Baryon Mass Spectrum in a Strong-Coupling Model with $O(3)\otimes SU(2)\otimes SU(3)$ Symmetry

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A model is presented in which an octet of pseudoscalar mesons interacts strongly with an octet of static spin-$\frac{1}{2}$ baryons. In addition to this, the baryons carry a central potential in which a hypothetical spinless meson χ is bound. The resulting Hamiltonian is symmetric under $O(3)\otimes SU(2)\otimes SU(3)$ transformations. The lowest positive-parity multiplets then turn out to be $\{8\}^{1/2+}$ and $\{10\}^{3/2+}$; the lowest negative-parity multiplets are $\{1\}^{1/2-}$ and $\{1\}^{3/2-}$, followed by $\{8\}^{3/2-}$ and $\{8\}^{1/2-}$. A short discussion of symmetry breaking is given.

I. INTRODUCTION

It is the purpose of this article to give a short discussion of a hybrid strong-coupling[1] model which is able to reproduce the main characteristics of the baryon spectrum as it is known today.[2] These characteristics include the $SU(3)$ assignments, spins, and parities and also the approximate values of the baryon masses, insofar as they concern the average masses of $SU(3)$ multiplets.

It is not necessary to go into the details of the calculations, since these can be carried out in a straightforward way described by Goebel[3] and Dullemond and Von der Linden.[4] For that reason only the results are given.

A pure strong-coupling theory has the disadvantage that, at least in Hamiltonian models, the generation of nontrivial mixed-parity spectra is difficult and requires assumptions that look somewhat artificial. For example, the assumption of the existence of bare baryons of positive and negative parity cannot be avoided.

Now, the experimental spectrum unmistakably shows the characteristics of a broken $O(3)\otimes SU(2)\otimes SU(3)$ symmetry. As will become clear later, when unbroken $SU(2)\otimes SU(3)$ symmetry is used in a proper strong-coupling Hamiltonian model, the lowest levels fit either into a broken $\{56\}$ representation or into a broken $\{70\}$ representation of $SU(6)$,[5] with only one exception. This exception deals with isobars which either have not yet been found or have not yet been given a definite $SU(3)$ assignment.

By assuming that the bare baryons are not only strongly coupled to a set of mesons, but carry in addition an ordinary central potential, in which a spinless meson χ [an $SU(3)$ singlet with positive parity, not to be identified with any of the known mesons] is bound, one obtains isobars that form representations of $O(3)\otimes SU(2)\otimes SU(3)$. Moreover, we shall assume that the terms in the interaction Hamiltonian which describe the strong coupling between the mesons and the baryons contain a factor that depends on the wave function of the χ meson. These terms therefore describe interactions in which the meson and baryon multiplets and the χ meson interact simultaneously. The χ-dependent factor must be a scalar under $O(3)$ transformations. In order to obtain the desired isobar spectrum, it is sufficient to assume that it depends only on the parity of the orbital state of the χ meson, which is just the parity of the isobar. This is the simplest nontrivial assumption that can be made. We shall refer to these χ-dependent factors as the meson-baryon coupling constants, since they are indeed constants as long as one considers isobars with the same parity.

Now, χ can be in the ground state or in one of a large number of excited states, labeled by a radial quantum number k and orbital quantum numbers l and m. For each choice of the numbers k and l there exists a baryon spectrum consisting of an infinite number of baryons. These are the excitations of the strongly coupled meson–bare-baryon system. Such a subspectrum is characterized by a unique parity $(-)^l$ and since there are no $\mathbf{L}\cdot\mathbf{S}$ terms in the potential, there is a high degree of degeneracy. It follows from our assumptions that there must be a similarity between all the positive-parity subspectra. This is indeed what is observed. A similar phenomenon occurs in the quark model, where the positive-parity baryons all seem to fit into $\{56\}$ representations of $SU(6)$. Furthermore there must be a similarity between all the negative-parity subspectra.

When two subspectra differ only in k value, but not in l value, the spectra are identical except for a constant shift in the mass which corresponds to the energy difference of the two states of χ. Thus, the Roper resonance

[1] G. Wentzel, Helv. Phys. Acta **13**, 269 (1940); **14**, 633 (1941).
[2] A. H. Rosenfeld, N. Barash-Schmidt, A. Barbaro-Galtieri, L. R. Price, P. Söding, C. G. Wohl, M. Roos, and W. J. Willis, Rev. Mod. Phys. **40**, 77 (1968). See also *Proceedings of the Fourteenth International Conference on High-Energy Physics, Vienna, 1968* (CERN Scientific Information Service, Geneva, 1968).
[3] C. J. Goebel, Phys. Rev. Letters **16**, 1130 (1966); in *Non-Compact Groups in Particle Physics*, edited by Yutze Chow (W. A. Benjamin, Inc., New York, 1966), p. 143.
[4] C. Dullemond and F. J. M. von der Linden, Ann. Phys. (N. Y.) **41**, 372 (1967).
[5] See J. J. J. Kokkedee, *The Quark Model* to be published in the series "Frontiers in Physics" (W. A. Benjamin, Inc., New York, 1969); R. H. Dalitz, in *High Energy Physics, Ecole d'Eté de Physique Theorique, Les Houches, 1965* (Gordon and Breach, Science Publishers, Inc., New York, 1966); R. H. Dalitz, in *Proceedings of the Thirteenth International Conference on High-Energy Physics, Berkeley, 1966* (University of California Press, Berkeley, 1967), p. 215.

and the nucleons could belong to two subspectra identical except for an additive constant mass value. Also, the $l=2$ and $l=0$ subspectra are sufficiently similar to speak of a shift in the masses of the two spectra. This mass shift can again be interpreted as an energy difference between two states of χ, etc.

In Sec. III, a short discussion will be given of some mechanisms which break $SU(3)$ symmetry, and also the influence of a simple spin-orbit term in the potential of meson χ will be considered. This discussion is by no means exhaustive and rather serves as an illustration of how some of the results can be modified.

Also, the introduction of the meson χ is heuristic; other models incorporating the strong-coupling Hamiltonians used here may equally well lead to $O(3)\otimes SU(2)\otimes SU(3)$ symmetry with exactly the same theoretical predictions.

II. STRONG-COUPLING MODEL

The system of strongly interacting particles will be supposed to consist of an octet of bare, static baryons, with spin $\frac{1}{2}$ and positive parity, and an octet of pseudoscalar mesons, which are coupled to the baryons by means of derivative Yukawa coupling. Although such a Hamiltonian model has been worked out only for the extended-source approximation,[6] there are good reasons to believe that the same spectrum will be found in the small-source models, a belief strengthened by the results of Pauli and Dancoff,[7] which show that this is the case for an $SU(2)\otimes SU(2)$ model.

The octet-octet interaction involves two coupling constants g_f and g_d which represent f-type and d-type coupling,[8] and one would expect a continuous variation of the form of the spectrum when g_f and g_d are varied. The results of Ref. 4 shows a much more rigid structure of the isobar spectrum. When $g_f/(g_f+g_d)=\alpha$ lies between 0 and 0.725, the spectrum contains an octet as the ground state and a decuplet as the first excited state, with spins $\frac{1}{2}$ and $\frac{3}{2}$, respectively.

In order to describe the details of the spectrum, let us introduce the integers p and $q\geq 0$ such that (p,q) represents the $SU(3)$ representation.[9] For example, (0,0) would mean a singlet, (1,1) an octet, (3,0) a decuplet, etc. The $SU(2)\otimes SU(3)$ multiplets to which the isobars belong can be labeled by a quantum number i_0 (the "internal isospin quantum number"), which is equal to the isospin of one of the $Y=1$ isomultiplets. Apparently, only those $SU(3)$ representations having $Y=1$ isomultiplets occur,[3] so a singlet is excluded and so are representations without an occupied center (the {3}, {3*}, {6}, {6*}, etc.). For each allowed set of quantum numbers p, q, and i_0, there exists one and only one isobar multiplet. The isobars have a spin s equal to i_0 and satisfy a mass formula, first found by Goebel[3]:

$$M=M_0+c[F^2-\tfrac{5}{8}s(s+1)], \tag{2.1}$$

where

$$F^2=\tfrac{1}{3}(p^2+pq+q^2)+p+q$$

and where M_0 and $c>0$ are constants to be adjusted. We shall call this spectrum I, which has an octet with spin $\frac{1}{2}$ as the ground state and a decuplet with spin $\frac{3}{2}$ as the first excited state. These states fit into a {56} representation of $SU(6)$, but this is an accident. The higher levels cannot be fitted into $SU(6)$ representations, but there is also no convincing experimental evidence for their existence.

When $\alpha<0$ or $\alpha>1.613$, the spectrum is the same as before, except that every representation is replaced by its R conjugate. In the above description, $Y=1$ should be replaced by $Y=-1$. An octet with spin $\frac{1}{2}$ occupies the ground state, followed by an antidecuplet with spin $\frac{3}{2}$. We shall call this spectrum II, but there are no indications that it occurs in nature.

Finally, when $0.725<\alpha<1.613$ we find a completely different situation. The $SU(2)\otimes SU(3)$ representations can again be labeled by an "internal isospin quantum number" i_0, which is equal to the isospin of one of the $Y=0$ isomultiplets. Thus a singlet can now occur. The spins s of the isobars are equal to $i_0\pm\frac{1}{2}$. For each allowed set of quantum numbers p, q, i_0, and s, there exists one and only one isobar multiplet. The mass formula is[3]

$$M=M_0+c[F^2-\tfrac{5}{8}\{ai_0(i_0+1)+(1-a)s(s+1)\}]. \tag{2.2}$$

Equation (2.2) transforms into Eq. (2.1) when $i_0=s$. The parameter a, occurring when $i_0\neq s$ cannot be determined by an algebraic method, but without knowledge of a the mass formula is not very useful. Fortunately, a turns out to be a function of α which can be determined with the method of Dullemond and von der Linden.[4] For α to lie in the correct interval, we find the following interval for a:

$$1.774<a\leq 1.8, \tag{2.3}$$

where $\alpha=1$ (pure f-type coupling) corresponds to $a=1.8$ (see Appendix). The interval is very narrow and allows only small variations of M. We shall therefore substitute $a=1.8$, and we shall call the result spectrum III. We find an $SU(3)$ singlet with spin $\frac{1}{2}$ as the ground state, followed by an octet with spin $\frac{1}{2}$. Then follow an octet with spin $\frac{3}{2}$ and a {27} representation with spin $\frac{3}{2}$.

Note that the singlet and the two octets fit into a {70} representation of $SU(6)$ if a decuplet with spin $\frac{1}{2}$ and sufficiently low mass can be found. However, spectrum III is self-conjugate under R transformations, while the {70} representation is not. Therefore, the missing decuplet must be degenerate with its R-conjugate counterpart. Such a decuplet indeed occurs, but its mass is rather high, and there exists a {27} representation with lower mass. Here, the predictions of the $SU(6)$ model and the strong-coupling model apparently differ.

[6] G. Wentzel, Phys. Rev. **125**, 771 (1962).
[7] W. Pauli and S. M. Dancoff, Phys. Rev. **62**, 85 (1942).
[8] M. Gell-Mann, Phys. Rev. **125**, 1067 (1962).
[9] See M. Gell-Mann and Y. Ne'eman, *The Eightfold Way* (W. A. Benjamin, Inc., New York, 1964).

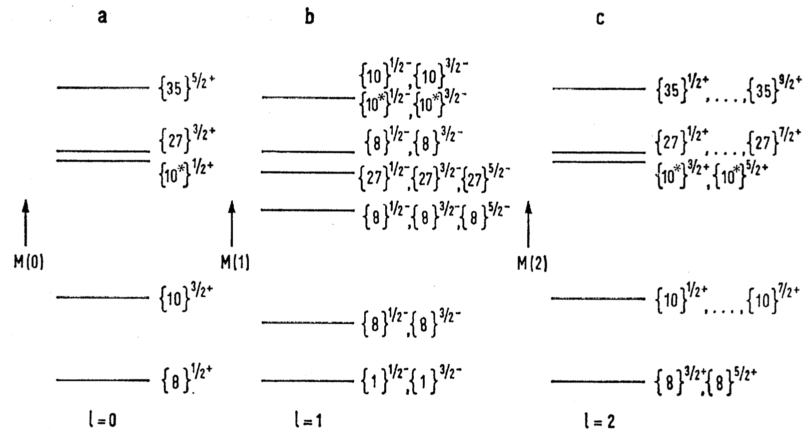

FIG. 1. Forms of the $l=0$, $l=1$, and $l=2$ subspectra. Mass units are different for the cases (a) and (b), the same for (a) and (c). The spectra can be shifted with respect to each other by arbitrary amounts.

We shall not consider the exceptional cases $\alpha=0$, $\alpha=0.725$, and $\alpha=1.613$.

III. INFLUENCE OF THE BOUND χ MESON

So far, no $SU(3)$ singlet baryon resonances with positive parity are known, while the negative-parity baryon resonances with lowest mass are singlets (the 1405- and 1520-MeV Λ resonances). We shall therefore assume that the meson-baryon coupling constants g_f and g_d are such that spectrum I occurs when the meson χ is in a positive-parity state and spectrum III occurs when χ is in a negative-parity state. Therefore, essentially four coupling constants play a part, namely, $g_f(+)$, $g_d(+)$, $g_f(-)$ and $g_d(-)$. Now let us call $\mathbf{J}=\mathbf{L}+\mathbf{S}$ the spin of the isobar, where $\mathbf{S}$ is the spin of the meson-baryon system without χ and $\mathbf{L}$ is the angular momentum of meson χ. Because of the absence of spin-orbit forces, the isobar mass will not depend on j, only on l and s. For $l=0$ the spectrum is unchanged. For $l=1$ we obtain the following modification: The singlet with spin $\frac{1}{2}$ generates two singlets with the same mass and with spins $\frac{1}{2}$ and $\frac{3}{2}$.

In the same way, the octet with spin $\frac{1}{2}$ generates two octets, one with spin $\frac{1}{2}$, the other with spin $\frac{3}{2}$. The octet with spin $\frac{3}{2}$ generates octets with spins $\frac{1}{2}$, $\frac{3}{2}$, and $\frac{5}{2}$, etc. The forms of the resulting subspectra are displayed for $l=0$, 1, and 2 in Fig. 1. The mass units are the same for $l=0$ and $l=2$, but different for $l=0$ and $l=1$. When the binding energy of χ is taken into account, the subspectra are shifted with respect to each other. Let us now make a comparison with experiment. The crucial test is to see whether the negative-parity baryon resonances fit into the picture. In Fig. 2, a plot is made of the masses of the known negative-parity baryon resonances together with their most likely $SU(3)$ assignments. A comparison with Fig. 1 shows that they all fit into an $l=1$ subspectrum. Not only the singlets are in agreement with this, but also the octets.

One notices also that the average experimental singlet and octet masses roughly follow the pattern of the theoretical spectrum. There are indications for the presence of another $\{8\}^{1/2-}$ almost coinciding with the $\{8\}^{5/2-}$, but then still an $\{8\}^{3/2-}$ is missing which must be in the neighborhood of the $\{8\}^{5/2-}$. This is not alarming, since the identification of spin-$\frac{3}{2}$ resonances in that mass region is very difficult.

Apparently refinements must be made which include spin-orbit interactions, $SU(3)$-breaking interactions, and a scaling down of the large coupling constants g_f and g_d. The spin-orbit interactions then lift the degeneracy of the $\{1\}^{1/2-}$ and $\{1\}^{3/2-}$, etc.

Note that the Δ resonance at 1670 MeV, which presumably has spin $\frac{1}{2}$ and negative parity, can be placed in a $\{27\}^{1/2-}$.

IV. NOTE ABOUT SYMMETRY-BREAKING FORCES AND SPIN-ORBIT COUPLING

The simplest way to lift the degeneracy of the $\{1\}^{1/2-}$ and $\{1\}^{3/2-}$ is by assuming a linear spin-orbit interaction between the bare baryons and the meson χ. If this is treated only in first-order perturbation theory, we need the matrix elements of the bare-baryon spins, i.e., the operators $\bar{B}^{\alpha}{}_{\beta i}B^{\beta}{}_{\alpha}{}^{j}$ in the notation of Ref. 4 (see Appendix). For the positive-parity states, these matrix elements turn out to be identically zero in the strong-coupling limit, but this is not the case for the negative-parity states. Now let us look at experiment. There are probably three Δ states with positive parity and spins $\frac{7}{2}$, $\frac{5}{2}$, and $\frac{1}{2}$ very close together at about 1930 MeV.[10] We may assume that a Δ with spin $\frac{3}{2}$ is really there at about the same mass, but this has not yet been found. In the scheme presented here, they have $l=2$ and there should be a complete mass degeneracy also when a simple spin-orbit interaction is taken into account. Experimentally, the mass differences are indeed very small.

For the negative-parity baryons, the degeneracy between the singlets and between the octets is lifted. The mass difference between the lowest-lying $\{8\}^{3/2-}$ and $\{8\}^{1/2-}$ turns out to be just $-\frac{1}{3}$ times the mass dif-

[10] C. Lovelace, in *Proceedings of the Heidelberg International Conference on Elementary Particles* (North-Holland Publishing Co., Amsterdam, 1968).

ference between the $\{1\}^{3/2-}$ and $\{1\}^{1/2-}$. Except for the Λ states, this is in agreement with experiment. Only the Λ states of the octets have the wrong order, which might be the result of representation mixing. For higher representations the agreement is not as good.

One could think of three ways in which the $SU(3)$ symmetry can be broken. The first place, the breaking of the baryon mass degeneracy can be caused by the fact that the meson mass degeneracy is broken. Second, it could be that the bare baryon masses are not all equal, and in the third place, the coupling between the mesons and the bare baryons may show deviations from exact $SU(3)$ symmetry, or other kinds of interactions may be present. If only the first mechanism is present, then the masses of the positive-parity baryon octets satisfy[3]

$$M_\Xi - M_\Sigma : M_\Sigma - M_\Lambda : M_\Lambda - M_N = 1:2:2. \quad (4.1)$$

This is satisfied in nature only as far as the sequence of the baryons is concerned. The distance between the Σ and Λ masses is too large.

For the negative-parity baryon octets we always find $M_\Xi = M_N$, because the breaking pattern of the mesons is R-symmetric, but so is the negative-parity baryon subspectrum in our model. (The renormalized meson-baryon coupling is then pure f-type and thus R-invariant.[3]) This condition is not at all satisfied experimentally, and thus we must conclude that other mechanisms are active in breaking the baryon masses.

Now, a breaking of the bare mass degeneracy can be introduced in a way which satisfies the Gell-Mann–Okubo (GMO) formula. We then find that for the positive-parity octets, the formula (4.1) still holds.

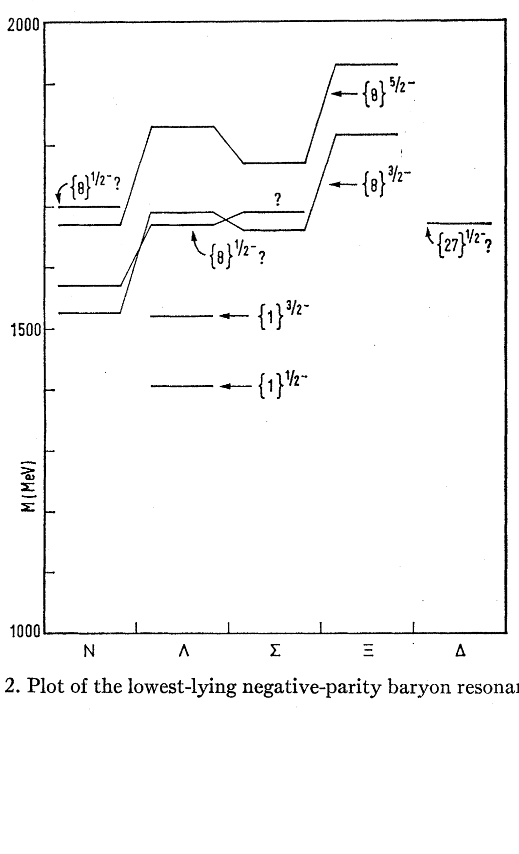

FIG. 2. Plot of the lowest-lying negative-parity baryon resonances.

For the negative-parity octets, there is much more flexibility, and one could obtain a Λ mass between that of the Σ and the Ξ, which is experimentally the case. However, one finds that if the mass of the Ξ is larger than that of the N for the lowest-lying negative-parity octets (spin $\frac{1}{2}$ and $\frac{3}{2}$), then the mass of the Ξ must be *smaller* than that of the N for the negative-parity octets which follow (spin $\frac{1}{2}$, $\frac{3}{2}$, and $\frac{5}{2}$). This of course, is not satisfied in nature, so that the breaking of the bare baryon and meson masses is not sufficient to explain the breaking pattern of the physical octets.

The third possibility, namely, the breaking of the $SU(3)$-invariant meson-baryon coupling, may be the answer to the above problems. We shall not discuss this here.

V. DISCUSSION

A simplified model describing meson-baryon interactions has been presented here. Some of the interactions are Yukawa-like and very strong; others occur via ordinary central potentials. In this way, a mixed-parity baryon spectrum has been generated whose lowest members can be identified with baryons and baryon resonances found in nature. The higher members of the theoretical spectrum are still a mystery, because they require the existence of isobars with positive strangeness, including the so-called Z particles. None of these particles is definitely established. So far, no low-lying baryon states are found which have no natural place in the theoretical spectrum. There are, however, numerous open spaces, not only of $SU(3)$ multiplets which are not completely filled, but also of low-lying $SU(3)$ multiplets of which no member has been found yet. They all occur in a region which is very crowded and where identification of resonances is difficult.

The theoretical mass spectrum is roughly equivalent with the experimental one. More cannot be expected, owing to the oversimplification of the model. Nevertheless, there are several roads open to make adjustments, and an intermediate-coupling model based on the model presented here may have several surprises in store.

ACKNOWLEDGMENTS

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APPENDIX

We shall consider a Hamiltonian, describing the interaction of an octet of pseudoscalar mesons with an octet of bare baryons in the strong-coupling limit.

We follow the notation of Ref. 4 and introduce greek indices running from 1 to 3 to indicate the tensor character under $SU(3)$ transformations and roman indices running from 1 to 2 to indicate the tensor character under rotations in space. The bound meson field is a P-wave octet with 24 components, and is represented by a traceless tensor $q^{\alpha}{}_{\beta}{}^{i}{}_{j}$ which could be considered as a point in a 24-dimensional meson space. The corresponding momenta are $p^{\alpha}{}_{\beta}{}^{i}{}_{j}$, satisfying the condition

$$[p^{\alpha}{}_{\beta}{}^{i}{}_{j},q^{\gamma}{}_{\delta}{}^{k}{}_{l}]=(1/i)(\delta^{\alpha}{}_{\delta}\delta^{\gamma}{}_{\beta}-\tfrac{1}{3}\delta^{\alpha}{}_{\beta}\delta^{\gamma}{}_{\delta})\times(\delta^{i}{}_{l}\delta^{k}{}_{j}-\tfrac{1}{2}\delta^{i}{}_{j}\delta^{k}{}_{l}). \quad \text{(A1)}$$

The bare baryons belong to an octet with spin $\frac{1}{2}$. These can be created out of a "bare baryon vacuum" $|\,\rangle$ by traceless operators $\bar{B}^{\alpha}{}_{\beta i}$, or they can be annihilated by the corresponding traceless operators $B^{\gamma}{}_{\delta}{}^{j}$ satisfying the condition

$$B^{\gamma}{}_{\delta}{}^{j}\bar{B}^{\alpha}{}_{\beta i}|\,\rangle=(\delta^{\gamma}{}_{\beta}\delta^{\alpha}{}_{\delta}-\tfrac{1}{3}\delta^{\gamma}{}_{\delta}\delta^{\alpha}{}_{\beta})\delta^{j}{}_{i}|\,\rangle. \quad \text{(A2)}$$

The sequence of the operators B and $\bar{B}$ is essential here. The form of the Hamiltonian in the case of a large source size becomes[4]

$$H=\tfrac{1}{2}p^2+\tfrac{1}{2}\mu^2q^2+g_1\bar{B}^{\gamma}{}_{\delta i}q^{\delta}{}_{\epsilon}{}^{i}{}_{j}B^{\epsilon}{}_{\gamma}{}^{j}+g_2\bar{B}^{\delta}{}_{\epsilon i}q^{\gamma}{}_{\delta}{}^{i}{}_{j}B^{\epsilon}{}_{\gamma}{}^{j}, \quad \text{(A3)}$$

where $p^2=p^{\alpha}{}_{\beta}{}^{i}{}_{j}p^{\beta}{}_{\alpha}{}^{j}{}_{i}$, $q^2=q^{\alpha}{}_{\beta}{}^{i}{}_{j}q^{\beta}{}_{\alpha}{}^{j}{}_{i}$, and μ^2 is the mass squared of the mesons. The meson masses are taken to be equal for all mesons of the octet. The Einstein summation convention is adopted, both for the $SU(3)$ and the $SU(2)$ indices. The coefficients g_1 and g_2 are large real coupling constants. We shall introduce the parameters g and γ as follows:

$$\begin{aligned} g_d&=g_1+g_2=g\sqrt{3}\cos\gamma, \quad (g>0)\\ g_f&=g_1-g_2=g\sqrt{3}\sin\gamma. \end{aligned} \quad \text{(A4)}$$

It turns out that a "potential minimum" in q space occurs when $q^{\alpha}{}_{\beta}{}^{i}{}_{j}$ has a high symmetry, namely, such that for every rotation in ordinary space there exists an $SU(3)$ transformation such that all the components of $q^{\alpha}{}_{\beta}{}^{i}{}_{j}$ stay unchanged. This is valid for all γ. Here, the potential minimum is defined as the lowest eigenvalue which an eigenvector of $H-\frac{1}{2}p^2$ can possibly have for given g_1 and g_2. Although the position of the potential minimum is only trivially dependent on γ, the eigenvector which has the lowest eigenvalue is not always the same.

One possible choice for the value of $q^{\alpha}{}_{\beta}{}^{i}{}_{j}$ at which the potential reaches its minimum is

$$\begin{aligned} &\frac{q_0}{\sqrt{12}}\sum_{m=1}^{3}(\tau_m)^{\alpha}{}_{\beta}(\sigma_m)^{i}{}_{j} \quad \text{when } \alpha \text{ and } \beta\neq 3, \quad \text{(A5)}\\ &0 \qquad\qquad \text{when } \alpha \text{ or } \beta=3. \end{aligned}$$

Here $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$ are Pauli spin matrices, α and i row indices, and β and j column indices. The proper value of q_0 depends on g and γ. We shall call this specific choice a "standard form." All other points where the potential reaches a minimum can be obtained from this standard form by applying rotations and $SU(3)$ transformations. In that way, an "orbit" in q space is described.

It is convenient to define a unit vector in the $q^{\alpha}{}_{\beta}{}^{i}{}_{j}$ direction which we call $\hat{q}^{\alpha}{}_{\beta}{}^{i}{}_{j}$, and which has the property

$$\hat{q}^{\alpha}{}_{\beta}{}^{i}{}_{j}\hat{q}^{\beta}{}_{\alpha}{}^{j}{}_{i}=1.$$

Moreover, let us introduce a traceless tensor $C^{\alpha}{}_{\beta}$ defined by

$$C^{\alpha}{}_{\beta}=(\sqrt{6})\hat{q}^{\alpha}{}_{\gamma}{}^{i}{}_{j}\hat{q}^{\gamma}{}_{\beta}{}^{j}{}_{i}-(\tfrac{1}{3}\sqrt{6})\delta^{\alpha}{}_{\beta}. \quad \text{(A6)}$$

Then, when $\hat{q}^{\alpha}{}_{\gamma}{}^{i}{}_{j}$ points in the direction of the potential minimum and has the standard form, $C^{\alpha}{}_{\beta}$ is diagonal and $C^1{}_1=C^2{}_2=\sqrt{\frac{1}{6}}$ so that $C^{\alpha}{}_{\beta}C^{\beta}{}_{\alpha}=1$.

Finally, let us introduce the parameter α such that[11]

$$\alpha=\frac{\tan\gamma}{1+\tan\gamma}=\frac{g_f}{g_f+g_d}. \quad \text{(A7)}$$

Then, for $0<\alpha<0.725$ we obtain a spectrum which gives the best description of the positive-parity baryons. This case is extensively discussed in Ref. 4. The eigenstate of $H-\frac{1}{2}p^2$ with the lowest eigenvalue is

$$S=\zeta(q)d^{\alpha}{}_{\beta}{}^{i}\bar{B}^{\beta}{}_{\alpha i}|\,\rangle, \quad \text{(A8)}$$

where $\zeta(q)$ is a function of the invariants of $q^{\alpha}{}_{\beta}{}^{i}{}_{j}$ which is zero everywhere except when $\sqrt{(q^2)}\approx q_0$, i.e., in the vicinity of the potential minimum. Furthermore, when $\hat{q}$ has the standard form we have

$$d^1{}_3{}^2=-d^2{}_3{}^1=(1/\sqrt{2})e^{i\epsilon},$$

all other components being zero. The phase ϵ is arbitrary and will change when transformations are carried out which leave $q^{\alpha}{}_{\beta}{}^{i}{}_{j}$ at the potential minimum unchanged. The eigenstates of H (where p^2 must now be interpreted as minus the Laplacian in 24-dimensional q space) must stay invariant under such transformations. Then a singlet cannot be constructed, but an octet with spin $\frac{1}{2}$ can be found:

$$d^{\dagger\gamma}{}_{\delta i}S \quad \text{(A9)}$$

and also a decuplet with spin $\frac{3}{2}$ can be constructed, etc.

For $0.725<\alpha<1.613$, another eigenvector takes over, namely,

$$S_j=\zeta(q)\{[\tfrac{1}{2}(1+\sin\gamma)]^{1/2}\hat{q}^{\alpha}{}_{\beta}{}^{i}{}_{j}\bar{B}^{\beta}{}_{\alpha i}+\tfrac{1}{2}(1-\sin\gamma)^{1/2}C^{\alpha}{}_{\beta}\bar{B}^{\beta}{}_{\alpha j}\}|\,\rangle. \quad \text{(A10)}$$

This state has no arbitrary phase and is an eigenstate of H. It is to be interpreted as a physical $SU(3)$ singlet baryon with spin $\frac{1}{2}$. Two octets with spin $\frac{1}{2}$ can be constructed as eigenstates of H. One of them is

$$C^{\gamma}{}_{\delta}S_j \quad \text{(A11)}$$

with mass

$$M_1(\{8\}^{1/2})=M(\{1\}^{1/2})+6/q_0^2, \quad \text{(A11')}$$

where $M(\{1\}^{1/2})$ is the mass of the singlet. The second

[11] J. J. de Swart, Rev. Mod. Phys. 35, 916 (1963).

spin-$\frac{1}{2}$ octet has the form

$$\hat{q}^{\gamma}{}_{\delta}{}^{k}{}_{j}S_{k} \tag{A12}$$

with mass

$$M_2(\{8\}^{1/2})=M(\{1\}^{1/2})+(5-2\sin\gamma)/2q_0^2. \tag{A12$'$}$$

The only octet with spin $\frac{3}{2}$ is

$$\hat{q}^{\gamma}{}_{\delta}{}^{k}{}_{(l}S_{m}\epsilon_{n)k}, \tag{A13}$$

where symmetrization takes place over the indices between parentheses and where ϵ_{nk} is the Levi-Civita tensor $\epsilon_{12}=-\epsilon_{21}=1$, $\epsilon_{11}=\epsilon_{22}=0$. The mass becomes

$$M(\{8\}^{3/2})=M(\{1\}^{1/2})+(8+\sin\gamma)/2q_0^2. \tag{A13$'$}$$

The mass values found in this way all satisfy formula (2.2) when for a the following value is substituted:

$$a=7/5+\tfrac{2}{5}\sin\gamma. \tag{A14}$$

This leads to the allowed interval, Eq. (2.3), for a.

In order to compute spin-orbit splittings due to the interaction of meson χ with the bare nucleon, one must know the matrix elements of the baryon spin operators $\mathbf{s}_{\text{bare}}$ which are defined as follows:

$$\begin{aligned} s_{\text{bare }1}&=\tfrac{1}{2}(\bar{B}^{\alpha}{}_{\beta 1}B^{\beta}{}_{\alpha}{}^{2}+\bar{B}^{\alpha}{}_{\beta 2}B^{\beta}{}_{\alpha}{}^{1}),\\ s_{\text{bare }2}&=\tfrac{1}{2}i(\bar{B}^{\alpha}{}_{\beta 1}B^{\beta}{}_{\alpha}{}^{2}-\bar{B}^{\alpha}{}_{\beta 2}B^{\beta}{}_{\alpha}{}^{1}),\\ s_{\text{bare }3}&=\tfrac{1}{2}(\bar{B}^{\alpha}{}_{\beta 1}B^{\beta}{}_{\alpha}{}^{1}-\bar{B}^{\alpha}{}_{\beta 2}B^{\beta}{}_{\alpha}{}^{2}). \end{aligned} \tag{A15}$$

The results must be proportional to the matrix elements of the physical isobar spins.

For α between 0 and 0.725, we obtain for the matrix elements of s_{bare} between different spin states of the physical octet the expression

$$N\int d\mathbf{q}\; S^{\dagger}d^{\mu}{}_{\nu}{}^{k}\mathbf{s}_{\text{bare}}d^{\dagger\nu}{}_{\mu l}S, \tag{A16}$$

where N is a normalization constant.[12] The result turns out to be zero in this case, and this is true also for the other members of the spectrum. Therefore, no spin-orbit splitting can occur for any of the positive-parity states.

When α lies between 0.725 and 1.613 the results are different. For the singlet, one must compute

$$N\int d\mathbf{q}\; S^{\dagger k}\mathbf{s}_{\text{bare}}S_{l} \tag{A17}$$

and for the lowest-lying octet with spin $\frac{1}{2}$,

$$N\int d\mathbf{q}\; S^{\dagger i}\hat{q}^{\mu}{}_{\nu}{}^{k}{}_{i}\mathbf{s}_{\text{bare}}\hat{q}^{\nu}{}_{\mu}{}^{j}{}_{l}S_{j}. \tag{A18}$$

The results are nonzero. The octet matrix elements are $-\frac{1}{3}$ times the corresponding singlet matrix elements, independent of γ. It follows then that the mass difference between the negative parity $\{8\}^{1/2}$ and $\{8\}^{3/2}$ is $-\frac{1}{3}$ times the mass difference between the $\{1\}^{1/2}$ and the $\{1\}^{3/2}$ if spin-orbit terms in the potential are treated in first-order perturbation theory.

If the bare masses are broken according to the GMO formula, the matrix elements of the following operators must be computed in the same way as described above, namely,

$$\bar{B}^{3}{}_{\alpha i}B^{\alpha}{}_{3}{}^{i} \quad\text{and}\quad \bar{B}^{\alpha}{}_{3i}B^{3}{}_{\alpha}{}^{i}. \tag{A19}$$

A suitable linear combination of these operators must be taken. The computations are again straightforward.

[12] See F. J. M. von der Linden and C. Dullemond, Nuovo Cimento 43, 615 (1966).