ij}(\eta_i s_j+\eta_j s_i)]\,, \quad (2.18)$$

$$(\eta s-m^2+i\epsilon)^{-1}=-i\int_0^\infty d\xi\,\exp[i(\eta s-m^2+i\epsilon)\xi]. \quad (2.19)$$

The exponential factor containing the variable s in (2.17) is then

$$\exp\{is[\eta[\xi+2(\lambda_{11}+\lambda_{22}-\lambda_{12})]+\sum_{j>2}\eta_j(\lambda_{1j}-\lambda_{2j})]\}\,. \quad (2.20)$$

Thus, the s integral in (2.17) will produce a $\delta(\eta)$ only if all η_i, $i=3,\ \cdots,\ k$ are zero. In other words, $\delta(\eta)$ will not appear unless the diagram A has no external line or is a piece of a vacuum diagram. Thus, one may ignore the point $\eta=0$ unless one deals with a vacuum diagram, which is in general nonzero due to the factor $\delta(\eta)$. In Ref. 3, the point $\eta=0$ is always ignored, and the contribution from vacuum diagrams is thus lost.

III. INFINITE-MOMENTUM LIMIT

The infinite-momentum frame, i.e., the reference frame in which all particles have infinite momenta, provides a physical interpretation of our new rules. We begin by reviewing briefly Weinberg's derivation of rules at infinite momentum.[3]

The momentum $\mathbf{p}$ of a given particle may be written

$$\mathbf{p}=\eta\mathbf{P}+\mathbf{q}\,,\quad \mathbf{P}\cdot\mathbf{q}=0\,, \quad (3.1)$$

where P approaches infinity. If we take $\mathbf{P}=(0,0,P)$ to be the total momentum of the particles in a given state, we have, summing over all the particles

$$\sum_k \eta_k=1\,. \quad (3.2)$$

The energy of a particle with momentum $\mathbf{p}$ is

$$p^0=(\mathbf{p}^2+m^2)^{1/2}=|\eta|P+(1/2|\eta|P)(m^2+\mathbf{q}^2)+O(P^{-2})\,. \quad (3.3)$$

[4] By symmetry, terms like $\epsilon_{\alpha\beta\gamma\sigma}p_i{}^\alpha p_j{}^\beta p_m{}^\gamma p_n{}^\sigma$ will not appear.

Now, one applies the rules of the old-fashioned perturbation theory with (3.3) serving as the expression of the single-particle energy. For a given intermediate state γ, the energy denominator is

$$E_\alpha-E_\gamma=E_\alpha-P\sum_k|\eta_k|+\frac{1}{2P}\sum_k\frac{1}{|\eta_k|}(\mathbf{q}_k{}^2+m^2)+O(P^{-2})\,. \quad (3.4)$$

The integration over $\mathbf{p}$ becomes an integral over $\mathbf{q}$ and η. By a careful counting of the power of P, it is observed in Ref. 3 that only those intermediate states with all $\eta>0$ contribute, and

$$E_\alpha-E_\gamma=(2P)^{-1}(S_\alpha-S_\gamma)\,, \quad (3.5)$$

where

$$S_\gamma=\sum_k\frac{1}{|\eta_k|}(\mathbf{q}_k{}^2+m^2)\,. \quad (3.6)$$

The final expression for the contribution to the invariant amplitude does not involve P. We may thus identify (3.6) with (2.10), and (3.2) with (2.11). Thus, the η variable defined by (3.1) is proportional to the absolute value of our previously defined η variable, i.e., p^0+p^3. To make the correspondence clearer, we observe that under a Lorentz transformation in the z direction

$$p^0+p^3\to e^\lambda(p^0+p^3)\,,\quad \mathbf{q}\to\mathbf{q}\,,\quad p^0-p^3\to e^{-\lambda}(p^0-p^3)\,. \quad (3.7)$$

In the infinite-momentum frame, by (3.3)

$$p^0+p^3=|\eta|2P\,,\quad p^0-p^3=(1/|\eta|)(\mathbf{q}^2+m^2)(2P)^{-1}. \quad (3.8)$$

Thus, p^0+p^3 and p^0-p^3 are, respectively, the *large* and *small* components of the four momentum in the infinite-momentum frame. If we apply a Lorentz transformation defined by

$$e^{-\lambda}=2P\,, \quad (3.9)$$

we have, in the new frame, which we call the "standard decelerated frame,"

$$p^0+p^3=|\eta|\,,\quad p^0-p^3=(1/|\eta|)(\mathbf{q}^2+m^2)\,, \quad (3.10)$$

which may be regarded as the on-shell values of our variables η and s. (On-shell means $p^2=\eta s-\mathbf{q}^2=m^2$.)

Therefore, our variables η and s may be, respectively, interpreted as the large and small components of the four-momentum at infinite momentum, apart from the infinite multiplicative constant $2P$, which may be viewed as the $e^{-\lambda}$ factor of a Lorentz transformation.

When an invariant quantity is calculated, a Lorentz transformation is irrelevant, and the factor $2P$ must drop out. When a vector quantity is calculated at

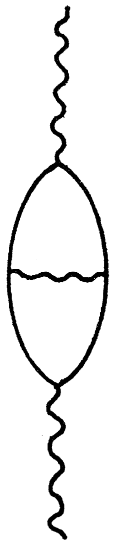

FIG. 3. Fourth-order polarization diagram.

infinite momentum, the large (i.e., 0+3) component is proportional to P while the small (i.e., 0−3) component is proportional to P^{-1}. In other words, when a standard frame [i.e., a finite-momentum frame given by (3.10)] is boosted to the infinite-momentum frame, we have

$$\eta \rightarrow \eta(2P), \quad s \rightarrow s(2P)^{-1}, \quad \eta s \rightarrow \eta s. \tag{3.11}$$

Similar conclusions may be drawn for any component of a tensor.

Thus, the tensor matrix elements, such as the photon self-energy and the anomalous magnetic moment analyzed in Sec. IV, separate into large and small components as $P\rightarrow\infty$. By going to the infinite-momentum frame, these large components, referred to by Gell-Mann as "good components," usually become simpler and easier to compute, while the small components become more complicated and difficult to handle. For example, assume that the original matrix elements need two subtractions. By separating these matrix elements into large and small components, the large components may have the advantage that they need one subtraction, but the small components may now need three subtractions. In other words, the simplification in one part of the calculation is often associated with the complication generated in the other part of the calculation. The real advantages, however, are realized in some simple cases in which we only need to deal with these good components. In Sec. V, we shall compute certain matrix elements by evaluating only the large components. The full matrix elements can be recovered in these cases by the requirements of Lorentz covariance.

Thus, we have given in this section the physical interpretation of s and η in terms of the small and large components in the infinite-momentum frame. We have also shown that Weinberg's rules are the rules of the old-fashioned perturbation theory in terms of the new variable η for τ-ordered diagrams. In the infinite-momentum frame, τ approaches t, η approaches p^3 (apart from an irrelevant infinite factor), and Weinberg's rules become identical to the old-fashioned rules.

Finally, we notice that under the reflection through the xy plane, p^3 changes sign, and s and η are interchanged. All our previous discussion may be carried out with s and η interchanged.

IV. NEW RULES APPLIED TO QUANTUM ELECTRODYNAMICS (QED)

In terms of the variables $(s, \eta, \mathbf{q})$, the electron propagator is

$$S_F(p)=[\tfrac{1}{2}s(\gamma^0+\gamma^3)+\tfrac{1}{2}\eta(\gamma^0-\gamma^3)-\boldsymbol{\gamma}\cdot\mathbf{q}+m] \times(s\eta-m^2-\mathbf{q}^2+i\epsilon)^{-1}. \tag{4.1}$$

The photon propagator is

$$D_{\mu\nu}(p)=-g_{\mu\nu}(s\eta-\mathbf{q}^2+i\epsilon)^{-1}. \tag{4.2}$$

The appearance of s, η, and $\mathbf{q}$ in the numerator of the electron propagator indicates that one would encounter integrals that are more divergent than those in the ϕ^3 model. Thus, one expects that some of the conclusions of the previous sections will not be valid for QED. We now examine this point.

The main conclusion of Sec. II is that the s integrals automatically set limits for the η integrals, and these limits can be easily determined by inspecting the Feynman diagram. This simple feature is a consequence of the fact that the s integrals define the directions of particle lines and restrict $\eta>0$ ($\eta<0$) lines to point forward (backward) in τ. Equation (2.9) may be viewed as the basic formula. Therefore, to see the implications of the numerator of $S_F(p)$, we Fourier transform (4.1) as we did for $G(p)$

$$\begin{aligned} S_F(\tau,\eta,\mathbf{q}) &= -\int (ds/2\pi i)S_F(s,\eta,\mathbf{q})e^{-i\tau s} \\ &= \tfrac{1}{2}(\gamma^0+\gamma^3)i(\partial/\partial\tau)G(\tau,\eta,\mathbf{q}) \\ &\quad +[\tfrac{1}{2}\eta(\gamma^0-\gamma^3)-\boldsymbol{\gamma}\cdot\mathbf{q}+m]G(\tau,\eta,\mathbf{q}), \end{aligned} \tag{4.3}$$

where $G(\tau,\eta,\mathbf{q})$ is given by (2.9).

As a function of τ, the first term in (4.3) behaves quite differently from $G(\tau,\eta,\mathbf{q})$. By differentiating (2.9), one obtains

$$\begin{aligned} i(\partial/\partial\tau)&G(\tau,\eta,\mathbf{q}) \\ &= (1/\eta^2)(m^2+\mathbf{q}^2)e^{-i(m^2+\mathbf{q}^2)\tau/\eta}\,\theta(\tau\eta)\,\mathrm{sgn}\eta+(1/\eta)i\delta(\tau), \quad \text{for } \eta\neq0 \\ &= [\delta'(\tau)/(m^2+\mathbf{q}^2)], \quad \text{for } \eta=0. \end{aligned} \tag{4.4}$$

The first term defines a direction of propagation in τ according to the sign of η. However, the second term, which is absent in the ϕ^3 model, is nonzero for all values of η. Thus, for this term there is no longer any limit for the value η; consequently, the Z diagrams as in Fig. 1(b) are no longer forbidden.

On the other hand, the other two terms in (4.3) are well behaved. As will be seen explicitly in the following sections, the first term of (4.3) can be avoided as far as second-order calculations are concerned, and the new rules make the calculations very simple. For some higher-order diagrams, $\partial G/\partial\tau$ can also be avoided entirely by special choices of components. For example, the fourth-order polarization diagram (Fig. 3) can be calculated without $\partial G/\partial\tau$ by choosing the $0+3$ component as the polarization of the external lines. This is because $(\gamma^0\pm\gamma^3)^2=0$. In these cases, all our previous conclusions remain valid. We shall discuss some of the features of the new rules involving higher-order diagrams after applying them to the second-order diagrams.

V. SECOND-ORDER DIAGRAMS IN QUANTUM ELECTRODYNAMICS (QED)

We shall evaluate the second-order self-energies and the anomalous magnetic moment to illustrate the new rules. In the following, we shall take advantage of the fact that as $P^3\to\infty$, certain leading terms dominate the processes. As we shall see, these leading terms are unambiguous and easy to compute. After we obtain the leading terms, we can express them easily in the covariant form.

A. Electron and Photon Self-Energies

For the electron self-energy, the diagram given by Fig. 4(a) leads to

$$\Sigma(p)=e^2\int\frac{d^4p'}{i(2\pi)^4}\frac{\gamma_\mu(p'+m)\gamma^\mu}{(p'^2-m^2+i\epsilon)[(p-p')^2+i\epsilon]}. \quad (5.1)$$

We choose $p=(s,\eta,0)$ and $p'=(s',\eta',\mathbf{q}')$. The numerator of the integrand may be reduced to

$$\begin{aligned}\gamma_\mu(p'+m)\gamma^\mu&=-2p'+4m\\&=-2[\tfrac{1}{2}s'(\gamma^0+\gamma^3)+\tfrac{1}{2}\eta'(\gamma^0-\gamma^3)-\boldsymbol{\gamma}\cdot\mathbf{q}']+4m. \quad (5.2)\end{aligned}$$

The term $\boldsymbol{\gamma}\cdot\mathbf{q}'$ will not contribute because of the cylindrical symmetry we have chosen. We can write

$$\begin{aligned}\Sigma(p)=\tfrac{1}{2}(\gamma^0+\gamma^3)F_1(s,\eta)&\\+\tfrac{1}{2}(\gamma^0-\gamma^3)F_2(s,\eta)&+F_3(s,\eta). \quad (5.3)\end{aligned}$$

Under reflection through the xy plane, s and η are interchanged and $\gamma^3\to-\gamma^3$. We thus conclude that

$$F_1(s,\eta)=F_2(\eta,s). \quad (5.4)$$

F_2 is the leading term at $p^3=\infty$, and F_3 remains finite at the same limit. For our purpose, we only need to compute F_2 and F_3. They will give us the corresponding coefficients of p and 1 in (5.1). To calculate F_2, we take the second term in (5.2) for the numerator of (5.1). The s' integral is well defined and simply gives an old-fashioned denominator; one finds

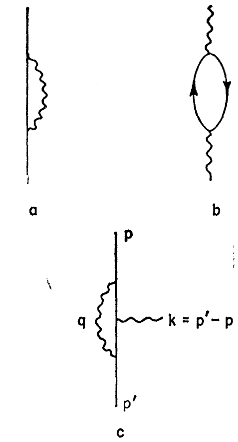

FIG. 4. (a) Electron self-energy. (b) Photon self-energy. (c) Magnetic moment.

$$F_2(s,\eta)=-\int\frac{d^2q'}{2(2\pi)^3}\int_0^\eta d\eta'(-2\eta')[\eta'(\eta-\eta')]^{-1}\times\left(s-\frac{m^2+\mathbf{q}'^2}{\eta'}-\frac{\mathbf{q}'^2}{\eta-\eta'}\right)^{-1}. \quad (5.5)$$

Performing the q' integration and introducing the variable α through $\eta'=\alpha\eta$, one finds

$$F_2(s,\eta)=\tfrac{1}{2}(2\pi)^{-2}\eta\int_0^1\alpha d\alpha\ln\left[\frac{m^2-\alpha s\eta}{m^2(1-\alpha)}\right]+B\eta, \quad (5.6)$$

where B is a logarithmically divergent constant. $F_3(s\eta)$ is obtained by using the last term of (5.2) for the numerator of (5.1). Similar steps lead to

$$F_3(s\eta)=m(2\pi)^{-2}\int_0^1 d\alpha\ln\left[\frac{m^2-\alpha\eta s}{m^2(1-\alpha)}\right]+A-mB, \quad (5.7)$$

where A is a logarithmic divergent constant. Substituting (5.7) and (5.6) in (5.3), and taking (5.4) into account, we have

$$\Sigma(p)=A+B(p-m)+S_c,$$

with

$$S_c=\tfrac{1}{2}e^2(2\pi)^{-2}\int_0^1 d\alpha(\alpha p-2m)\ln\left(\frac{m^2-\alpha p^2}{m^2(1-\alpha)}\right), \quad (5.8)$$

where we have identified ηs with p^2 and $\frac{1}{2}(\gamma^0+\gamma^3)s+\frac{1}{2}(\gamma^0-\gamma^3)\eta$ with p. The infinite constants A and B are disposed of by renormalization.

As the second example, we consider the photon self-energy diagram shown in Fig. 4(b). The diagram shown in Fig. 4(b) gives

$$\begin{aligned}\Pi^{\mu\nu}(k)&=e^2\int\frac{d^4p}{(2\pi)^4 i}\frac{\mathrm{Tr}\gamma^\mu(p+k+m)\gamma^\nu(p+m)}{(p^2-m^2)[(p+k)^2-m^2]}\\&\qquad-\text{regulator terms}\\&=(k^\mu k^\nu-g^{\mu\nu})[\Pi(k^2)-\text{regulator terms}]. \quad (5.9)\end{aligned}$$

We let $k=(s,\eta,0)$, $p=(s',\eta\alpha,\mathbf{q})$. The computation is greatly simplified if we calculate only the leading term at $p^3=\infty$. This leading term is specified by $\mu=\nu=0+3$, so that $\Pi(k^2)$ is obtained by finding the coefficient of η^2 in (5.9). The numerator of the integrand is then

$$\mathrm{Tr}\{(\gamma^0+\gamma^3)(\tfrac{1}{2}(\gamma^0-\gamma^3)\eta(1-\alpha)+\boldsymbol{\gamma}\cdot\mathbf{q}+m)(\gamma^0+\gamma^3)\times[\tfrac{1}{2}(\gamma^0-\gamma^3)\eta\alpha-\boldsymbol{\gamma}\cdot\mathbf{q}+m]\}, \quad (5.10)$$

where we have made use of the fact that $(\gamma^0+\gamma^3)^2=0$, so that terms proportional to s', $s-s'$ do not appear. The coefficient of η^2 is easily read off from (5.10).

The s' integral provides the denominator for the electron-positron intermediate state. We have

$$\Pi(k^2)=4e^2(2\pi)^{-3}\int d^2q\int_0^1 d\alpha \times\left(k^2-\frac{\mathbf{q}^2+m^2}{\alpha}-\frac{\mathbf{q}^2+m^2}{1-\alpha}\right)^{-1}. \quad (5.11)$$

We have written k^2 for ηs and counted for the fact that

$$\mathrm{Tr}[(\gamma^0+\gamma^3)(\gamma^0-\gamma^3)(\gamma^0+\gamma^3)(\gamma^0-\gamma^3)]=32. \quad (5.12)$$

Performing the q integral, one obtains

$$\Pi(k^2)=\tfrac{1}{2}e^2\pi^{-2}\int_0^1 d\alpha(1-\alpha)\alpha\ln\left[1-\frac{k^2\alpha(1-\alpha)}{m^2}\right]+C, \quad (5.13)$$

where C is again a logarithmically divergent constant and can be removed by a charge renormalization. Notice that our final result is only logarithmically divergent and we have used no additional regulators other than the ordinary cutoff.

As a last example, we compute the anomalous magnetic moment of an electron. The lowest-order diagram is Fig. 4(c) and leads to

$$M^\mu=e^3\int\frac{d^4q}{i(2\pi)^4}\frac{\bar{u}(p')\gamma_\lambda(\not{p}'-\not{q}+m)\gamma^\mu(\not{p}-\not{q}+m)\gamma^\lambda u(p)}{[(p'-q)^2-m^2+i\epsilon][q^2-m^2+i\epsilon][(p-q)^2-m^2+i\epsilon]}=e\bar{u}(p')(F_e(k^2)\gamma^\mu+\frac{i}{2m}F_M(k^2)\sigma^{\mu\nu}k_\nu)u(p). \quad (5.14)$$

We choose for simplicity, $p'=(s,\eta,\tfrac{1}{2}\mathbf{k})$, $p=(s,\eta,-\tfrac{1}{2}\mathbf{k})$, $k=(0,0,\mathbf{k})$, and $q=(s',\eta'=\alpha\eta,\mathbf{q})$, where p' and p are on the mass shell. Since we are only interested in the magnetic moment, we may ignore all "charge terms," which are of the form $\bar{u}(p')\gamma^\mu u(p)$.

Let us first simplify the numerator. Making use of the Dirac equations

$$(\not{p}-m)u(p)=0,\qquad \bar{u}(p')(\not{p}'-m)=0, \quad (5.15)$$

we have

$$(\not{p}-\not{q}+m)\gamma^\lambda u(p)=(2p^\lambda-\not{q}\gamma^\lambda)u(p),\qquad \bar{u}(p')\gamma_\lambda(\not{p}'-\not{q}+m)=\bar{u}(p')(2p_\lambda'-\gamma_\lambda\not{q}). \quad (5.16)$$

Then the numerator of the integrand in (5.14) can be reduced to

$$-4mq^\mu\bar{u}(p')u(p)+4(p'+p-q)^\mu\bar{u}(p')\not{q}u(p). \quad (5.17)$$

In the infinite-momentum frame, we evaluate only the leading term of M^μ, i.e., M^{0+3}. For this leading term, we have

$$p'^\mu=p^\mu,\quad q^\mu=\alpha p^\mu=\alpha p'^\mu. \quad (5.18)$$

If we ignore the charge term and make use of the cylindrical symmetry and the fact that $k_\lambda M^\lambda=0$, it is straightforward to verify that

$$\bar{u}(p')\not{q}u(p)=\alpha m\bar{u}(p')u(p). \quad (5.19)$$

Then, the leading term in the numerator reduces to

$$2m\alpha(1-\alpha)(p'+p)^\mu\bar{u}(p')u(p)+\text{charge term}. \quad (5.20)$$

Now, we apply our new rules and integrate over s'. We find

$$\begin{aligned}M^\mu&=-e^3(p'+p)^\mu\bar{u}(p')u(p)\int_0^\infty\frac{d^2q}{2(2\pi)^3}\int_0^1\frac{d\alpha}{\alpha(1-\alpha)^2}\frac{2m\alpha(1-\alpha)}{\{(m^2+\frac{1}{4}\mathbf{k}^2)-[m^2+(\frac{1}{2}\mathbf{k}+\mathbf{q})^2]/(1-\alpha)-\mathbf{q}^2/\alpha\}}\\&\quad\times\frac{1}{\{(m^2+\frac{1}{4}\mathbf{k}^2)-[m^2+(\frac{1}{2}\mathbf{k}-\mathbf{q})^2]/(1-\alpha)-\mathbf{q}^2/\alpha\}}+\text{charge term}\\&=-e^3(p'+p)^\mu\bar{u}(p')u(p)\int_0^\infty\frac{d^2q}{2(2\pi)^3}\int_0^1 2m\alpha^2(1-\alpha)d\alpha\frac{1}{[\mathbf{q}^2+\alpha^2(m^2+\frac{1}{4}\mathbf{k}^2)]^2-\alpha^2(\mathbf{k}\cdot\mathbf{q})^2}.\end{aligned} \quad (5.21)$$

After the q^2 integration, which is elementary, we have

$$M^\mu=-e^3\frac{(p'+p)^\mu}{m}\bar{u}(p')u(p)\int_0^1\frac{d\alpha}{4(2\pi)^2}\frac{2\alpha^2(1-\alpha)}{\alpha^2}F_2(k)=-\frac{e^3}{4(2\pi)^2}\frac{(p'+p)^\mu}{m}\bar{u}(p')u(p)F_2(k), \quad (5.22)$$

where

$$F_2(k)=\frac{m^2}{k[m^2+\frac{1}{4}\mathbf{k}^2]^{1/2}}\ln\frac{1+k/(4m^2+\mathbf{k}^2)^{1/2}}{1-k/(4m^2+\mathbf{k}^2)^{1/2}},$$
$$F_2(0)=1. \tag{5.23}$$

Since

$$(p'+p)_\mu\bar{u}(p')u(p)=-i\bar{u}(p')\sigma_{\mu\nu}k^\nu u(p)+2m\bar{u}(p')\gamma_\mu u(p)$$
$$=-i\bar{u}(p')\sigma_{\mu\nu}k^\nu u(p)+\text{charge term}, \tag{5.24}$$

we then have

$$F_M(k)=(\alpha/2\pi)F_2(k), \tag{5.25}$$

which is the well-known result for the second-order magnetic moment.

In the above examples, the simplicity of algebra is achieved by judicious choices of the particular terms which are easy to calculate. All the other terms are inferred by symmetry.

VI. REMARKS ON HIGHER-ORDER TERMS

After seeing the remarkable simplicity, as compared to the conventional method, in the second-order QED calculations with the new rules, one naturally asks if similar simplicity would remain in higher-order calculations. As the following discussion will indicate, the answer seems to be no.

There are mainly two complications that appear in the higher-order QED terms, the multiple numerator and the multiple denominator. Let us consider the latter first with an example.

Consider the self-energy diagram shown in Fig. 3. In the conventional method, one combines the five denominators by the Feynman technique of introducing four variables $0<\alpha_i<1$. One then "completes the square" in the denominator and performs the two four-dimensional integrals with the help of the Wick rotation. Finally, there is the four-dimensional integral over α_i to do. With the new technique, one writes down the three denominators for the three intermediate states by inspection. There are two η integrals and two 2-dimensional q integrations to do. The question now is whether one can do the two q integrations quickly. In the absence of any special trick, the q integrations can always be performed at the high cost of introducing two Feynman parameters to combine the three denominators. Integrating over these two Feynman parameters and then over the two η variables, is expected to involve complexities similar to the conventional integration over the four Feynman parameters. It thus appears that we have gained nothing using the new technique unless a shortcut is found to evaluate the q integrals, which are 2-dimensional integrals and are responsible for much of the simplicity observed in Sec. V. Such a shortcut is not in sight so far.

The second complication of higher-order diagrams in QED is the complicated numerator involving many γ matrices as well as powers of momenta, which cause divergence. In the second-order calculations, we were able to avoid the occurrence of s or q in the numerator. Since the η integrals are over a finite range, we got around the regularization required in the conventional calculation. For higher-order diagrams, this cannot be done in general. For example, the numerator of the self-energy diagram shown in Fig. 3 contains four powers of momenta. By fixing the external lines to the $0+3$ components, one removes all powers of s, but two powers of q remain. In the conventional method, one also pulls out two powers of momentum outside the integral. In general, s, η, $\mathbf{q}$ will all appear in the numerator. When a product of s and η appears in the numerator, the undesirable features mentioned previously will appear [see the discussion around (4.4)]. The s integral tends to bring more powers of q to the numerator and more η to the denominator. Consequently, the q integrals become more divergent and the η integrals may blow up in the lower limits. One thus needs more regulators than the conventional method. In addition, there will be terms with an unlimited range of η. These features are readily verified even in calculating the second-order photon self-energy if one chooses the $0-3$ components (i.e., the small component rather than the large $0+3$ component) which would bring two powers of s' to the numerator, and insists on performing the s' integral first.

We thus conclude that, in calculating higher-order terms in QED, our new rules are not practical until further progress in technique is made. This conclusion, however, by no means excludes the possibility that the new Feynman rules will be useful in other applications.

Note added in proof. We have found that the infinite-momentum techniques developed in the present paper can be used to compute certain leading terms of the high-energy limit of the electron-electron, electron-positron, Compton, and photon-photon scattering amplitudes. The difficulties discussed above do not appear in these terms. The lowest-order amplitudes, as computed by Cheng and Wu,[5] can be reproduced in a much simpler manner.[6] We have also computed and summed up the amplitudes with an arbitrary number of photons exchanged.

ACKNOWLEDGMENT

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[5] H. Cheng and T. T. Wu, Phys. Rev. Letters **22**, 666 (1969).
[6] This simpler method is also known to Cheng and Wu. We are indebted to Dr. S. Adler for communicating to us the unpublished results of Cheng and Wu.