Axial-Vector Current in Spinor Electrodynamics*

RICHARD A. BRANDT
Center for Theoretical Physics, Department of Physics and Astronomy, University of Maryland, College Park, Maryland 20742

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Adler has shown that in perturbation theory for spinor electrodynamics, suitably regularized, the divergence of the unrenormalized axial-vector current contains, in addition to the expected mass term, a term of the form $\epsilon_{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta}$. Motivated by this result, we study here the renormalized axial-vector current $j_\mu{}^5$. We explicitly construct the essentially unique finite local $j_\mu{}^5$ and find it to be invariant under local gauge transformations. Taking the divergence $\partial_\mu j_\mu{}^5$ gives the renormalized analog of Adler's additional term. Similar "noncanonical" operator terms are seen to occur in equal-time commutators involving $j_\mu{}^5$. Although all matrix elements of $\partial_\mu j_\mu{}^5$ are finite, we find that the off-shell Green's function $\langle T\partial_\mu j_\mu{}^5(x)\psi(y)\bar{\psi}(z)\rangle$ is divergent and, correspondingly, that equal-time commutators involving $j_\mu{}^5$ are in general singular. We show that Ward identities can nevertheless be given a meaning. The algebraic properties of $j_\mu{}^5$ are seen to be reflected in the scattering amplitudes of the theory. Renormalized integral representations of axial-vector vertices are constructed, and radiative corrections to weak interaction are discussed. We conclude with a discussion of the axial-vector currents in other spinor models.

I. INTRODUCTION

ADLER[1] has recently studied the axial-vector vertex in spinor electrodynamics and found that in perturbation theory, *suitably regularized*, the divergence $\partial_\mu j_{0\mu}{}^5$ of the *unrenormalized* axial-vector current $j_{0\mu}{}^5$ contains, in addition to the expected $2im_0j_0{}^5$ term, the term[2] $(\alpha_0/4\pi)\epsilon_{\alpha\beta\gamma\delta}F_{0\alpha\beta}F_{0\gamma\delta}$. This rather surprising result shows that $\partial_\mu j_{0\mu}{}^5$ does not vanish when $m_0=0$ and gives rise to divergences in the usual weak-interaction theory of lepton-neutrino scattering. In the context of the σ model, the additional term resolves a discrepancy between perturbation theory and partially conserved axial-vector current (PCAC) and gives a good account of the $\pi^0\to 2\gamma$ decay.

Motivated by Adler's results, we study here the *renormalized* axial-vector current $j_\mu{}^5$ in spinor electrodynamics. We explicitly construct[3] the essentially unique finite local $j_\mu{}^5$ and find it to be invariant under local gauge transformations. Thus all of the Green's functions $\langle Tj_\mu{}^5\psi\cdots\psi\bar{\psi}\cdots\bar{\psi}A\cdots A\rangle$ are finite and gauge-invariant[4] without subtractions or regularizations. Taking the divergence $\partial_\mu j_\mu{}^5$ of this $j_\mu{}^5$ gives the renormalized analog of Adler's additional term. Although all matrix elements of $\partial_\mu j_\mu{}^5$ are finite, we find that the off-shell Green's function $\langle T\partial_\mu j_\mu{}^5(x)\psi(y)\bar{\psi}(z)\rangle$ is *divergent* and, correspondingly, that equal-time commutators involving $j_\mu{}^5$ are in general singular. We show that Ward identities can nevertheless be given a meaning.

Our work shows that the results of a calculation in renormalized perturbation theory will not contradict the local formulation of the theory *provided* this formulation employs the proper definition of current operators. In order to obtain a finite local axial-vector current $j_\mu{}^5$, one must subtract explicit functions of the electromagnetic field from the product $\bar{\psi}(x)\gamma_\mu\gamma_5\psi(x)$. It is these subtraction terms which give rise to the extra "noncanonical" operator terms in both equal-time commutators of $j_\mu{}^5$ and in its divergence $\partial_\mu j_\mu{}^5$.[5] The $\pi^0\to 2\gamma$ decay indicates the empirical significance of these subtractions. It is the algebraic properties of the resulting renormalized current which are reflected in the scattering amplitudes of the theory.

In Sec. II we review and simplify some results of the local formulation[6] of spinor electrodynamics for later reference. In Sec. III we derive the properties of $j_\mu{}^5$ stated above. In part A we exhibit the general form of $j_\mu{}^5$. In part B we give integral representations for the primitively divergent axial-vector vertex functions and use them to define a particular finite $j_\mu{}^5$. In part C we show that the $j_\mu{}^5$ of part B is unique apart from an over-all finite multiplicative constant, and in part D we show that it is gauge-invariant and obtain a more explicit form for it. In part E we define and derive equal-time commutation relations involving $j_\mu{}^5$. In part F we calculate $\partial_\mu j_\mu{}^5$, first for an external electromagnetic field and then in the general case. We then use this result together with one of part E to conclude that $\langle T\partial_\mu j_\mu{}^5\psi\bar{\psi}\rangle$ is divergent but that the corresponding Ward identity is still meaningful. In part G, we present some speculations concerning radiative corrections to weak interactions. In Sec. IV, we discuss the extension of our results to other spinor models.

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[1] S. L. Adler, Phys. Rev. **177**, 2426 (1969).

[2] We designate unrenormalized quantities with a subscript 0.

[3] Expressions for some finite local-current operators have been previously derived by R. A. Brandt [Ann. Phys. (N.Y.) **44**, 221 (1967)] and W. Zimmermann [Commun. Math. Phys. **6**, 161 (1967)]. We refer to the former paper as I. References to earlier work can be found in these papers.

[4] That is, they satisfy the conditions of gauge invariance.

[5] The extra terms in each are, of course, related by Ward identities.

[6] R. A. Brandt, Ann. Phys. (N. Y.) **52**, 122 (1969).

II. FINITE LOCAL FORMULATION OF SPINOR ELECTRODYNAMICS

The local formulation[6] of spinor electrodynamics is based on the field equations[7]

$$(i\gamma\cdot\partial - m)\psi(x) = ef(x) = e\lim_{\eta\to 0} f(x;\eta)\,, \tag{2.1}$$

$$\Box A_\mu(x) = ej_\mu(x) = e\lim_{\xi\to 0} j_\mu(x;\xi)\,, \tag{2.2}$$

and the current definitions

$$f(x;\eta) = TA(x+\eta)\psi(x) - D_1(\eta)\psi(x) - D_{2\mu}(\eta)\partial_\mu\psi(x) - D_3(\eta)f(x)\,, \tag{2.3}$$

$$\begin{aligned} j_\mu(x;\xi) = {} & T\bar\psi(x)\gamma_\mu\psi(x+\xi) - C_{1\mu}(\xi) - C_{2\mu\nu}(\xi)A_\nu(x) \\ & - C_{3\mu\nu\kappa}(\xi)A_{\nu,\kappa}(x) - C_{4\mu\nu\kappa\lambda}(\xi)A_{\nu,\kappa\lambda}(x) \\ & - C_{5\mu\nu\kappa\lambda}(\xi):A_\nu(x)A_\kappa(x)A_\lambda(x): \\ & - C_{6\mu ij}(\xi):\bar\psi_i(x)\psi_j(x):. \end{aligned} \tag{2.4}$$

All of the parameters and field operators are the renormalized ones. The functions $D_i(\eta)$ have singularities for $\eta\to 0$ which compensate those of the local product $A(x)\psi(x)$ so that the limit $\eta\to 0$ in (2.1) exists. The operators occurring in (2.3) are all those in the theory with dimensions (in mass units) $\frac{3}{2}$ or less and with appropriate transformation properties.[8] Since the leading singularities in perturbation theory are mass-independent, one has $D_1(\eta)\sim\eta^{-1}$, $D_{2\mu}(\eta)\sim 1$, and $D_3(\eta)\sim 1$, to within logarithmic factors, for $\eta\sim 0$. Analogous remarks apply to (2.4). Here the "generalized Wick products" $:A^3(x):$ and $:\bar\psi(x)\psi(x):$ must be defined by similar expansions. In general there is a one-one correspondence between free-field Wick products and "generalized Wick products" which transform in the same way and which are finite in perturbation theory. Possible arbitrariness in defining such products is just the usual arbitrariness of choosing basis vectors in a vector space and corresponds to the usual renormalization invariance.

The functions D_i, C_i can be essentially uniquely determined by imposing the usual normalization conditions on the "primitively divergent" proper part functions[9]

$$\Pi_{\mu\nu}(0) = \Pi_{\mu\nu}'(0) = \Pi_{\mu\nu}''(0) = 0\,, \tag{2.5}$$

$$\Sigma(\not p = m) = \Sigma'(\not p = m) = 0\,, \tag{2.6}$$

$$\Gamma_\mu(p,p')\,|_{\not p=\not p'=m,\,p=p'} = \gamma_\mu\,, \tag{2.7}$$

$$X_{\alpha\beta\gamma\delta}(0,0,0,0) = 0. \tag{2.8}$$

[7] We employ the notation and conventions of J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill Book Co., Inc., New York, 1965), except that we do not raise indices when employing the summation convention, and we quantize the electromagnetic field in the Lorentz gauge. We write

$$\not p = \gamma\cdot p = \gamma_\mu p_\mu, \quad \Box = \partial^2,$$
$$(\partial/\partial x^\mu)F(x) = \partial_\mu F(x) = F_{,\mu}(x).$$

[8] The rules describing such expansions were first clearly formulated by K. Wilson (unpublished). See I for more details concerning the general case.

[9] The primes in (2.5) and (2.6) indicate differentiation.

Here Π and Σ are the proper self-energy parts defined in terms of the photon and electron Green's functions D and G by

$$[k^2 g_{\mu\lambda} + \Pi_{\mu\lambda}(k)]D_{\lambda\nu}(k) = -g_{\mu\nu}\,, \tag{2.9}$$

$$[\not p - m - \Sigma(p)]G(p) = 1\,. \tag{2.10}$$

Γ is the proper vertex part, and X the proper photon-photon scattering amplitude.

The conditions (2.5)–(2.8) are implemented by imposing them on integral equations relating all the proper functions of the theory. An example of such an equation is[10]

$$\Gamma_{ij}{}^\mu(p,p') = \gamma_{ij}{}^\mu + i\int_k [\mathrm{tr}\gamma^\mu G(k)\mathrm{H}_{ij}(k,p,p')G(k-p+p') + iC_{5\alpha\beta\gamma}{}^\mu(k)\bar\Gamma_{ij}{}^{\alpha\beta\gamma}(p,p') + iC_{6kl}{}^\mu(k)\Gamma_{lkij}'(p,p')]\,, \tag{2.11}$$

where $\mathrm{H}_{ijkl}(k,p,p')$ is the proper electron-electron scattering amplitude and $\bar\Gamma$ and Γ' are electron vertex functions corresponding to vertices $:AAA:$ and $:\psi\bar\psi:$. The iteration of this infinite set of coupled integral equations yields perturbation expansions (in terms of the renormalized charge e) for all the Green's functions of the theory and these expansions have been shown to be the same as those given by the usual diagrammatic renormalization prescription of Feynman and Dyson,[11] Bogoliubov and Parasiuk,[12] and Hepp.[13] Thus, the limits in (2.1) and (2.2) exist and yield the correct finite local current operators.

A major advantage of the above formalism is that it enables a direct imposition of local gauge invariance. It is shown in II that the requirement that the field equations are invariant under the local gauge transformations

$$\begin{aligned}\psi(x) &\to \exp[-ie\Lambda(x)]\psi(x)\,,\\ A_\mu(x) &\to A_\mu(x) + \partial_\mu\Lambda(x)\,,\end{aligned} \tag{2.12}$$

is equivalent to the requirement that the theory satisfies all of the generalized Ward identities and divergence conditions. It was further shown that the theory determined by the normalizations (2.5)–(2.8) satisfies these conditions. For later reference we exhibit here the simplest of these conditions:

$$k_\mu \Pi_{\mu\nu}(k) = 0\,, \tag{2.13}$$

$$k_{1\alpha}X_{\alpha\beta\gamma\delta}(k_1,k_2,k_3,k_4) = k_2\cdot X = k_3\cdot X = k_4\cdot X = 0\,, \tag{2.14}$$

$$-\partial_\mu G(p) = G(p)\Gamma_\mu(p,p)G(p)\,, \tag{2.15}$$

$$e^2\partial_\mu\partial_\nu G(k) = G(k)\Theta_{\mu\nu}(k,0,0)G(k)\,. \tag{2.16}$$

Here $\Theta_{\mu\nu}(k,p,q)$ is the proper electron-photon scattering amplitude $[e(k)+\gamma(p,\mu)\to e(k-p-q)+\gamma(q,\nu)]$ which

[10] For momentum-space integrations we use the notation $\int_k \equiv \int d^4k/(2\pi)^4$. We denote the Fourier transform of a function $C(x)$ by $C(k) = \int d^4x\,\exp[ik\cdot x]C(x)$.

[11] See Ref. 7.

[12] N. Bogoliubov and O. Parasiuk, Acta Math. **97**, 227 (1957).

[13] K. Hepp, Commun. Math. Phys. **2**, 301 (1966).

satisfies the Bose symmetry condition

$$\Theta_{\mu\nu}(k,p,q)=\Theta_{\nu\mu}(k,q,p) \qquad (2.17)$$

and the on-shell divergence conditions

$$p_\mu\Theta_{\mu\nu}(k,p,q)=q_\nu\Theta_{\mu\nu}(k,p,q)=0 \quad \text{(on electron mass shell)}. \qquad (2.18)$$

Equations (2.13)–(2.15) are consistent with (2.5)–(2.8). Equation (2.14) implies further that

$$\mathrm{X}(0,k_2,k_3,k_4)=\mathrm{X}(k_1,0,k_3,k_4)=\cdots=0. \qquad (2.19)$$

Now the limit $\xi\to 0$ in (2.2) can be taken in any direction. Since the resulting current operator is covariant, however, it is convenient to explicitly take the limit in a covariant way.[14] It is also convenient to explicitly impose on the subtraction functions the analyticity properties of perturbation theory which prohibit the occurrence of expression such as $\sqrt{\xi^2}$. Then, for example, $C_{6\mu}(\xi)=C(\xi^2)\gamma_\mu$, with $C(\xi^2)$ logarithmically divergent, since a $\xi_\mu\gamma\cdot\xi/\xi^2$ term is equivalent and a $\xi_\mu/(\xi^2)^{1/2}$ term is forbidden. Also, we formally have $C(0)=Z_1^{-1}-1$. In this way, (2.4) can be formally simplified to

$$ej_\mu(x;\xi)=\tfrac{1}{2}eZ_1Z_3^{-1}[\bar\psi(x)\gamma_\mu\psi(x+\xi)-\gamma_\mu\psi(x)\bar\psi(x+\xi)] + (1-Z_3^{-1})\partial_\mu\partial_\nu A_\nu(x)+e^2Z_1Z_3^{-1}[\mathrm{tr}\gamma_\mu G(\xi)] \times\{\xi\cdot A(x)+\tfrac{1}{6}(\xi\cdot\partial)^2\xi\cdot A(x)-\tfrac{1}{6}e^2:[\xi\cdot A(x)]^3:\}. \qquad (2.20)$$

We shall work with such simplified expressions throughout this paper. For this reason, some of our results may be of only formal validity.

Equal-time current commutation relations in electrodynamics can be defined and calculated by interchanging the $\xi\to 0$ limit in (2.2) with the equal-time limit.[15] In order to use the field equal-time commutation relations in this connection, the $\xi\to 0$ limit must be taken in a spacelike direction, say $\xi_0=0$. Then the appropriate simplified expressions for $j_\mu(x;\xi)$ are given by Eqs. (III 5.39) and (III 5.40). In this way, one finds, for example,[16]

$$e^2[j_k(x),j_0(x')] = \infty\,\partial_k\delta(\mathbf{x}-\mathbf{x}')-iZ_3^{-2}(e^2/12\pi^2)\partial_k\nabla^2\delta(\mathbf{x}-\mathbf{x}')+iZ_3^{-2} \times(e^4/12\pi^2)(:\mathbf{A}^2:\partial_k+2:A_kA_l:\partial_l)\delta(\mathbf{x}-\mathbf{x}'). \qquad (2.21)$$

This expression was explicitly shown to be correct through fourth order in the sense that it is the same result as that obtained by defining the equal-time commutator as the limit of the ordinary commutator smeared with smooth testing functions $f_n(x_0-x_0')$ converging to $\delta(x_0-x_0')$ in $\mathcal{S}'$.

[14] By a Lorentz covariant limit we mean simply that

$$\lim_{\xi\to 0}\frac{\xi_\alpha\xi_\beta}{\xi^2}=\frac{g_{\alpha\beta}}{4}.$$

[15] R. A. Brandt, Phys. Rev. **166**, 1795 (1968). We refer to this paper as III and to its Eq. (A.B) as (III A.B).

[16] Here $x_0=x_0'$.

III. AXIAL-VECTOR CURRENT

A. General Form of the Current

Let us now construct the most general local axial-vector operator with dimension three. The basis vectors are in one-one correspondence with the free-field Wick products[17]

$$:\bar\psi^0\gamma_\mu\gamma_5\psi^0:, \quad \epsilon_{\mu\alpha\beta\gamma}\xi_\alpha\partial_\beta A_\gamma{}^0, \quad \epsilon_{\mu\alpha\beta\gamma}:A_\alpha{}^0\partial_\beta A_\gamma{}^0:. \qquad (3.1)$$

Correspondingly, we have the expansion (see I, II)

$$T\bar\psi(x)\gamma_\mu\gamma_5\psi(x+\xi)\sim R_1(\xi)\epsilon_{\mu\alpha\beta\gamma}\xi_\alpha\partial_\beta A_\gamma(x) +R_2(\xi)\epsilon_{\mu\alpha\beta\gamma}:A_\alpha(x)\partial_\beta A_\gamma(x): +R_3(\xi):\bar\psi(x)\gamma_\mu\gamma_5\psi(x): \qquad (3.2)$$

for $\xi\sim 0$, for some singular functions R_i and some corresponding definitions of $:A\partial A:$ and $:\psi\gamma_\mu\gamma_5\psi:$. $\xi^2R_1(\xi)$, $R_2(\xi)$, and $R_3(\xi)$ will be at worse logarithmically divergent for $\xi\to 0$. Denoting the (yet to be uniquely defined) axial-vector current by

$$j_\mu{}^5(x)\equiv:\bar\psi(x)\gamma_\mu\gamma_5\psi(x):, \qquad (3.3)$$

we can invert (3.2) and write

$$j_\mu{}^5(x)=\lim_{\xi\to 0}[T\bar\psi(x)\gamma_\mu\gamma_5\psi(x+\xi)+R_1'(\xi)\epsilon_{\mu\alpha\beta\gamma}\xi_\alpha\partial_\beta A_\gamma(x) +R_2'(\xi)\epsilon_{\mu\alpha\beta\gamma}:A_\alpha(x)\partial_\beta A_\gamma(x):+R_3'(\xi)j_\mu{}^5(x)]. \qquad (3.4)$$

The generalized Wick product $:A\partial A:$ is itself an expression of the form (3.4). It is convenient (and possible) to consider simultaneously the behavior (3.2) and that of $A(x)\partial A(x+\xi)$. Thus we write

$$j_\mu{}^5(x)=\lim_{\xi\to 0}j_\mu{}^5(x;\xi), \qquad (3.5)$$

with

$$j_\mu{}^5(x;\xi)=T\bar\psi(x)\gamma_\mu\gamma_5\psi(x+\xi)+E_1(\xi)\epsilon_{\mu\alpha\beta\gamma}\xi_\alpha\partial_\beta A_\gamma(x) +E_2(\xi)\epsilon_{\mu\alpha\beta\gamma}A_\alpha(x)\partial_\beta A_\gamma(x+\xi)+E_3(\xi)j_\mu{}^5(x). \qquad (3.6)$$

The existence and uniqueness of the quantities involved in (3.6) will be discussed below.

B. Vertex Functions and Subtraction Functions

We define the axial-vector–photon vertex $\Pi_{\mu\nu}{}^5(p)$ (which actually vanishes), the axial-vector–photon-photon vertex $F_{\mu\nu\kappa}{}^5(p,q)$, and the axial-vector–electron-electron vertex $\Gamma_\mu{}^5(p,p')$ from the Fourier transforms (symbolically denoted by $\mathfrak{F}$) of Green's functions as follows:

$$\mathfrak{F}\langle Tj_\mu{}^5(x)A_\nu(y)\rangle=\Pi_{\mu\rho}{}^5(p)D_{\rho\nu}(p), \qquad (3.7)$$

$$\mathfrak{F}\langle Tj_\mu{}^5(x)A_\nu(y)A_\kappa(z)\rangle=F_{\mu\rho\sigma}{}^5(p,q)D_{\rho\nu}(p)D_{\sigma\kappa}(q), \qquad (3.8)$$

$$\mathfrak{F}\langle Tj_\mu{}^5(x)\psi(y)\bar\psi(z)\rangle=G(p)\Gamma_\mu{}^5(p,p')G(p') +\Pi_{\mu\rho}{}^5(p+p')D_{\rho\sigma}(p+p')G(p) \times\Gamma_\sigma(p,p')G(p'). \qquad (3.9)$$

[17] We are using here the simplifying assumptions mentioned above Eq. (2.20).

Equations (3.5) and (3.6) then lead to the integral representations

$$\Pi_{\mu\nu}{}^5(p)=\int_k [ie\,\mathrm{tr}\gamma_\mu\gamma_5 G(k)\Gamma_\nu(k,\,k-p)G(k-p) -\partial_\alpha E_1(k)\epsilon_{\mu\alpha\beta\nu}p_\beta+E_3(k)\Pi_{\mu\nu}{}^5(p)], \quad (3.10)$$

$$F_{\mu\nu\kappa}{}^5(p,q)=\int_k [i\,\mathrm{tr}\gamma_\mu\gamma_5 G(k)\Theta_{\nu\kappa}(k,p,q)G(k+p+q) +i\int_{k'} E_2(k+k')\epsilon_{\mu\alpha\beta\gamma}H_{\alpha\gamma\nu\kappa}(k',p,q)k_\beta' +E_3(k)F_{\mu\nu\kappa}{}^5(p,q)], \quad (3.11)$$

$$\Gamma_\mu{}^5(p,p')=\gamma_\mu\gamma_5+\int_k [i\,\mathrm{tr}\gamma_\mu\gamma_5 G(k)\mathrm{H}(k,p,p')G(k-p+p') +i\int_{k'} E_2(k+k')\epsilon_{\mu\alpha\beta\gamma}K_{\alpha\gamma}(k',p,p')k_\beta' +E_3(k)\Gamma_\mu{}^5(p,p')], \quad (3.12)$$

where[18]

$$H_{\alpha\gamma\nu\kappa}(k',p,q)=g_{\alpha\nu}g_{\gamma\kappa}\delta(k'+q)+g_{\alpha\kappa}g_{\gamma\nu}\delta(k'+p) +D(k')\mathrm{X}_{\alpha\gamma\nu\kappa}(-k'-p-q,\,k',\,p,\,q)D(k'+p+q) \quad (3.13)$$

and

$$K_{\alpha\gamma}(k',p,p')=D_{\alpha\rho}(p-p'-k')\Theta_{\rho\sigma} \times(p',\,p-p'-k',\,k')D_{\sigma\gamma}(k'). \quad (3.14)$$

The vertex functions (3.10)–(3.12) correspond to the primitive divergencies involving one primitive axial-vector–spinor vertex $\gamma_\mu\gamma_5$. The function (3.11) also contains, via the first two terms in (3.13), a primitive axial-vector–photon vertex which can formally be written

$$-i\int_k [E_2(k-q)\epsilon_{\mu\nu\beta\kappa}q_\beta+E_2(k-p)\epsilon_{\mu\kappa\beta\nu}p_\beta]. \quad (3.15)$$

The expression (3.15) will be explicitly evaluated below.

The vertex functions in any order can be found from (3.10)–(3.12) by substituting the appropriate pure electrodynamic functions G, Γ_ν, $\Theta_{\nu\kappa}$, $D_{\mu\nu}$, $\mathrm{X}_{\alpha\beta\gamma\delta}$, and H [obtained by iterating the integral Eqs. (2.11), etc.] as well as the appropriate lower-order axial vertices in the right-hand sides. The as yet unspecified subtraction functions E_i are to be chosen in each order so that the resulting integrands yield finite integrals. They are, *a priori*, otherwise arbitrary. As in I and II, it can be shown to follow from renormalization theory that such E_i exist. In this context the arbitrariness in the E_i's is the arbitrariness in the points at which the renormalization subtractions are made.

Let us now choose a specific set $\{E_i\}$ by fixing the subtraction points. This amounts to placing normalization conditions in the functions (3.10)–(3.12). We take

$$\Pi_{\mu\nu}{}^5(0)=0, \quad (3.16)$$

$$\partial_\alpha\Pi_{\mu\nu}{}^5(p)\,|_{p=0}=0, \quad (3.17)$$

$$\partial_\alpha\partial_\beta\Pi_{\mu\nu}{}^5(p)\,|_{p=0}=0, \quad (3.18)$$

$$F_{\mu\nu\kappa}{}^5(0,0)=0, \quad (3.19)$$

$$(\partial/\partial p^\alpha)F_{\mu\nu\kappa}{}^5(p,0)\,|_{p=0} =(\partial/\partial q^\alpha)F_{\mu\nu\kappa}{}^5(0,q)\,|_{q=0}=0, \quad (3.20)$$

and

$$\Gamma_\mu{}^5(p,p')\,|_{p\!\!\!/=p\!\!\!/'=m,\,p=p'}=\gamma_\mu\gamma_5. \quad (3.21)$$

Thus we are subtracting as mass-shell values of the external momenta, $p_{\text{photon}}=0$ and $p\!\!\!/_{\text{electron}}=m$ (in the usual sense). The number of subtractions is given by the superficial divergences ν of the functions (i.e., by naive power counting): $\nu_\Pi=3$, $\nu_F=2$, $\nu_\Gamma=1$.

Since the well-defined quantities $\Pi_{\mu\nu}{}^5(p)$ and $F_{\mu\nu\kappa}{}^5(0,0)$ must vanish by symmetry,[19] the conditions (3.16)–(3.19) are clearly required ones. Equation (3.20) is also required if we want to maintain the usual consequence

$$F_{\mu\nu\kappa}{}^5(p,0)=F_{\mu\nu\kappa}{}^5(0,q)=0 \quad (3.22)$$

of gauge invariance.[20] Finally, the condition (3.21) will be seen to simply fix the over-all normalization of $j_\mu{}^5$.

We proceed to impose the conditions (3.16)–(3.21) on Eqs. (3.10)–(3.14) in order to determine the subtraction functions. We begin with Eq. (3.10) which we note is consistent with (3.16) since $\mathrm{tr}\gamma_\mu\gamma_5 G(k)\Gamma_\nu(k,k)G(k)$ vanishes by symmetry [or, explicitly, by the Ward identity (2.15)]. The condition (3.17) imposed on (3.10) implies

$$\partial_\alpha E_1(k)\epsilon_{\mu\alpha\beta\nu}=ie(\partial/\partial p^\beta)\,\mathrm{tr}\gamma_\mu\gamma_5 G(k)\Gamma_\nu \times(k,\,k-p)G(k-p)\,|_{p=0}. \quad (3.23)$$

The consistency of (3.11) and (3.19) follows from the vanishing of $\mathrm{tr}\gamma_\mu\gamma_5 G(k)\Theta_{\nu\kappa}(k,0,0)G(k)$ by symmetry[21] [since, by the Bose symmetry (2.17), the expression is symmetric in ν and κ] together with the relation (2.19) in (3.13). We see, in fact, that (3.11) implies (3.19). The condition (3.20) imposed on (3.11), using (3.13) and (2.19), implies

$$E_2(k)\epsilon_{\mu\kappa\alpha\nu}=-(\partial/\partial p^\alpha)\,\mathrm{tr}\gamma_\mu\gamma_5 G(k)\Theta_{\nu\kappa} \times(k,p,0)G(k+p)\,|_{p=0}. \quad (3.24)$$

The equivalence of the two conditions (3.20) follows from the Bose symmetry relation (2.17).

We finally consider Eq. (3.12). We define the functions X and Y by

$$i\,\mathrm{tr}\gamma_\mu\gamma_5 G(k)\mathrm{H}(k,p,p')G(k)\,|_{p\!\!\!/=p\!\!\!/'=m}=X(k)\gamma_\mu\gamma_5 \quad (3.25)$$

[18] We have written here $D_{\mu\nu}(k)\equiv g_{\mu\nu}D(k^2)+k_\mu k_\nu D'(k^2)$ and have used (2.14).

[19] We use this expression in place of "there exists no axial tensor depending on a single four-vector p_μ," etc.

[20] We see below that we have no choice but to maintain (3.22).

[21] The vanishing explicitly follows from the Ward identity (2.16).

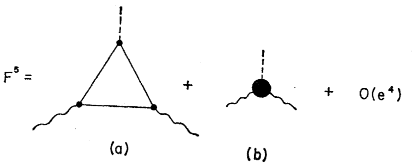

FIG. 1. Diagrams contributing to the lowest-order axial-vector–photon–photon proper vertex.

and

$$i\int_{k'} E_2(k+k')\epsilon_{\mu\alpha\beta\gamma}k_\beta' K_{\alpha\gamma}(k',p,p')|_{p=p'=m} = Y(k)\gamma_\mu\gamma_5\,, \quad (3.26)$$

the equalities holding in the sense of distributions on invariant[22] testing functions $f(k^2)$ of sufficiently fast decreases for $k^2\to\infty$. Then the condition (3.21) imposed on (3.12) leads to

$$-E_3(k)=X(k)+Y(k)\,. \quad (3.27)$$

We have the formal identity

$$X(\xi=0)=\int_k X(k)=Z_1^{-1}-1\,, \quad (3.28)$$

where Z_1^{-1} is the usual vertex renormalization constant which is 1 in order e^0 and logarithmically divergent in order e^2 and higher. The function Y is first nonvanishing in order e^4 and is logarithmically divergent.

The methods of I and II can be used to show that, with the values of E_1-E_3 specified by (3.23), (3.24), and (3.27), the expressions (3.10)–(3.12) are finite. The expressions are, furthermore, guaranteed to satisfy (3.16)–(3.21). In particular, since $\Pi_{\mu\nu}{}^5(p)$ is finite, it must vanish.[23] It will be shown in part C that the E_i are essentially unique. Explicit forms for E_1 and E_2 will be given in part D.

Back in position space, we have the result that, with $E_i(\xi)$ given by the Fourier transforms of (3.23), (3.24), and (3.27), the limit (3.5) exists and yields a finite local axial-vector field operator. In particular, all of the renormalized Green's functions

$$\langle Tj_\mu{}^5(x)\psi(x_1)\cdots\psi(x_n)\bar\psi(y_1)\cdots\bar\psi(y_n)A_{\nu_1}(z_1)\cdots A_{\nu_m}(z_m)\rangle \quad (3.29)$$

are finite in each order of perturbation theory.

We conclude this subsection by illustrating Eqs. (3.11) and (3.12) in low orders of perturbation theory. Equation (3.11) in order e^2 is given in Fig. 1. Diagram (a) represents the first term[24]

$$\mathfrak{F}_{\mu\nu\kappa}{}^{(2)}(p,q)\equiv i\int \mathrm{tr}\gamma_\mu\gamma_5 G^{(0)}\Theta_{\nu\kappa}{}^{(2)}G^{(0)} \quad (3.30)$$

[22] This restriction is a consequence of our taking a covariant limit.
[23] This then implies (3.23), which has not yet been used.
[24] We indicate by $T^{(n)}$ the term of order e^n in the perturbation expansion of the function T.

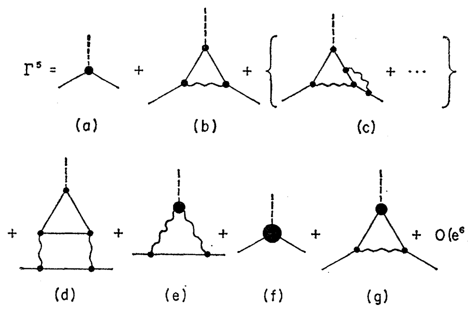

FIG. 2. Diagrams contributing to the axial-vector–electron–electron proper vertex in orders e^2 and e^4.

in (3.11), and (b) represents the second term in (3.11), given by (3.15), which is formally [by (3.24)]

$$-iE_2{}^{(2)}(\xi=0)\epsilon_{\mu\nu\beta\kappa}(q_\beta-p_\beta) \equiv(q_\beta-p_\beta)[(\partial/\partial p^\beta)\mathfrak{F}_{\mu\nu\kappa}{}^{(2)}(p,0)]_{p=0}\,. \quad (3.31)$$

If the integration in (3.30) is done symmetrically, the finite result

$$E_2{}^{(2)}(\xi=0)=e^2/8\pi^2 \quad (3.32)$$

is obtained. The resultant function (3.11) is that given by Rosenberg[25] and studied by Adler.[1]

Equation (3.12) is illustrated in Fig. 2 through order e^4. Diagram (a) represents the lowest-order vertex given by the first term on the right-hand side of (3.12). Diagrams (b)–(d), (e), and (f)–(g) represent, respectively, the first, second, and third terms in the integral in (3.12). The omitted diagrams (c) represent other radiative corrections to (b). The blob in (e) represents (3.32) and the rest of (e) corresponds to (3.14) in second order. The blob in (f) represents the sum of

$$E_3{}^{(4)}=-X^{(4)}-Y^{(4)}=(1-Z_1^{-1})^{(4)}-Y^{(4)}$$

and $E_3{}^{(2)}=-X^{(2)}=(1-Z_1^{-1})^{(2)}$. The blob in (g) represents $E_3{}^{(2)}$ and the rest of the diagram represents $\Gamma_\mu{}^{5(2)}$. The $(1-Z_1^{-1})$ terms renormalize diagrams (b)–(c) in the usual way. Adler has shown that (d)+(e) diverges like $\gamma_\mu\gamma_5(-\frac{3}{4})(\alpha/\pi)^2\ln\Lambda^2$, as does (e) alone, so that (d) is finite. According to (3.26), this divergence is canceled by the $Y^{(4)}$ piece of $E_3{}^{(4)}$ in (f)[26]:

$$Y^{(4)}(\xi)=+\tfrac{3}{4}(\alpha/\pi)^2\ln\xi^2\,. \quad (3.33)$$

C. Uniqueness of the Subtraction Functions

In part B we exhibited subtraction functions E_1-E_3 such that (3.10)–(3.12) are finite. We now search for all other functions with this property. These will give all other axial-vector currents (3.6). We shall find that our

[25] L. Rosenberg, Phys. Rev. **129**, 2786 (1963).
[26] We here simply replace the momentum cutoff Λ^2 by $1/\xi^2$. A more accurate description of $Y(\xi)$ is that it satisfies $\int d\xi\, Y(\xi)\phi(\xi) = +\int dk\, F(k)\hat\phi(k)$ for every smooth function $\phi(\xi)$ of fast decrease, where $\hat\phi(k)$ is the Fourier transform of $\phi(\xi)$, and

$$\int dk\, F(k)[-\Lambda^2/(k^2-\Lambda^2)]\sim -\tfrac{3}{4}(\alpha/\pi)^2\ln\Lambda^2$$

is the appropriate cutoff-dependent piece of $(d)+(e)$ [Eq. (3.26)].

previous expression for (3.6) is unique (in a sense we shall specify) to within an over-all constant factor.

From the renormalization theory viewpoint we need essentially simply make subtractions at points other than those used to define E_1-E_3. It is more convenient, however, to work directly with the integrals (3.10)–(3.12). This involves no loss of generality since there is a one-one correspondence between conventional renormalization theories and theories defined by integrals of the type (3.10)–(3.12).[27]

Let us suppose there exist other subtraction functions $E_1'-E_3'$ which yield finite expressions for the vertex functions via (3.10)–(3.14). We call the resulting vertex functions $\Pi_{\mu\nu}{}^{5\prime}$, $F_{\mu\nu\kappa}{}^{5\prime}$, and $\Gamma_\mu{}^{5\prime}$. As discussed in part B, we must have

$$\Pi_{\mu\nu}{}^{5\prime}(p)=0 \tag{3.34}$$

and

$$\Gamma_{\mu\nu\kappa}{}^{5\prime}(0,0)=0. \tag{3.35}$$

The analogs of (3.20) and (3.21), however, are not specified and so we define the constants Q and R by

$$(\partial/\partial p^\alpha)F_{\mu\nu\kappa}{}^{5\prime}(p,0)\,|_{p=0}=Q\epsilon_{\mu\kappa\alpha\nu} \tag{3.36}$$

and

$$\Gamma_\mu{}^{5\prime}(p,p')\,|_{p=p'=m}=R\gamma_\mu\gamma_5. \tag{3.37}$$

The relation between the new and the old integral equations can be simply expressed in terms of the current definition (3.6) which can be written (with suppressed arguments)

$$\bar\psi\gamma_\mu\gamma_5\psi\sim(1-E_3)j_\mu{}^5-E_1\epsilon_{\mu\alpha\beta\gamma}\xi_\alpha\partial_\beta A_\gamma -E_2\epsilon_{\mu\alpha\beta\gamma}A_\alpha\partial_\beta A_\gamma \tag{3.38}$$

for $\xi\sim0$. Now (3.38) does not uniquely define the set

$$j_\mu{}^5,\quad E_1,\quad E_2,\quad E_3, \tag{3.39}$$

since if (3.39) satisfies (3.38), than formally so does

$$j_\mu{}^{5\prime}\equiv aj_\mu{}^5+b_\alpha\epsilon_{\mu\alpha\beta\gamma}\partial_\beta A_\gamma+b\epsilon_{\mu\alpha\beta\gamma}A_\alpha\partial_\beta A_\gamma, \tag{3.40}$$

with a, b_α, and b constants, for suitable new E_i'. As shown in I, this way of generating new E_i' is equivalent to that of the preceding paragraph. In the present case, however, we must have $b_\alpha=0$ for covariance and also $b=0$ for $j_\mu{}^{5\prime}$ to be finite since matrix elements of $A\partial A$ are divergent, in general [e.g., Fig. 2(e)].[28] Thus, the only possible change which can be made in (3.6) is to multiply it by a finite constant a.[29]

It is nevertheless instructive to consider (3.40) with $b\neq0$ (but $b_\alpha=0$) in order to see what goes wrong in terms of the integral equations. The E_i corresponding to (3.40) are

$$E_1'=E_1, \tag{3.41}$$

$$E_2'=E_2+(b/a)(1-E_3), \tag{3.42}$$

$$E_3'=E_3+[1-(1/a)](1-E_3). \tag{3.43}$$

Equation (3.41) follows directly from (3.34). Equations (3.11) and (3.12) give, symbolically,[30]

$$F'-F=\int_k\left[\frac{b}{a}\int_{k'}(\delta-E_3)\epsilon Hk' +\left(1-\frac{1}{a}\right)\delta F'+\frac{E_3}{a}F'-E_3F\right], \tag{3.44}$$

$$\Gamma'-\Gamma=\int_k\left[\frac{b}{a}\int_{k'}(\delta-E_3)\epsilon Kk' +\left(1-\frac{1}{a}\right)\delta\Gamma'+\frac{E_3}{a}\Gamma'-E_3\Gamma\right]. \tag{3.45}$$

Applying $\partial/\partial p^\alpha$ to (3.44) and putting $p=q=0$ gives [using (3.36)]

$$Q=b, \tag{3.46}$$

and evaluation of (3.45) on the mass-shell gives [using (3.37)]

$$R=a+bY/E_2. \tag{3.47}$$

We immediately have an inconsistency unless $b=0$, since Y/E_2 is divergent, whereas Q, R, a, and b must be finite.

Let us finally consider Eqs. (3.44) and (3.45) in second order:

$$F_{\mu\nu\kappa}{}^{(2)\prime}=a^{(0)}F_{\mu\nu\kappa}{}^{(2)}-[b^{(2)}+b^{(0)}(Z_1{}^{-1})^{(2)}] \times\epsilon_{\mu\nu\beta\kappa}(q_\beta-p_\beta), \tag{3.48}$$

$$\Gamma^{(2)\prime}=a^{(0)}\Gamma^{(2)}+b^{(0)}\int_k\epsilon kK^{(2)}. \tag{3.49}$$

The divergence of the integral in (3.49) again gives $b^{(0)}=0$. Equation (3.48) is then finite, and so it alone

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[27] See I, Sec. V.

[28] Thus, this uniqueness is a consequence of our using the ordinary separated product $A(x)\partial A(x+\xi)$ in (3.6). Had we instead used a generalized Wick product $:A(x)\partial A(x):$, then (3.6) would not have been explicitly unique, but this nonuniqueness would have been illusory, amounting simply to the nonuniqueness of $:A\partial A:$.

[29] The reason for only considering finite a and b in (3.40) is that, as discussed in Ref. 3, only then does one have an axial-vector current $j_\mu{}^{5\prime}$ which is local with respect to the fields in the sense that it is the limit of $\bar\psi(x)\gamma_\mu\gamma_5\psi(x+\xi)$ and $A_\mu(x)\partial_\nu A_\kappa(x+\xi)$ for $\xi\to0$, so that an equation of the form (3.38) is valid. It may be possible to construct a finite $j_\mu{}^{5\prime}$ by using an infinite a which, if it could be written

$$j_\mu{}^{5\prime}(x)=\lim_{\eta\to0}[a(\eta)j_\mu{}^5(x)+b\epsilon_{\mu\alpha\beta\gamma}A_\alpha(x)\partial_\beta A_\gamma(x+\eta)],$$

would be local, but not in the above sense (since two limits are involved). In this case, a is fixed by (3.47). Our uniqueness claim is with respect to local currents in the sense of (3.38). For the case of an external c-number electromagnetic potential, R. Jackiw and K. Johnson (to be published) have defined nongauge-invariant axial-vector currents. If their definitions can be extended to define currents finite to all orders, then these currents will not be local in our sense. Put differently, as they remark, their nongauge-invariant currents can *only* be defined by Lorentz-covariant limits.

[30] The δ's below signify four-dimensional momentum-space delta functions.

does not *a priori* require that $b^{(2)}=0$. Adler[1] has pointed out, however, that (3.48) is consistent with the requirement that an axial-vector meson cannot decay into two photons only if $b^{(2)}=0$. We have derived the stronger result that the theory can be *finite* only if $b=0$.[29]

In summary then, we have obtained the result that a finite local axial-vector current of dimension three exists in electrodynamics if and only if it is a (finite) constant multiple of (3.5), with E_1-E_3 given by (3.23), (3.24), and (3.27). The "if" part of this statement was obtained in part B and the "only if" part was established above by showing that any other set $E_1'-E_3'$ is consistent with finiteness only if essentially $E_i=E_i'$, $i=1, 2, 3$. In the remainder of this section we shall investigate some of the properties of (3.5).

D. Gauge Invariance

We have constructed the unique local axial-vector current (3.5) essentially using only the requirements of finiteness and Lorentz covariance. Further conditions cannot be imposed but can only serve as checks of the usefulness of (3.5). One such condition is local gauge invariance; i.e., invariance under (2.12). If the current is to be an observable quantity,[31] it must be gauge-invariant, and so this invariance should be considered as a consistency check on the above formalism. We show in this subsection that (3.5) is indeed gauge-invariant and, conversely, that the requirement of gauge invariance is sufficient to determine E_1 and E_2. In the course of the analysis we construct explicit expressions for $E_1(\xi)$ and $E_2(\xi)$.

We make the mild assumption that the transformations (2.12) can be taken inside of the $\xi\to 0$ limit in (3.5), at least if this limit is taken in a Lorentz-covariant way.[14] The covariant transformation (2.12) should commute with the covariant limit.[32]

Let us now determine the consequences of gauge invariance. Under the transformation (2.12), the first term in (3.6) acquires the factor

$$e^{ie[\Lambda(x)-\Lambda(x+\xi)]}=1-ie\xi\cdot\partial\Lambda(x)+O(\xi^2). \quad (3.50)$$

Since no matrix element of $\bar\psi(x)\gamma_\mu\gamma_5\psi(x+\xi)$ behaves worse than ξ^{-1} (within logarithmic factors) for $\xi\to 0$, only the first two terms in (3.50) will contribute for $\xi\to 0$. The second term in (3.6) is gauge-invariant because of the antisymmetry of $\epsilon_{\mu\alpha\beta\gamma}$. Under (2.12), the third term in (3.6) goes into the sum of itself, and

$$E_2(\xi)\epsilon_{\mu\alpha\beta\gamma}\partial_\alpha\Lambda(x)\partial_\beta A_\gamma(x+\xi)=E_2(\xi)\epsilon_{\mu\alpha\beta\gamma}\partial_\alpha\Lambda(x)\partial_\beta A_\gamma(x)+O[\xi E_2(\xi)]. \quad (3.51)$$

Since $E_2(\xi)$ is at worst logarithmically divergent, only the first term in (3.51) will contribute for $\xi\to 0$. Thus we find that, apart from terms which vanish for $\xi\to 0$, the transformation (2.12) induces in $j_\mu{}^5(x;\xi)$ the change

$$\delta j_\mu{}^5(x;\xi)=\bar\psi(x)\gamma_\mu\gamma_5\psi(x+\xi)[-ie\xi\cdot\partial\Lambda(x)]+E_2(\xi)\epsilon_{\mu\alpha\beta\gamma}\partial_\alpha\Lambda(x)\partial_\beta A_\gamma(x). \quad (3.52)$$

Our task now is to determine the restrictions imposed on E_1 and E_2 by the requirement that

$$\lim_{\xi\to 0}\delta j_\mu{}^5(x;\xi)=0. \quad (3.53)$$

The vacuum expectation value of (3.52) vanishes identically, and so we consider the $\langle\gamma,p|\cdots|0\rangle$ matrix element, where $\langle\gamma,p|$ is a (covariantly normalized) one-photon state of momentum p and polarization ϵ_ν. We define $I_{\mu\nu}$ by

$$\epsilon_\nu I_{\mu\nu}(\xi)=\langle\gamma,p|\bar\psi(x)\gamma_\mu\gamma_5\psi(x+\xi)|0\rangle, \quad (3.54)$$

and, for clarity, we first calculate $I(\xi)$ for $\xi\sim 0$ in lowest order[33]:

$$I_{\mu\nu}{}^{(1)}(\xi)=(-ie)e^{-ip\cdot x}\int_k e^{ik\cdot\xi}\,\mathrm{tr}\gamma_\mu\gamma_5 \times\frac{k+m}{k^2-m^2}\gamma_\nu\frac{k-p+m}{(k-p)^2-m^2}. \quad (3.55)$$

Evaluation of the trace and integration by parts gives

$$I_{\mu\nu}{}^{(1)}(\xi)=(-4e)e^{-ip\cdot x}\epsilon_{\mu\alpha\nu\beta}p_\beta(-i\partial_\alpha)\times\int_k e^{ik\cdot\xi}\frac{1}{k^2-m^2}\frac{1}{(k-p)^2-m^2}. \quad (3.56)$$

For $\xi\to 0$ only the $O(\xi^{-1})$ part of (3.56) will contribute to (3.52). This comes from the leading term in the integral in (3.56), which is easily[34] found to be $[(2\pi)^2 4i]^{-1}\ln\xi^2$. Thus

$$I_{\mu\nu}{}^{(1)}(\xi)=[2e/(2\pi)^2]e^{-ip\cdot x}\epsilon_{\mu\alpha\nu\beta}p_\beta(\xi_\alpha/\xi^2), \quad (3.57)$$

apart from terms which do not contribute to (3.52).

Using (3.57) and

$$\langle\gamma,p|F_{\beta\gamma}(x)|0\rangle=\epsilon_\nu(-ip_\gamma g_{\beta\nu}+ip_\beta g_{\gamma\nu})e^{-ip\cdot x}, \quad (3.58)$$

the requirement that the photon-vacuum matrix element of (3.52) vanish for $\xi\to 0$ in second order uniquely determines the second-order E_2 to be

$$E_2{}^{(2)}(0)=e^2/8\pi^2. \quad (3.59)$$

Thus $E_2{}^{(2)}(\xi)$ is nonsingular for $\xi\to 0$ and we essentially have

$$E_2{}^{(2)}(\xi)=e^2/8\pi^2. \quad (3.60)$$

[31] According to the Gell-Mann current-algebra philosophy, the axial-vector current is an observable, measurable via the weak interactions.

[32] This is in contrast to the equal-time commutator operation for which the limit $\xi=(0,\boldsymbol{\xi})\to 0$ is clearly appropriate.

[33] The indicated Fourier transform in (3.55) is strictly meaningful only as a distribution.

[34] A quick way to obtain this result is to use the expansion for the Feynman propagator $\Delta_F(\xi)$ for $\xi^2\sim 0$.

It is not much more difficult to determine E_2 to all orders. The definition (3.54) gives, in general,

$$I_{\mu\nu}(\xi)=(-ie)e^{-ip\cdot x}\int_k e^{ik\cdot\xi}\,\mathrm{tr}\gamma_\mu\gamma_5 G(k)\Gamma_\nu \times(k,\,k-p)G(k-p)\,. \quad (3.61)$$

Let us write

$$G(k)=a(k^2)\mathbf{k}+b(k^2)\,, \quad (3.62)$$

and

$$\Gamma_\nu(k,\,k-p)=a'(k^2,\,k\cdot p)\gamma_\nu+b_\nu'(k,p)\mathbf{k}+c_\nu'(k,p) +d_\nu'(k,p)\mathbf{p}+e_{\nu\alpha\beta}'(k,p)\sigma_{\alpha\beta}\,. \quad (3.63)$$

We then have the relevant high-k behaviors

$$G(k)\rightarrow a(k^2)\mathbf{k}\,, \qquad \Gamma_\nu(k,\,k-p)\rightarrow a'(k^2,0)\gamma_\nu\equiv a'(k^2)\gamma_\nu\,. \quad (3.64)$$

Equations (3.62)–(3.64) imply that

$$\mathrm{tr}\gamma_\mu\gamma_5 G(k)\Gamma_\nu(k,\,k-p)G(k+p) \rightarrow aa'a\;\mathrm{tr}\gamma_\mu\gamma_5\mathbf{k}\gamma_\nu\mathbf{p} \quad (3.65)$$

for $k\rightarrow\infty$. They also imply that

$$\mathrm{tr}\gamma_\mu\gamma_5 G(k)\Gamma_\nu(k,k)G(k)\mathbf{k}\mathbf{p}(k^2-m^2)^{-1} \rightarrow aa'a\;\mathrm{tr}\gamma_\mu\gamma_5\mathbf{k}\gamma_\nu\mathbf{p}\,. \quad (3.66)$$

Hence, using the Ward identity (2.15), we have

$$I_{\mu\nu}(\xi)\rightarrow(ie)e^{-ip\cdot x}\int_k e^{ik\cdot\xi}\,\mathrm{tr}\gamma_\mu\gamma_5\partial_\nu G(k)\mathbf{k}\mathbf{p} \times(k^2-m^2)^{-1} \quad (3.67)$$

for $\xi\rightarrow 0$.

We next use the spectral representation

$$G(k)=\int_{-\infty}^{\infty}d\kappa\,\sigma(\kappa)G_0(k;\kappa)\,, \quad (3.68)$$

where

$$G_0(k;\kappa)=(\mathbf{k}+\kappa)/(k^2-\kappa^2) \quad (3.69)$$

and, formally,

$$\int_{-\infty}^{\infty}d\kappa\,\sigma(\kappa)=Z_1^{-1}\,. \quad (3.70)$$

Insertion of (3.68) in (3.67) and integration by parts (only the $\partial_\nu\mathbf{k}=\gamma_\nu$ term contributes) gives

$$I_{\mu\nu}(\xi)\rightarrow(-ie)e^{-ip\cdot x}\int_{-\infty}^{\infty}d\kappa\,\sigma(\kappa)\int_k e^{ik\cdot\xi}\,\mathrm{tr}\gamma_\mu\gamma_5\mathbf{k}\gamma_\nu\mathbf{p} \times(k^2-\kappa^2)^{-1}(k^2-m^2)^{-1}\,. \quad (3.71)$$

The k integral in (3.71) is essentially the same as that in (3.55) and so, using (3.70), we obtain[35]

$$I_{\mu\nu}(\xi)\rightarrow Z_1^{-1}[2e/(2\pi)^2]e^{-ip\cdot x}\epsilon_{\mu\alpha\nu\beta}p_\beta(\xi_\alpha/\xi^2)\,, \quad (3.72)$$

apart from terms which do not contribute to (3.52) for $\xi\rightarrow 0$. The result (3.72), which is valid to all orders in e, reduces to (3.57) in lowest order. It follows that

$$E_2(\xi)=(e^2/8\pi^2)Z_1^{-1} \quad (3.73)$$

in each order of perturbation theory. More precisely, we should write

$$E_2(\xi)=(e^2/8\pi^2)E(\xi)\,, \quad (3.74)$$

where $E(\xi)$ has a logarithmic singularity at $\xi=0$, formally given by $E(0)=Z_1^{-1}$. Thus,

$$E(\xi)=1-(e^2/8\pi^2)\ln\xi^2+O(e^4)\,. \quad (3.75)$$

We have determined $E_2(\xi)$ from the requirement that the photon-vacuum matrix element of (3.52) vanish. We next determine $E_1(\xi)$ directly from the operator condition (3.53). We shall use Eq. (3.6) to rewrite the first term in (3.52). It follows from the existence of the limit (3.5) and the fact that $E_2(\xi)A(x)\partial A(x+\xi)$ is only logarithmically divergent for $\xi\rightarrow 0$ that

$$\bar\psi(x)\gamma_\mu\gamma_5\psi(x+\xi)\xi_\nu\rightarrow -E_1(\xi)\xi_\alpha\xi_\nu\epsilon_{\mu\alpha\beta\gamma}\partial_\beta A_\gamma(x) \quad (3.76)$$

for $\xi\rightarrow 0$. Thus, by (3.52),

$$\delta j_\mu{}^5(x;\,\xi)\rightarrow\{-E_1(\xi)\xi_\alpha\xi_\nu\epsilon_{\mu\alpha\beta\gamma}[-ie\partial_\nu\Lambda(x)] +E_2(\xi)\epsilon_{\mu\alpha\beta\gamma}\partial_\alpha\Lambda(x)\}\partial_\beta A_\gamma(x)\,.$$

The requirement (3.53) now gives

$$E_1(\xi)=-\frac{4}{ie}\frac{1}{\xi^2}E_2(\xi)=\frac{ie}{2\pi^2}\frac{1}{\xi^2 Z_1}=\frac{ie}{2\pi^2}\frac{E(\xi)}{\xi^2}\,, \quad (3.77)$$

which exhibits the expected quadratic singularity of $E_1(\xi)$.[36]

We have seen that gauge invariance requires that E_1 and E_2 have the rather explicit values (3.73) and (3.77). Are these the same values as those given by the finiteness requirements of parts B and C? It is not difficult to show that they are indeed equivalent.

We consider first E_1. In position space, Eq. (2.23) becomes

$$\xi_\alpha E_1(\xi)\epsilon_{\mu\alpha\beta\nu}=-e\int_k e^{ik\cdot\xi}\,\mathrm{tr}\gamma_\mu\gamma_5\frac{\partial}{\partial p^\beta}G(k)\Gamma_\nu \times(k,\,k-p)G(k-p)\big|_{p=0}\,. \quad (3.78)$$

Now the contraction of the integrand in (3.78) with p_β gives the leading high-k contribution to the integrand in (3.61) so that, by (3.72), we have

$$\xi_\alpha E_1(\xi)\epsilon_{\mu\alpha\beta\nu}=Z_1^{-1}[2ie/(2\pi)^2]\epsilon_{\mu\alpha\nu\beta}(\xi_\alpha/\xi^2)\,,$$

which is completely equivalent to (3.77).[36]

We can similarly show that Eqs. (3.24) and (3.73) are equivalent. We omit the details and only remark that the result follows from use of the forms (3.62)–(3.64) and the Ward identity (2.16) in (3.24).

[35] A quick formal way of obtaining this result is to use $G\rightarrow Z_1^{-1}\mathbf{k}/k^2$ and $\Gamma_\nu\rightarrow Z_1\gamma_\nu$ in (3.61).

[36] The derivation above would allow $E_1(\xi)$ to have, in addition, a logarithmically divergent piece. Such a piece would never contribute to anything, however, since $E_1(\xi)$ occurs in the current definition as $E_1(\xi)\xi_\alpha$.

We thus have the result that the unique[29] finite current of part B is gauge-invariant. In fact, we have seen that gauge invariance uniquely determines E_1 and E_2.[37] We have also derived the simple forms (3.73) and (3.77) for the subtraction functions.[38]

We conclude this subsection by remarking that the vertex function (3.11) can be shown to satisfy the expected consequence of the established gauge invariance:

$$p_\mu F_{\mu\nu\kappa}{}^5(p,q)=q_\kappa F_{\mu\nu\kappa}{}^5(p,q)=0. \qquad (3.79)$$

These equations imply (3.19) and (3.20), and so we see that the choice of the conditions (3.19) and (3.20), which was shown in part C to be the only one consistent with finiteness, is also the only one consistent with gauge invariance.

E. Commutation Relations

Let us write the axial-vector current symbolically as

$$j_\mu{}^5=\bar\psi\gamma_\mu\gamma_5\psi_+ + E_1\epsilon_{\mu\alpha\beta\gamma}\xi_\alpha\partial_\beta A_\gamma + E_2\epsilon_{\mu\alpha\beta\gamma}A_\alpha\partial_\beta A_{\gamma+}+E_3 j_\mu{}^5, \qquad (3.80)$$

where E_i means $E_i(\xi)$, a $+$ subscript means evaluation at $x+\xi$, and a limit $\xi\to 0$ is understood. We have seen above formally that

$$E_1=(ie/2\pi^2\xi^2)Z_1^{-1},\quad E_2=(e^2/8\pi^2)Z_1^{-1},\quad E_3=-X-Y,\quad X=Z_1^{-1}-1. \qquad (3.81)$$

We shall work with these formal expressions for the remainder of this section. Our first task is to write (3.80) in a form convenient for discussion of its algebraic properties.

In order to express the final term in (3.80) as a multiplicative renormalization, we define the function $E_0=E_0(\xi)$ by

$$(1-E_3)^{-1}=E_0Z_1. \qquad (3.82)$$

Thus,

$$E_0=Z_1^{-1}(1-E_3)^{-1} =(1+Z_1Y)^{-1}=1-Y^{(4)}+O(e^6), \qquad (3.83)$$

and (3.80) becomes

$$j_\mu{}^5=E_0Z_1(\bar\psi\gamma_\mu\gamma_5\psi_+ + E_1\epsilon_{\mu\alpha\beta\gamma}\xi_\alpha\partial_\beta A_\gamma+E_2\epsilon_{\mu\alpha\beta\gamma}A_\alpha\partial_\beta A_{\gamma+}) \qquad (3.84)$$

$$=E_0(Z_1\bar\psi\gamma_\mu\gamma_5\psi_+ +(ie/2\pi^2\xi^2)\epsilon_{\mu\alpha\beta\gamma}\xi_\alpha\partial_\beta A_\gamma +(e^2/8\pi^2)\epsilon_{\mu\alpha\beta\gamma}A_\alpha\partial_\beta A_{\gamma+}). \qquad (3.85)$$

In terms of the unrenormalized quantities

$$\psi_0\equiv Z_1^{1/2}\psi,\quad A_0\equiv Z_3^{1/2}A,\quad e_0\equiv Z_3^{-1/2}e, \qquad (3.86)$$

we have[39]

$$j_\mu{}^5=E_0[\bar\psi_0\gamma_\mu\gamma_5\psi_{0+}+(ie_0/2\pi^2\xi^2)\epsilon_{\mu\alpha\beta\gamma}\xi_\alpha\partial_\beta A_{0\gamma} +(e_0^2/8\pi^2)\epsilon_{\mu\alpha\beta\gamma}A_{0\alpha}\partial_\beta A_{0\gamma+}]. \qquad (3.87)$$

We define equal-time commutators as in III, where the equal-time and $\xi\to 0$ limits are interchanged and the field commutation relations (III 5.1)–(III 5.6) are employed. Owing mainly to the logarithmically divergent factor E_0 in (3.85), we shall find the equal-time commutators to be generally ill defined. We find, for example,[16]

$$[j_\mu{}^5(x),\psi(x')]=-E_0\gamma_0\gamma_\mu\gamma_5\psi(x)\delta(\mathbf{x}-\mathbf{x}'). \qquad (3.88)$$

In particular,

$$[j_0{}^5(x),\psi(x')]=-E_0\gamma_5\psi(x)\delta(\mathbf{x}-\mathbf{x}'). \qquad (3.89)$$

Thus, these commutators diverge in fourth order. As discussed in III (Appendix), such expressions can nevertheless be given a meaning. Similarly,[40]

$$[j_\mu{}^5(x),A_\nu(x')]=iE_0Z_3^{-1}[(ie/2\pi^2\xi^2)\epsilon_{\mu\alpha0\nu}\xi_\alpha +(e^2/8\pi^2)\epsilon_{\mu\alpha0\nu}A_\alpha(x)]\delta(\mathbf{x}-\mathbf{x}'). \qquad (3.90)$$

The nonvanishing of this commutator arises from the presence of the "noncanonical" subtraction terms in (3.85), i.e., from the finiteness and/or gauge invariance of $j_\mu{}^5$.

The commutator of $j_\mu{}^5$ with itself behaves similarly. We note here only the fact that the $[j_0{}^5,j_k{}^5]$ commutator contains, in addition to derivative of δ-function (Schwinger) terms with operator coefficients involving $\mathbf{A}$ and $\mathbf{A}^2$, pure δ-function terms

$$[j_0{}^5(x),j_k{}^5(x')]=-iZ_3^{-1}E_0^2(\alpha/2\pi)[(ie/2\pi^2\xi^2)\times\epsilon_{0mij}\epsilon_{0mkl}\xi_l\partial_iA_j(x+\xi)+(e^2/8\pi^2)\epsilon_{0mij}\epsilon_{0mkl}A_l(x)\partial_iA_j\times(x+\xi)]\delta(\mathbf{x}-\mathbf{x}')+\text{derivatives of }\delta. \qquad (3.91)$$

Lee and Zumino[41] have shown that gauge invariance requires that a term of the form $A_k\delta(\mathbf{x}-\mathbf{x}')$ be present in the commutator $[J_0,J_k{}^\dagger]$ involving a charged current J_μ. It can be said that the similar terms in (3.91) arise as a more subtle consequence of gauge invariance.

[37] Although, as we have seen above, this statement is logically correct, the situation *can* be described in another way. Namely, Eq. (3.6) and the known behaviors of the E_i give immediately $\bar\psi(x)\gamma_\mu\gamma_5\psi(x+\xi)\xi_\nu+E_1(\xi)\epsilon_{\mu\alpha\beta\gamma}\xi_\alpha\xi_\nu\partial_\beta A_\gamma(x)\sim 0$ and this equation determines E_1 (see Ref. 36). The value of E_2 then follows from gauge invariance, as in either (3.73) or (3.77). From this viewpoint, finiteness alone determines E_1 and gauge invariance determines just E_2. Clearly, *either* finiteness *or* gauge invariance determines E_1, and these determinations are consistent with each other.

[38] E_3 could be given a correspondingly simple form, but such a description does not seem illuminating.

[39] In this language, Adler's use of regularized perturbation theory amounts to the statement that the current he considered is the coefficient of E_0 in (3.87).

[40] Strictly speaking, since (3.87) is only valid for a covariant limit $\xi\to 0$, it cannot be directly used to compute equal-time commutation relations. This will not affect the above commutators but might change those given below. We shall nevertheless work with (3.87), however, and so the following relations should only be considered formally. S. Adler (private communication) has found, in fact, that in lowest order the commutators defined by the Bjorken limit differ from ours in that (3.90) vanishes and (3.92) is multiplied by 4. Hopefully, our use of a sequence that converges to $j_\mu{}^5(x)$ for spacelike $\xi\to 0$ would resolve this discrepancy.

[41] T. D. Lee and B. Zumino, Phys. Rev. **163**, 1667 (1967).

Finally, using (2.4), we find

$$[j_0(x),j_0{}^5(x')]_T = (ie/8\pi^2)E_0Z_3{}^{-1}\epsilon_{0\alpha\beta\gamma}\partial_\beta A_\gamma\partial_\alpha\delta(\mathbf{x}-\mathbf{x}'). \quad (3.92)$$

Here we see an operator Schwinger term which is finite in lowest order. The charge-current commutators, however, vanish as expected:

$$[Q,j_0{}^5(x')]=[j_0(x),Q^5(x)_0]=0. \quad (3.93)$$

We conclude from the above examples, that although all matrix elements (3.9) involving $j_\mu{}^5$ are finite, equal-time commutators involving $j_\mu{}^5$ are divergent. This state of affairs analogs the difficulties generally encountered when one tries to work with nonconserved currents in quantum field theory.[42]

F. Divergence of the Current

In this subsection we shall derive and discuss the divergence $\partial_\mu j_\mu{}^5$ of the axial-vector current. We shall find that the terms in (3.80) explicitly involving the electromagnetic potential imply the presence of a term of the form $\epsilon_{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta}$ in $\partial_\mu j_\mu{}^5$ in addition to the usual in $\bar\psi\gamma_5\psi$ term. We will thus understand Adler's divergence condition directly in terms of the local formulation of electrodynamics and the local expression (3.80) for $j_\mu{}^5$.

We consider first the trivial case when $A_\mu(x)$ is a prescribed external c-number field. Then all radiative connections vanish ($m=m_0$, $Z_1=Z_3=E_0=1$) and the Dirac equation [(2.1)] becomes

$$(i\gamma\cdot\partial-m)\psi=eA\psi, \quad (3.94)$$

the field product here now, of course, being well defined. The current becomes

$$j_\mu{}^5=\bar\psi\gamma_\mu\gamma_5\psi_+ +(ie/2\pi^2\xi^2)\epsilon_{\mu\alpha\beta\gamma}\xi_\alpha\partial_\beta A_\gamma +(e^2/8\pi^2)\epsilon_{\mu\alpha\beta\gamma}A_\alpha\partial_\beta A_\gamma. \quad (3.95)$$

Equation (3.94) gives

$$\partial_\mu(\bar\psi\gamma_\mu\gamma_5\psi_+)=2im\bar\psi\gamma_5\psi_+ - ie\bar\psi\gamma_\nu\gamma_5\psi_+ \times[A_\nu(x+\xi)-A_\nu(x)]. \quad (3.96)$$

Multiplication of (3.95) by ξ_κ gives for $\xi\to 0$

$$\bar\psi\gamma_\mu\gamma_5\psi_+\xi_\kappa\to -(ie/2\pi^2\xi^2)\xi_\alpha\xi_\kappa\epsilon_{\mu\alpha\beta\gamma}\partial_\beta A_\gamma \to -(ie/8\pi^2)\epsilon_{\mu\kappa\beta\gamma}\partial_\beta A_\gamma, \quad (3.97)$$

so that, upon writing

$$A_\nu(x+\xi)-A_\nu(x)=\xi\cdot\partial A_\nu(x)+O(\xi^2), \quad (3.98)$$

only the $O(\xi)$ term contributes to (3.96) for $\xi\to 0$. Thus,

$$\partial_\mu(\bar\psi\gamma_\mu\gamma_5\psi_+)\to 2im\bar\psi\gamma_5\psi+(e^2/8\pi^2)\epsilon_{\mu\alpha\beta\gamma}\partial_\mu A_\alpha\partial_\beta A_\gamma.$$

The divergence of the second term in (3.95) vanishes, and the divergence of the third term is $(e^2/8\pi^2)\epsilon_{\mu\alpha\beta\gamma}\times\partial_\mu A_\alpha\partial_\beta A_\gamma$. Thus we have

$$\partial_\mu j_\mu{}^5=2im\bar\psi\gamma_5\psi+(e^2/4\pi^2)\epsilon_{\mu\alpha\beta\gamma}\partial_\mu A_\alpha\partial_\beta A_\gamma =2imj^5+(\alpha/4\pi)\epsilon_{\mu\alpha\beta\gamma}F_{\mu\alpha}F_{\beta\gamma}, \quad (3.99)$$

in agreement with the result of Adler.[43]

The same method *cannot* be used to derive $\partial_\mu j_\mu{}^5$ for the case of an interacting electromagnetic field because of ambiguities arising from the two different limits [$\eta\to 0$ in (2.1) and $\xi\to 0$ in (3.6)] involved. In particular, an expansion of the form (3.98) is not effective because of the increasing degree of divergence of the local products $\bar\psi\psi\partial^n A$. All that can be said is that the divergence has the form

$$\partial_\mu j_\mu{}^5=2imVj^5+2WE_2\epsilon_{\mu\alpha\beta\gamma}\partial_\mu A_\alpha\partial_\beta A_\gamma \quad (3.100)$$

for some constants V and W. Here j^5 is the finite local pseudoscalar current operator

$$j^5=Z_1\bar\psi\gamma_5\psi=\bar\psi_0\gamma_5\psi_0. \quad (3.101)$$

It is normalized as usual so that

$$\langle p|j^5|p'\rangle=\bar u(p)\gamma_5u(p')F((p-p')^2),\quad F(0)=1. \quad (3.102)$$

We shall derive the values of V and W directly from (3.84) and (3.100) at the end of this subsection. For now we simply observe that a diagrammatic analysis similar to that given by Adler[1] gives the expected results

$$V=E_0,\quad W=E_0Z_1, \quad (3.103)$$

so that

$$\partial_\mu j_\mu{}^5=E_0Z_1(2im\bar\psi\gamma_5\psi_+ +2E_2\epsilon_{\alpha\beta\gamma\delta}\partial_\alpha A_\beta\partial_\gamma A_{\delta+}) \quad (3.104)$$

$$=E_0(2imj^5+(\alpha/4\pi)\epsilon_{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta+}) \quad (3.105)$$

$$=E_0(2im\bar\psi_0\gamma_5\psi_0+(\alpha_0/4\pi)\epsilon_{\alpha\beta\gamma\delta}F_{0\alpha\beta}F_{0\gamma\delta+}). \quad (3.106)$$

The divergence condition (3.106) can be used together with the commutation relation (3.89) to derive generalized Ward identities. We formally define the vertex function $R(p,p')$ by

$$\mathfrak{F}\langle T\partial_\mu j_\mu{}^5(x)\psi(y)\bar\psi(z)\rangle=G(p)R(p,p')G(p') \quad (3.107)$$

and obtain in the usual way

$$\Delta_\mu\Gamma_\mu{}^5(p,p') =R(p,p')+E_0[G^{-1}(p)\gamma_5+\gamma_5G^{-1}(p')], \quad (3.108)$$

where

$$\Delta_\mu=p_\mu-p_\mu'. \quad (3.109)$$

Since $\Gamma_\mu{}^5$ and G^{-1} are finite functions, and E_0 is diver-

[42] For a survey see C. A. Orzalesi, University of Maryland Technical Report No. 833, 1968 (unpublished).

[43] A derivation of (3.99) from the exponential form

$$\psi(x')\beta\gamma_\mu\gamma_5\exp\left[ieq\int_{x''}^{x'}dx\cdot A\right]\psi(x'')$$

has been given by C. R. Hagen, Phys. Rev. **177**, 2622 (1969), and also by R. Jackiw and K. Johnson, Ref. 29, and J. Schwinger, Phys. Rev. **82**, 664 (1951). This form is only valid, however, for an external potential, since in the interacting case increasing powers of A in the expansion of the exponential cause increasingly worse divergences for $\xi\to 0$.

gent, we learn from (3.108) the surprising result that *the time-ordered product*

$$\langle T\partial_\mu j_\mu{}^5(x)\psi(y)\bar{\psi}(z)\rangle \qquad (3.110)$$

of renormalized field operators is divergent. That is, although $j_\mu{}^5$ and all the Green's functions (3.29), in particular (3.9), are finite, the product $\partial_\mu j_\mu{}^5(x)\psi(y)\bar{\psi}(z)$ is too singular at equal times to be multiplied by the θ functions necessary to define (3.110). Of course, since $j_\mu{}^5$ is finite, all matrix elements of $\partial_\mu j_\mu{}^5$ [e.g., $\langle p|\partial_\mu j_\mu{}^5|p'\rangle$, the electron-electron element] must be finite. But such matrix elements cannot be extended off the mass shell by (3.110). This is consistent with (3.108) which becomes, on shell, the finite relation

$$\Delta_\mu \bar{u}\Gamma_\mu{}^5(p,p')u=\bar{u}R(p,p')u. \qquad (3.111)$$

The divergence of (3.110), of course, is not an inconsistency and might be expected in view of the singular commutators encountered in part E and the fact that $\partial_\mu j_\mu{}^5$, on dimensional grounds alone, should be a more singular object than $j_\mu{}^5$. In particular, the relation (3.108) is a meaningful one.[44] To directly check this in fourth order, we formally define further vertex functions Γ^5 and M by

$$\mathfrak{F}\langle Tj^5(x)\psi(y)\bar{\psi}(z)\rangle=G(p)\Gamma^5(p,p')G(p') \qquad (3.112)$$

and

$$\mathfrak{F}\langle T\epsilon_{\alpha\beta\gamma\delta}F_{\alpha\beta}(x)F_{\gamma\delta}(x)\psi(y)\bar{\psi}(z)\rangle =G(p)M(p,p')G(p'). \qquad (3.113)$$

Then, in view of (3.105), (3.108) becomes

$$\Delta_\mu\Gamma_\mu{}^5(p,p')=2mE_0\Gamma^5(p,p')-iE_0(\alpha/4\pi)M(p,p') +E_0[G^{-1}(p)\gamma_5+\gamma_5G^{-1}(p')]. \qquad (3.114)$$

In fourth order, using (3.83), this becomes

$$\Delta_\mu\Gamma_\mu{}^{5(4)}=\underset{(A)}{2m\Gamma^{5(4)}}-\underset{(B)}{2mY^{(4)}\gamma_5}-\underset{(C)}{i(\alpha/4\pi)M^{(2)}} +\underset{(D)}{(G^{-1}\gamma_5+\gamma_5G'^{-1})^{(4)}}-Y^{(4)}(\underset{(E)}{\gamma\cdot\Delta\gamma_5}-\underset{(F)}{2m\gamma_5}). \qquad (3.115)$$

Let us observe how the various divergent terms in (3.115) conspire to produce a finite result. Terms (A) and (D) are finite and, in view of (3.33), terms (B), (E), and (F) are divergent. Adler[1] has shown that the term (C) satisfies

$$-i(\alpha/4\pi)M^{(2)}(p,p') =-\tfrac{3}{4}(\alpha/\pi)^2(\ln\Lambda^2)\gamma\cdot\Delta\gamma_5+\text{finite}. \qquad (3.116)$$

We see that term (E) exactly cancels the divergent piece of term (C), and that term (F) exactly cancels the divergent term (B), so that (3.115) is finite. Note that, as mentioned above, the term $(A)+(B)+(C)$ corresponding to (3.110) is in general divergent. On shell, however, as must be the case, $(A)+(B)+(C)$ is finite since (B) cancels the divergent piece of (C) on shell and, of course, (E) and (F) cancel on shell. Adler has emphasized that (C) is not multiplicatively renormalizable. In the above language, this corresponds to the fact that the term (E), which cancels its divergent piece, is not a multiple of a lower-order vertex (3.110) but rather a multiple of a lower-order commutator term $(G^{-1}\gamma_5+\gamma_5G'^{-1})$.

It is interesting to see how the various terms in (3.115) arise from the fourth-order diagrams of Fig. 2. We have, symbolically,

$$\Delta_\mu[(c)+(f_X)+(g)]=(A_1)+(D),$$
$$\Delta_\mu[(d)+(e)]=(A_2)+(C) =[(A_2)+(C)+(B)]+(F),$$
$$\Delta_\mu(f_Y)=(E),$$

where $(A_{1,2})$ denotes the contributions to (A) arising from (c), (d)+(e) so that $(A)=(A_1)+(A_2)$, and where $(f_X)=-X^{(4)}\gamma_\mu\gamma_5$, $f_Y=-Y^{(4)}\gamma_\mu\gamma_5$. Note that diagrams (d)+(e) contribute not only to the vertex (3.110) terms $(A)+(B)+(C)$, but also to the commutator terms $(D)+(E)+(F)$. Diagrammatic reasoning of this type can, in fact, be used to derive (3.105).

We conclude this subsection by presenting the promised derivation of (3.104) directly from (3.100) and (3.84). We first sandwich (3.100) between one-electron states $\langle p|$ and $|p'\rangle$. For $\Delta\equiv p-p'\to 0$ each term in (3.100) vanishes, but equating terms of order Δ, using (3.21), (3.102), and (3.26), gives the relation

$$1=V+WY. \qquad (3.117)$$

To obtain more information, we use (3.100) and (3.89) [which follows from (3.84)] in (3.107) and (3.108) to write

$$\Delta_\mu\Gamma_\mu{}^5=2mV\Gamma^5-2iWE_2M+E_0(G^{-1}\gamma_5+\gamma_5G'^{-1}).$$

Putting $\Delta=0$ (note that since $\epsilon_{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta}$ is a total derivitive, $M\propto\Delta$), we get

$$0=2mV\Gamma^5(p,p)+E_0[-2m\gamma_5-2\Sigma_2(p^2)\gamma_5],$$

where we wrote

$$\Sigma(p)=\Sigma_1(p^2)\not{p}+\Sigma_2(p^2).$$

Since E_0 is divergent, this immediately gives $V\propto E_0$ and evaluation at $p^2=m^2$ gives equality:

$$V=E_0. \qquad (3.118)$$

Substitution of (3.118) into (3.117), using (3.83), gives $W=Z_1E_0$. Thus, (3.103) and hence (3.104) is derived.

G. Vector–Axial-Vector Symmetry

When A_μ is simply an external c-number potential, we can consider the quantity

$$\bar{\psi}_0(x)\gamma_\mu(1-\gamma_5)\psi_0(x)+\cdots, \qquad (3.119)$$

[44] In order to convert (3.108) into a strictly meaningful mathematical relation, one should not take the $\xi\to 0$ limits of the individual terms on the right-hand side but only of their sum.

where the omitted terms involve the electromagnetic potential. This term exhibits the V-A symmetry $\gamma_\mu \leftrightarrow \gamma_\mu\gamma_5$ familiar from weak-interaction phenomenology. When A_μ becomes the full quantized electromagnetic potential, however, (3.119) becomes divergent. The corresponding finite quantity is the renormalized current difference

$$j_\mu(x) - j_\mu{}^5(x) = \bar{\psi}_0(x)\gamma_\mu(1-E_0\gamma_5)\psi_0(x) + \cdots. \quad (3.120)$$

This expression still maintains V-A universality in the sense that (2.7) and (3.21) are valid. The physical principle here is that electromagnetism preserves the exact V-A symmetry of the (lowest-order) quantity (3.119).

Put differently, we can say that radiative corrections to (3.119) are not finite since, by (3.87), $Z_2\tilde{\Gamma}^5 = E_0{}^{-1}\Gamma_\mu{}^5$, where $\tilde{\Gamma}^5$ is the unrenormalized proper axial-vector vertex function, and $E_0{}^{-1} = 1 + Y^{(4)} + O(e^6)$ is divergent in order e^4.[45] Radiative corrections to (3.120), on the other hand, are finite.[46]

The expression (3.120) exhibits a $\gamma_\mu \leftrightarrow E_0\gamma_\mu\gamma_5$ symmetry for neutral currents rather than the usual $\gamma_\mu \leftrightarrow \gamma_\mu\gamma_5$ symmetry. The former symmetry in the currents implies the latter symmetry for matrix elements in the sense of (2.7) and (3.38). A more algebraic aspect of the symmetry is exhibited in the transformation properties of the currents under the axial gauge transformations

$$\psi(x) \to \exp[i\gamma_5\Lambda(x)]\psi(x), \quad A_\mu(x) \to A_\mu(x). \quad (3.121)$$

We find roughly

$$j_\mu(x) \to j_\mu(x) - (e^2/8\pi^2)\epsilon_{\mu\alpha\beta\gamma}\partial_\alpha\Lambda(x)\partial_\beta A_\gamma(x), \quad (3.122)$$

$$j_\mu{}^5(x) \to j_\mu{}^5(x) - E_0(e^2/8\pi^2) \times \tfrac{1}{3}\chi_{\mu\alpha\beta\gamma}\partial_\alpha\Lambda(x)\partial_\beta A_\gamma(x). \quad (3.123)$$

Since $\epsilon_{\mu\alpha\beta\gamma}$ and $\frac{1}{3}\chi_{\mu\alpha\beta\gamma}$ are equivalently normalized ($\epsilon_{0123} = \frac{1}{3}\chi_{0000} = 1$), Eqs. (3.122) and (3.123) show that j_μ and $E_0{}^{-1}j_\mu{}^5$ transform similarly under (3.121).

IV. DISCUSSION

In Sec. III we saw that the essentially unique finite local axial-vector current operator $j_\mu{}^5$ of dimension three in quantum electrodynamics is given by Eq. (3.85). The terms explicitly involving the electromagnetic potential were required and, indeed, fixed by the conditions of finiteness or gauge invariance. These subtractions gave rise to noncanonical terms in equal-time commutation relations involving $j_\mu{}^5$ [Eqs. (3.90)–(3.92)] and also to a noncanonical contribution to the divergence $\partial_\mu j_\mu{}^5$ [Eq. (3.105)]. The multiplicative renormalization constant E_0 in (3.84) was required for the finiteness of $j_\mu{}^5$ and was fixed by the normalization condition (3.21). Its presence enhanced the singularities of the equal-time commutators and was responsible for the divergence of the Green's function $\langle T\partial_\mu j_\mu{}^5\psi\bar{\psi}\rangle$ off the electron mass shells.

Results similar to the above ones hold in any model in which the axial-vector current involves local products of spinor fields. We comment on several such models from the viewpoint developed in Sec. III. We begin with the special case of the electrodynamic model in which the spinor particle is massless. Adler[1] has noted that due to the presence of the additional term in (3.99) the axial-vector current is *not* conserved when $m=0$, apparently contradicting the formal invariance of the Lagrangian under the axial gauge transformation (3.121). The resolution of this difficulty is simply the fact that massless electrodynamics is *not* really invariant under (3.121). We have not constructed the Lagrangian to directly check this, but it is sufficient to observe that, in view of (3.122), the equation of motion (2.2) is *not* invariant under (3.121). Thus $j_\mu{}^5$ cannot be conserved.[47]

We next consider the σ model.[48] Arguments similar to those of Sec. III are expected to show that the usual PCAC divergence condition[49] $\partial_\mu j_\mu{}^5 = c\phi(x)$ should be replaced by

$$\partial_\mu j_\mu{}^5 = c\phi(x) + (\alpha/4\pi)\epsilon_{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta}. \quad (4.1)$$

We emphasize that we have not derived (4.1). A rigorous derivation would be quite involved because of the large number of fields present in the electrodynamic σ model. The axial-vector current, in particular, is certainly *not* unique in this model as it was in pure electrodynamics.[50] Nevertheless, (4.1) seems theoretically reasonable and should hold if the usual PCAC does in the absence of electromagnetism.[51] More importantly, as Adler[1] has shown, (4.1) gives good agreement with the experimentally observed $\pi^0 \to 2\gamma$ decay rate in the soft-pion ($\Delta^2=0$) limit, whereas the usual PCAC does not.[52] We feel that this result is important because it shows that the operator product subtleties responsible for the additional term can have an empirical sig-

[45] Of course, according to (3.87), the subtractions involving the potential must be included in (3.119) in order that even $\Gamma_\lambda{}^5$ be finite. We assume these terms are present but do not explicitly exhibit them.

[46] We are, of course, making the radiative corrections finite by the usual method of adding counter terms to the electromagnetic interaction. Since, in any case, the subtraction terms (E_1 and E_2) in (3.80) are needed, the additional subtraction ($E_3 j_\mu{}^5$) we are proposing does not seem unreasonable—especially since a form of universality can be maintained.

[47] We are using here the fact that, for a current defined by limits to be conserved, it is necessary that it be invariant under *local* gauge transformations in addition to simple phase transformations (which are sufficient to insure conservation in the classical case). See, for example, K. Johnson, Phys. Letters **5**, 253 (1963); L. S. Brown, Nuovo Cimento **29**, 617 (1963); and Ref. 6.

[48] M. Gell-Mann and M. Lévy, Nuovo Cimento **16**, 705 (1960).

[49] For the remainder of this section, the fields are meant to be the unrenormalized ones and we work only to order α.

[50] For example, arbitrary finite multiples of $\partial_\mu\phi$ can always be added.

[51] Arguments exactly analogous to those used in Sec. III D can be used to prove this.

[52] D. G. Sutherland, Nucl. Phys. **B2**, 433 (1967); J. S. Bell and R. Jackiw (to be published).

nificance. From this viewpoint, the $\pi^0 \to 2\gamma$ decay serves as a probe of the short-distance structure of local fields and seems to require for its explanation a behavior not describable by the usual formal canonical approach.

We finally consider the general class of models in which

$$j_\mu{}^5 = \sum_j g_j \bar{\psi}_j \gamma_\mu \gamma_5 \psi_j + \text{meson terms}$$

in the absence of electromagnetism. Adler has suggested that the usual PCAC hypothesis should, in the presence of electromagnetism, be replaced by

$$\partial_\mu j_\mu{}^5 = c\phi + (\sum_j g_j Q_j{}^2)(\alpha/4\pi)\epsilon_{\alpha\beta\gamma\delta} F_{\alpha\beta} F_{\gamma\delta}, \quad (4.2)$$

where $Q_j e$ is the charge of the jth fermion. Our only comment here is that an obvious generalization of the gauge invariance argument of Sec. III D again shows that (4.2) is correct if $\partial_\mu j_\mu{}^5 = c\phi$ is for $e=0$.

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Time-Reversal Invariance in Semileptonic Weak Processes*

C. W. Kim
Department of Physics, The Johns Hopkins University, Baltimore, Maryland 21218

AND

H. Primakoff
Department of Physics, University of Pennsylvania, Philadelphia, Pennsylvania 19104

(Received 26 December 1968)

It is shown that no unambiguously interpretable experimental evidence is now available for the time-reversal invariance of the semileptonic weak Hamiltonian. Several experiments, decisive in this regard but as yet unperformed, are treated briefly.

THE discovery of the violation of CP invariance in the process $K_L{}^0 \to \pi^+ + \pi^-$,[1] together with the continuing belief in the universal validity of CPT invariance, has given a new lease on life to theories which incorporate violations of T invariance into one or another piece of the world Hamiltonian.[2] In the present discussion we show that no unambiguously interpretable experimental evidence is now available for the T invariance of the semileptonic weak Hamiltonian and treat briefly several as yet unperformed experiments which would go far toward settling the issue. Specifically, we consider the semileptonic strangeness-conserving weak Hamiltonian[3]

$$H_{\rm sl}{}^{\Delta S=0} = \frac{G}{\sqrt{2}} \int L_\lambda{}^\dagger(\mathbf{x},0) \times [V_\lambda(\mathbf{x},0) + A_\lambda(\mathbf{x},0)] d\mathbf{x} + \text{Herm. conj.}, \quad G = 10^{-5}/m_p{}^2, \quad (1)$$

where L_λ, V_λ, and A_λ are the lepton weak current and the polar and axial $\Delta S=0$ hadron weak currents. The explicit expression for L_λ in terms of the lepton field operators and the conserved vector current (CVC)-imposed identification of V_λ with the isospin current imply that L_λ and V_λ are "normal," i.e., odd, under time reversal, and that V_λ is "regular," i.e., odd, under charge symmetry. On the other hand, we show below that experiments so far performed do not exclude the possibility that A_λ is a sum of two terms, the first "normal" (n), i.e., odd, and the second "abnormal" (a), i.e., even, under time reversal. We then have, since A_λ can also be decomposed into a sum of two terms, the first "regular" (r), i.e., odd, and the second "irregular" (i), i.e., even, under charge symmetry,

$$A_\lambda = \sum_{x=\mathrm{n,a}} \sum_{y=\mathrm{r,i}} A_\lambda{}^{(x)(y)}, \quad (2)$$

$$T A_\lambda{}^{(x)(y)} T^{-1} = -a_{xy} (A_\lambda{}^{(x)(y)})^\dagger, \quad a_{\mathrm{nr}} = -a_{\mathrm{ar}} = a_{\mathrm{ni}} = -a_{\mathrm{ai}} = 1, \quad (3)$$

$$e^{i\pi I^{(2)}} A_\lambda{}^{(x)(y)} e^{-i\pi I^{(2)}} = -b_{xy} (A_\lambda{}^{(x)(y)})^\dagger, \quad b_{\mathrm{nr}} = b_{\mathrm{ar}} = -b_{\mathrm{ni}} = -b_{\mathrm{ai}} = 1, \quad (4)$$

so that $A_\lambda{}^{(\mathrm{n})(\mathrm{r})}$ and $A_\lambda{}^{(\mathrm{a})(\mathrm{i})}$ are "first-class," and

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[1] J. H. Christenson, J. W. Cronin, V. L. Fitch, and R. Turlay, Phys. Rev. Letters 13, 138 (1964).

[2] In addition, R. C. Casella [Phys. Rev. Letters 21, 1128 (1968); 22, 554 (1969)] has shown directly from an analysis of present data that T invariance is violated in $K_L{}^0 \to \pi + \pi$.

[3] We use the notation $L_\mu{}^\dagger \equiv$ (Hermitian conjugate of L_μ) $\times (1 - 2\delta_{\mu 4})$ and analogously for V_λ and A_λ, and [see Eq. (5) et seq.] $F^* \equiv$ complex conjugate of F.