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Mandelstam Symmetries in the Complex Angular Momentum Plane*

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A symmetry found by Mandelstam in the complex l plane for potential scattering is investigated. One consequence of the symmetry is given in the form of logarithmic finite-energy sum rules. A series of proofs of the symmetry for potential scattering is given, culminating in ones using methods that are applicable to off-energy-shell and relativistic scattering. In both these latter cases, this symmetry is disproven in contrast to the more well-known Mandelstam symmetry, which is seen to hold there.

I. INTRODUCTION

In order to make his modification of the Sommerfeld-Watson transformation, Mandelstam[1] proved a symmetry of the S matrix, which we call half-integer symmetry:

$$S(l,k)=S(-l-1,\,k)\,,\quad l=\tfrac{1}{2},\,\tfrac{3}{2},\,\cdots. \tag{1.1}$$

His proof was for potential scattering, as were later proofs by other workers.[2] All of these also prove a second symmetry, integer symmetry

$$S(l,k)=-S(-l-1,\,k)\,,\quad l=0,\,1,\,\cdots. \tag{1.2}$$

This latter symmetry has received almost no consideration in the literature. For example, it is straightforward to show that half-integer symmetry comes directly from the Froissart-Gribov continuation in both nonrelativistic and relativistic scattering[3]; integer symmetry for the latter case has not even been attacked, much less proven. This paper seeks to fill this gap by giving some consequences of integer symmetry and showing that it, in fact, cannot be extended to relativistic scattering.

Section II proceeds on the assumption that integer symmetry does hold. From this and the Froissart-Gribov continuation, certain logarithmic finite-energy sum rules (FESR) are obtained. They are given in terms of relativistic scattering, even though we know the symmetry fails there. This is justified in several ways.

First, FESR are very much more familiar for relativistic scattering, and it was thought desirable to present the new forms in that context. Second, the establishment of the form that the symmetry and then the FESR could take to be consistent with any relativistic theory is in itself an interesting question, though unfortunately rendered empty by the absence of the symmetry. Third, it seems desirable to express the FESR in a form suitable for experimental verification, in the admittedly faint hope that the symmetry does hold in the real world. Finally, of course, the restatement of the FESR for potential scattering is trivial.

Section III gives several proofs of the Mandelstam symmetries. The emphasis is on new features, giving both a new expression to the well-known proof and entirely new proofs designed to extend the results in certain ways. The culmination is a perturbative treatment based on the N/D method, applicable to both relativistic and nonrelativistic scattering, giving the only possible form of the Mandelstam symmetries for each and disproving one of the symmetries for the former. The crucial distinction is seen to be the different phase-space expressions.

II. CONSEQUENCES OF INTEGER SYMMETRY

In this section we take the known symmetry (1.2) of potential scattering and assume that it can be extended to relativistic scattering. The procedure is one frequently used, though admittedly it lacks rigor. It will be useful to rewrite our symmetry as

$$a(l,k)+a(-l-1,\,k)=i/k\,,\quad l=0,\,1,\,\cdots \tag{2.1}$$

$$S(l,k)=1+2ika(l,k)\,. \tag{2.2}$$

The relativistic analogs of these are

$$A(l,s)+A(-l-1,\,s) =-\pi[s/(s-4m^2)]^{1/2}/\ln\{[(s-4m^2)^{1/2}-s^{1/2}]/[(s-4m^2)^{1/2}+s^{1/2}]\}\,,\quad l=0,\,1,\,\cdots \tag{2.3}$$

$$S(l,s) =1+2i[(s-4m^2)/s]^{1/2}A(l,s)\,. \tag{2.4}$$

The justification for (2.3) will be deferred to Sec. III. For the moment, we will only note that the right-hand side has a square-root singularity at $s=0$ on the second sheet and not on the first sheet; this agrees with the singularities of $A(l,s)$. We also see that (2.3) and (2.4) together do not yield (1.2), and indeed that the latter cannot give the correct analytic behavior in the neighborhood of $s=0$. Equation (2.3) is the simplest form of the symmetry that is not in contradiction to known analytic properties; we will see in Sec. III that it is also the unique permissible form.

We must now consider the complication that signature introduces into the problem. For potential scatter-

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[1] S. Mandelstam, Ann. Phys. (N.Y.) **19**, 254 (1962).

[2] E. J. Squires, *Complex Angular Momentum and Particle Physics* (W. A. Benjamin, Inc., New York, 1964); R. G. Newton, *The Complex j Plane* (W. A. Benjamin, Inc., New York, 1964).

[3] S. M. Roy, Phys. Rev. **161**, 1575 (1967).

ing the problem is trivial: We obtain separate Schrödinger equations for each signature,[4] and then (2.1) holds separately for $a^+(l,k)$ and $a^-(l,k)$. For relativistic scattering, we will assume that the same situation holds, with one exception. The Froissart-Gribov contribution

$$A^{\pm}(l,s)=\frac{1}{\pi}\int_{t_0}^{\infty}\frac{2dt}{s-4m^2}[A_t(s,t)\pm A_u(s,t)]\times Q_l\left(1-\frac{2t}{s-4m^2}\right),\quad (2.5)$$

shows, as is well known, that Gribov-Pomeranchuk (G-P) singularities appear at wrong-signature nonsense points if the third double-spectral function is nonzero.[5] Thus, half of our symmetries would involve points with singularities on the left-hand side and are patently false. The remainder can be written

$$A^{\sigma}(l,s)+A^{\sigma}(-l-1,s)=-\pi[s/(s-4m^2)]^{1/2}/\ln\{[(s-4m^2)^{1/2}-s^{1/2}]/[(s-4m^2)^{1/2}+s^{1/2}]\},\quad l=0,1,\cdots\quad \sigma=-(-1)^l. \quad (2.6)$$

We now put (2.6) into (2.5), using the identity of Legendre polynomials,[6]

$$Q_{-l-1}(z)+Q_l(z)=-P_N(z)/(l-N)-P_N(z)\ln[\tfrac{1}{2}(z-1)]+R_N(z)+O(l-N),\quad (2.7)$$

in the neighborhood of $l=N$, where R_N is a polynomial of order N. The condition that a G-P singularity is absent gives the usual superconvergence relations

$$\int_{z_0}^{\infty}dz[A_t-(-1)^N A_u]P_N(z)=0,\quad N=0,1,\cdots. \quad (2.8)$$

The additional assertion of Mandelstam symmetry gives

$$\int_{z_0}^{\infty}dz[A_t-(-1)^N A_u]\left[P_N(z)\ln\left(\frac{z-1}{2}\right)-R_N(z)\right]=\pi^2\left(\frac{s}{s-4m^2}\right)^{1/2}\bigg/\ln\left(\frac{(s-4m^2)^{1/2}-s^{1/2}}{(s-4m^2)^{1/2}+s^{1/2}}\right),\quad N=0,1,\cdots \quad (2.9)$$

which are our new logarithmic sum rules.

As is well known, sum rules of the type (2.8) hold only in the presence of external spin. This is simply because the integrals in question do not converge. The same difficulty appears in (2.9). Rigorously, we should perform a correct analytic continuation of (2.5); instead we will simply rewrite (2.8) and (2.9) in a way that eliminates divergences and claim that we obtain the same result as by the former method. We assume Regge behavior for a single pole of definite signature

$$A_t(s,t)\pm A_u(s,t)\xrightarrow[t\to\infty]{}\frac{\beta(s)}{\Gamma(\alpha+1)}\left(\frac{2t}{s-4m^2}\right)^{\alpha(s)}. \quad (2.10)$$

If this is a good approximation for $t>L$, we can use it to evaluate the high-energy contribution to (2.8) and (2.9). For $N=0$ these become

$$\int_{t_0}^{L}dt[A_t(s,t)-A_u(s,t)]=\frac{\beta(s)L}{(\alpha+1)\Gamma(\alpha+1)}\left(\frac{2L}{s-4m^2}\right)^{\alpha(s)}, \quad (2.11)$$

$$\int_{t_0}^{L}dt[A_t(s,t)-A_u(s,t)]\ln\left(\frac{t}{s-4m^2}\right)=\frac{\beta(s)L}{(\alpha+1)\Gamma(l+1)}\left(\frac{2L}{s-4m^2}\right)^{\alpha(s)}\left[\ln\left(\frac{L}{s-4m^2}\right)-\frac{1}{\alpha+1}\right]+\tfrac{1}{2}\pi^2[s(s-4m^2)]^{1/2}\bigg/\ln\left(\frac{(s-4m^2)^{1/2}-s^{1/2}}{(s-4m^2)^{1/2}+s^{1/2}}\right). \quad (2.12)$$

Equation (2.11) is just the usual finite-energy sum rule (FESR); (2.12) is a logarithmic sum rule whose validity is equivalent to that of integer symmetry. Subtracting $\ln(L/s-4m^2)$ times (2.11) from it, we get the somewhat simpler form

$$\int_{t_0}^{L}dt[A_t(s,t)-A_u(s,t)]\ln\left(\frac{t}{L}\right)=\tfrac{1}{2}\pi^2[s(s-4m^2)]^{1/2}\bigg/\ln\left(\frac{(s-4m^2)^{1/2}-s^{1/2}}{(s-4m^2)^{1/2}+s^{1/2}}\right)-\frac{\beta(s)L}{(\alpha+1)^2\Gamma(\alpha+1)}\left(\frac{2L}{s-4m^2}\right)^{\alpha(s)}. \quad (2.13)$$

Such sum rules can be directly tested with experimental data. Conversely, if by some independent means they are shown to hold, the rules would act as a constrain on fitting data in much the same as that the usual FESR are. The existence of such verification is investigated in Sec. III.

Inasmuch as Mandelstam symmetry is known to be valid for potential scattering, the nonrelativistic form for our logarithmic FESR are equally true. This property of the scattering amplitude does not seem to have been previously known.

III. PROOFS OF INTEGER SYMMETRY

Here we give a sequence of proofs of both Mandelstam symmetries, starting with the well-known original proof

[4] E. J. Squires, see Ref. 2.

[5] See, for example, C. E. Jones and V. L. Teplitz, Nuovo Cimento **31**, 1079 (1964).

[6] This follows from H. Bateman, in *Higher Transcendental Functions*, edited by A. Erdélyi (McGraw-Hill Book Co., New York, 1953), Vol. I, 3.2(14), pp. 124–5; 3.7(2), p. 160.

and proceeding on to entirely new approaches. The goals are twofold: to uncover the differences between integer and half-integer symmetry and to try to extend the proofs to relativistic scattering.

A. Proofs Involving the Wave Function

We begin with the radial Schrödinger equation

$$\left(\frac{d^2}{dr^2}-\frac{l(l+1)}{r^2}+k^2\right)\phi(l,k;r)=V(r)\phi(l,k;r),\quad rV(r)=\sum_0^\infty v_i r^i, \tag{3.1}$$

and seek to show that its solutions have a symmetry analogous to (1.1) and (1.2), which we will call functional symmetry. Two facts about (3.1) are used in the proof below: (1) It is symmetric about $l=-\frac{1}{2}$; (2) $r=0$ is a regular singular point with indices $l+1$ and $-l$.

We start by defining the regular solution by the boundary condition

$$\lim_{r\to 0} r^{-l-1}\phi(l,k;r)=1, \tag{3.2}$$

for $\mathrm{Re}\,l>-\frac{1}{2}$ and by analytic continuation below that. Let $l=\frac{1}{2}N$, $N=0, 1, \cdots$. Property (1) shows that the analytic continuation to $l=-\frac{1}{2}N-1$ is also a solution, and (2) shows that they correspond to the solution with greater and lesser index, respectively.

The indices have integer spacing, and thus by the theory of ordinary differential equations,[7] the latter contains the former solution with some coefficient not fixed by either the differential equation or the boundary condition. It is, however, given by the analytic continuation. We thus have

$$\phi(-\tfrac{1}{2}N-1,k;r)=r^{-N/2}+C_1r^{-N/2+1}+\cdots+C_Nr^{N/2}+C_{N+1}\phi(\tfrac{1}{2}N,k,r)+\cdots. \tag{3.3}$$

It is a simple matter to verify that the series solution to (3.1) gives a pole in C_{N+1} at $l=\frac{1}{2}N-1$, so that we can write

$$\lim_{l\to-\frac{1}{2}N-1}(l+\tfrac{1}{2}N+1)\phi(l,k;r)=C(N)\phi(\tfrac{1}{2}N,k;r),\quad N=0,1,\cdots. \tag{3.4}$$

This is the desired symmetry of the wave function.[8] We have been able to get rid of the other terms in (3.3) by cancelling the pole with a zero; without the pole, no such symmetry would have been possible. We will need the particular result $C(0)=\frac{1}{2}V_0$.

[7] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge University Press, London, 1965), 4th ed., Sec. 10.3.

[8] E. J. Squires, Ref. 2, finds this result directly from the recursion relation for the series solution. The present method is adopted because it brings out explicitly the elements needed in the proof, and because it is equally suitable for more involved problems, such as the Bethe-Salpeter equation, where direct verification from the series is overly cumbersome.

We can now find the Mandelstam symmetries by means of the Jost function. We define the irregular or Jost solution to (3.1) by

$$e^{\pm ikr}f_\pm(l,k;r)\xrightarrow[r\to\infty]{}1. \tag{3.5}$$

Since the boundary condition does not depend on l, Poincare's theorem tells us that $f_\pm$ are entire in $l(l+1)$, hence

$$f_\pm(l,k;r)=f_\pm(-l-\mathbf{1},k;r). \tag{3.6}$$

With Jost functions defined by

$$f_\pm(l,k)=W(f_\pm,\phi)=(df_\pm/dr)\phi-f_\pm(d\phi/dr), \tag{3.7}$$

we have from (3.4) and (3.6) that

$$\lim_{l\to-\frac{1}{2}N-1}(l+\tfrac{1}{2}N+1)f_\pm(l,k)=C(N)f_\pm(\tfrac{1}{2}N,k). \tag{3.8}$$

Since

$$S(l,k)=e^{i\pi l}f_\pm(l,k)/f_-(l,k), \tag{3.9}$$

we find (1.1) and (1.2) immediately. An equivalent method is simply to use the asymptotic expansion

$$\phi(l,k,1)\xrightarrow[r\to\infty]{}C(l,k)\{e^{i\pi l/2}e^{ikr}-e^{-i\pi l/2}S(l,k)e^{-ikr}\} \tag{3.10}$$

inserted in (3.4).

This method can also be used to find functional symmetry for the Bethe-Salpeter equations (except at $l=-1$ and -2, where poles are lacking). However, it is not possible to go from this to Mandelstam symmetries: There are no Jost functions for the equation, and (3.10) is no longer the correct analytic continuation of S in l. The latter is easy to see: If it were the correct continuation, (1.2) would hold relativistically. However, we have said that this gives the wrong analytic behavior at $s=0$. Thus we must seek other means to find the relativistic generalization.

Further, the use of functional symmetry does not distinguish between integer and half-integer symmetry. To do this we turn to integral methods.

B. Proofs Involving the T Matrix

We have the equation for the off-shell T matrix[9]

$$S(l,k)=1+2iT(l,k,k), \tag{3.11}$$

$$T(l,k,k')=\frac{-i(\sqrt{\pi})(\frac{1}{2}k)^{l+1}}{k\Gamma(l+\frac{3}{2})F(l,k)}\int_0^\infty dr(\tfrac{1}{2}\pi k'r)^{1/2}\times J_{l+\frac{1}{2}}(k'r)V(r)\phi(l,k;r), \tag{3.12}$$

$$F(l,k)=1+\frac{i(\sqrt{\pi})(\frac{1}{2}k)^{l+1}}{k\Gamma(l+\frac{3}{2})}\int_0^\infty dr(\tfrac{1}{2}\pi kr)^{1/2}\times H_{l+\frac{1}{2}}^{(1)}(kr)V(r)\phi(l,k;r). \tag{3.13}$$

[9] V. De Alfaro and T. Regge, *Potential Scattering* (North-Holland Publishing Co., Amsterdam, 1965).

(F is a normalized Jost function.) We will analytically continue these equations to $l=-1$. We see from (3.3) that F has a pole there, with two contributions to the residue: the divergence at the origin from the r^{l+1} term and the pole in ϕ itself. The residue is

$$\tfrac{1}{2}v_0(i/k)+\tfrac{1}{2}v_0(e^{i\pi/2}/k)[F(0,k)-1] =\tfrac{1}{2}v_0(i/k)F(0,k). \quad (3.14)$$

Similarly, the integral in (3.12) has a residue

$$\tfrac{1}{2}v_0-\tfrac{1}{2}v_0\int_0^\infty dr(\tfrac{1}{2}\pi k'r)^{1/2}Y_{1/2}(k'r)V(r)\phi(0,k,r). \quad (3.15)$$

We now evaluate $T(-1,k,k')$ by a quotient of residues, $T(0,k,k')$ directly from (3.12) and find

$$T(0,k,k')+T(-1,k,k')=\frac{i}{F(0,k)}\left\{1+i\int_0^\infty dr(\tfrac{1}{2}\pi k'r)^{1/2} \times H_{1/2}^{(1)}(k'r)V(r)\phi(0,k,r)\right\}. \quad (3.16)$$

On the energy shell $k'=k$ the bracket in (3.16) is just (3.13), and we have a rather remarkable cancellation that gives integer symmetry

$$T(0,k,k)+T(-1,k,k)=i. \quad (3.17)$$

Half-integer symmetry comes much more easily, as we see below.

This method is not applicable to the Bethe-Salpeter equation. The analogies to (3.12) and (3.13) can be found, but the appearance of an extra variable, the relative time, complicates the manipulation and no simple cancellation has been found.

We now work with the Lippmann-Schwinger equation for T[9]

$$T(l,k,k')=B(l,k,k')-\int_0^\infty dh''K(l,k;k',k'')T(l,k,k''),$$

$$B(l,k,k')=-\frac{1}{k}\int_0^\infty dr(\tfrac{1}{2}\pi k'r)^{1/2}J_{l+\frac{1}{2}}(k'r)V(r) \times(\tfrac{1}{2}\pi kr)^{1/2}J_{l+\frac{1}{2}}(kr), \quad (3.18)$$

$$K(l,k;k',k'')=-\frac{2}{\pi}\frac{k}{k''-(k+i\epsilon)^2}B(l,k',k'').$$

We see immediately from $J_{-n}(z)=(-1)^nJ_n(z)$ that we have half-integer symmetry off the energy shell. (We neglect the problem of analytic continuation, which gives no difficulty here.) This generalization to off-shell scattering does not seem to be previously known. Further, we need not limit the potential to the sort indicated in (3.1); A far larger class is suitable for half-inteter symmetry.

We now work with a Yukawa potential $ge^{-\mu r}/r$ and perform a Fredholm expansion[10] of T, $T=N/D$. Both N and D have poles at negative integer l, which cancel in the quotient. To second order, keeping only the pole term at $l=-1$,

$$N(l,k,k')=-\frac{1}{l+1}\frac{g}{2k}\left[1+\frac{gi}{4k}\ln\left(\frac{k'^2-(k-i\mu)^2}{k'^2-(k+i\mu)^2}\right)+O(g^2)\right]+\cdots, \quad (3.19)$$

$$D(l,k,k')=\frac{1}{l+1}\frac{1}{2}\frac{gi}{k}\left[1+\frac{1}{2}\frac{gi}{k}\ln\left(1-2i\frac{k}{\mu}\right)+O(g^2)\right]+\cdots, \quad (3.20)$$

$$T(-1,k,k')=i+\frac{1}{4}\frac{g}{k}\ln\left[\left(1-2i\frac{k}{\mu}\right)^2\frac{k'^2-(k-i\mu)^2}{k'^2-(k+i\mu)^2}\right]+O(g^2). \quad (3.21)$$

We also have

$$T(0,k,k')=-\frac{1}{4}\frac{g}{k}\ln\left(\frac{(k+k')^2+\mu^2}{(k-k')^2+\mu^2}\right)+O(g^2), \quad (3.22)$$

and we find that on the energy shell the first-order terms in g cancel in the sum (3.17), as they must. This method is not a complete proof, since we would need to know all higher-order terms in (3.21) and (3.22). However, it does have two valuable points, both of which concern the limit $g=0$. (1) It shows the unique form that integer symmetry can have. For, in the limit $g=0$, all higher-order terms vanish; If the sum of $T(l)$ and $T(-l-1)$ is to be independent of the potential, it can only be the lowest-order term from the Fredholm expansion. (2) One could say from (3.18) that $T(l)=O(g)$ for all l. This would contradict (3.17). The resolution is that one must first go to $l=-1$ and then take the limit $g=0$ to get the proper result; the Fredholm expansion does this automatically. The poles in N and D are related to those in ϕ and $f_\pm$; all are necessary in the various proofs.

This method can be used with the Bethe-Salpeter Fredholm expansion given by Lee and Sawyer.[11] It yields

$$A(0,s)+A(-1,s) =-\pi\left(\frac{s}{s-4m^2}\right)^{1/2}\Big/\ln\left(\frac{(s-4m^2)^{1/2}-s^{1/2}}{(s-4m^2)^{1/2}+s^{1/2}}\right)+O(g). \quad (3.23)$$

This is one justification of our asserted form for rel-

[10] E. T. Whittaker and G. N. Watson, Ref. 7, Sec. 11.2.

[11] B. W. Lee and R. F. Sawyer, Phys. Rev. **127**, 2266 (1962).

ativistic integer symmetry. It has not been possible to explicitly evaluate the higher-order terms.

C. N/D Method

Our final approach is suggested by the Fredholm expansion, but is not equivalent to it. We write the usual N/D equations and find the lowest-order terms from them. Again, the appearance of poles at negative integer l is the crucial point. The advantage of this method is that it does not presuppose a wave equation and applies equally to relativistic and nonrelativistic scattering, differing only in the phase space. We will work in the absence of a third double-spectral function, as its effects are fully treated above.

Our equations are

$$B(l,s)=q^{-2l}A(l,s)$$
$$=N(l,s)/D(l,s)\,, \tag{3.24}$$

$$N(l,s)=V(l,s)+\frac{1}{\pi}\int_{s_0}^{\infty}ds'K(l;s',s)N(l,s')\,, \tag{3.25}$$

$$K(l;s',s)=\{[V(l,s)-V(l,s')]/(s-s')\}\rho(l,s')\,, \tag{3.26}$$

$$D(l,s)=1-\frac{1}{\pi}\int_{s_0}^{\infty}ds'\frac{\rho(l,s')}{s'-s}N(l,s')\,. \tag{3.28}$$

The "potential" $V(l,s)$ has only the left-hand cut of $B(l,s)$; $\rho(l,s)$ is two-body phase space times q^{2l}. Because the third double-spectral functions are absent, we have no poles at negative integer l in the left-hand discontinuity.[12] Thus, any pole in $V(l,s)$ must come from a divergence of the integral over the left-hand cut, and its residue will lack the left-hand cut. Since it cannot have a right-hand cut either, and has no essential singularity at infinity, it must be a rational function

$$V(l,s)=[1/(l+N)][P(s)/Q(s)]+V_N(l,s)\,, \tag{3.29}$$

where P and Q are polynomials. Thus the kernel (3.26) is

$$K(l;s',s)=\left[\frac{1}{l+N}\frac{1}{Q(s)Q(s')}\frac{P(s)Q(s')-P(s')Q(s)}{s-s'}+\frac{V_N(l,s)-V(l,s')}{s-s'}\right]\rho(l,s')\,. \tag{3.30}$$

$P(s)Q(s')-P(s')Q(s)$ is a polynomial with root $s=s'$, hence, we can cancel in the first term and obtain a separable residue for the pole in the kernel. Then, the resolvent of the kernel K has no pole at $l=-N$,[13] and (3.25) gives only a simple pole for $N(l,s)$.

Our simplest case is

$$gV(l,s)=g\{[1/(l+1)]+V_1(l,s)\}\,. \tag{3.31}$$

Here, the kernel has no pole at $l=-1$. The pole terms in N and D are

$$N(l,s)=\frac{g}{l+1}\left\{1+\frac{g}{\pi}\int_{s_0}^{\infty}\frac{ds'}{s'-s}\rho(-1,s)[V_1(-1,s')-V_1(-1,s)]+O(g^2)\right\}+\cdots\,, \tag{3.32}$$

$$D(l,s)=-\frac{g}{l+1}\frac{1}{\pi}\left\{\int_{s_0}^{\infty}\frac{ds'}{s'-s}\rho(-1,s')+\frac{g}{\pi}\int_{s_0}^{\infty}\frac{ds'}{s'-s}\rho(-1,s')\int_{s_0}^{\infty}\frac{ds''}{s''-s}\rho(-1,s'')\times[V_1(-1,s'')-V_1(-1,s')]+O(g^2)\right\}+\cdots\,. \tag{3.33}$$

From this we find that

$$A(-1,s)=-\left[\pi q^2\Big/\int_{s_0}^{\infty}\frac{ds'}{s'-s}\rho(-1,s')\right]-\left[gq^2\Big/\left[\int_{s_0}^{\infty}\frac{ds'}{s'-s}\rho(-1,s')\right]^2\right]\int_{s_0}^{\infty}\frac{ds'}{s'-s}\rho(-1,s')\times\int_{s_0}^{\infty}ds''\rho(-1,s'')\left[\frac{V_1(-1,s')-V_1(-1,s)}{s''-s}-\frac{V_1(-1,s'')-V_1(-1,s')}{s''-s'}\right]+O(g^2)\,. \tag{3.34}$$

We also see directly that

$$A(0,s)=gV(0,s)+O(g^2)\,. \tag{3.35}$$

We thus have the only possible form for integer symmetry

$$A(0,s)+A(-1,s)=-\pi q^2\Big/\left(\int_{s_0}^{\infty}\frac{ds'}{s'-s}\rho(-1,s')\right)\,. \tag{3.36}$$

[12] G. Chew, *The Analytic S Matrix* (W. A. Benjamin, Inc., New York, 1966), Sec. 9–4.

[13] One can see this by summing the Born series, where the fixed poles at each order sum to a moving pole.

We now need to put in the specific phase space for relativistic (R) and nonrelativistic (NR) scattering.

$$\rho_{\mathrm{NR}}(l,s)=q^{2l+1}, \tag{3.37a}$$

$$\rho_{\mathrm{R}}(l,s)=[\tfrac{1}{4}s-4m^2]^{2l}[(s-4m^2)/s]^{1/2}. \tag{3.37b}$$

Hence,

$$\int_{s_0}^{\infty}\frac{ds'}{s'-s}\rho_{\mathrm{NR}}(-1,s')=\frac{2\pi i}{q}, \tag{3.38a}$$

$$\int_{s_0}^{\infty}\frac{ds'}{s'-s}\rho_{\mathrm{R}}(-1,s')=4[s(s-4m^2)]^{-1/2}\times\ln\left(\frac{(s-4m^2)^{1/2}-s^{1/2}}{(s-4m^2)^{1/2}+s^{1/2}}\right), \tag{3.38b}$$

and we find our previous results again.

Now we look at our higher order terms and demand that they be zero. In particular, we insist on a zero right-hand cut to order g. $A(0,s)$ has no right-hand cut in this order. One can show that

$$\int_{s_0}^{\infty}\frac{ds'}{s'-s}\rho(-1,s')[V_1(-1,s')-V_1(-1,s)]$$

has no right-hand cut, and consequently neither does the double integral in (3.34). Thus, we need only look at the square of the integral over phase space. We see from (3.38) that nonrelativistically the square has no right-hand cut, and that relativistically it does. We know already that nonrelativistic scattering has the symmetry, so we have only shown consistency there. However, we have also shown that the relativistic symmetry *cannot* hold, for we have nonzero higher-order terms.

The one possible escape from this is the possibility of even higher-order terms cancelling the nonzero terms, which could occur if the amplitude were not analytic in g at $g=0$. As suggested in I, experiment could decide this last possibility.

IV. CONCLUSIONS

The positive results of this investigation are within potential scattering. First, a number of differences between integer and half-integer symmetry have been brought forth, particularly that the latter holds identically in such formulation as the Lippmann-Schwinger equation and is valid off the energy shell. The new methods of proof indicate that the symmetry of the wave functions need not be considered as central to the symmetry of the amplitude, but that the appearance of poles in the l plane is a more "basic" requirement. Indeed, the Bethe-Salpeter equation, which has functional symmetry but not integer symmetry (because of its relativistic dynamics), shows that the two phenomena need not appear simultaneously. Finally, we have a new series of sum rules that act as constraints on the amplitude.

The one negative result is very simple: There is no relativistic analog of integer Mandelstam symmetry. This in turn eliminates one possible way of looking at the left-hand l plane and a series of numerically interesting FESR. It is noteworthy that the difference lies solely in the different phase spaces.

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